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ALTERNATING 3-FORMS AND EXCEPTIONAL SIMPLE LIE GROUPS OF TYPE G_2.

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Preface. It is now customary to give concrete descriptions of the exceptional simple Lie groups of type G_2 as groups of automorphisms of the Cayley algebras. There is, however, a more elementary description. Let W be a complex 7-dimensional vector space. Among the alternating 3-forms on W there is a connected dense open subset $\Psi(W)$ of "maximal" forms. If $\psi \in \Psi(W)$ then the subgroup of $\mathrm{AUT}^{\mathbf{C}}(W)$ consisting of the invertible complex-linear transformations S such that $\psi(S\cdot, S\cdot, S\cdot) = \psi(\cdot, \cdot, \cdot)$ is denoted $G(\psi)$, and, in Proposition 3.6. we prove

$$G(\psi) = G_1(\psi) \times \{\epsilon I: \epsilon^3 = 1\}, \text{ direct product,}$$

where $G_1(\psi)$ is identified with the exceptional simple complex Lie group of dimension 14. Thus the complex Lie algebra $\mathfrak{g}(\psi)$ of type G_2 is defined in terms of the alternating 3-form ψ alone without the need to specify an invariant quadratic form. In the real case the result is more striking. The generic real alternating 3-form ψ on a 7-dimensional real vector space V extends to an element of $\Psi(W)$ for $W = \mathbf{C} \otimes_{\mathbf{R}} V$, but in the real domain there are two cases. The case where ψ is "maximal" gives rise to a real form $G^{\theta}(\psi)$ of $G(\psi)$ which is compact! Then there is an invariant positive-definite quadratic form φ, but this is determined by ψ. In the other case ψ is "pseudo-maximal" and the corresponding group $G^{\gamma}(\psi)$ is a version of the doubly-connected, connected, non-compact, exceptional simple Lie group of dimension 14.

Most of the results about groups of type G_2 may be found in the works of Elie Cartan. He defines the complex group [**3**, p. 297] in terms of an alternating 3-form ψ and a non-degenerate symmetric quadratic form φ as

$$G_1(\varphi, \psi) = \text{component of the identity in } \{S \in \mathrm{Aut}^{\mathbf{C}}(W): S\psi = C(S)\psi,\ S\varphi = \varphi\}$$

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where

$$S\psi(\cdot, \cdot, \cdot) = \psi(S^{-1}\cdot, S^{-1}\cdot, S^{-1}\cdot), \quad S\varphi(\cdot, \cdot) = \varphi(S^{-1}\cdot, S^{-1}\cdot).$$

In Section 1 we consider some generalities about groups of the form

$$G(\varphi, \psi) = \{S \in \mathrm{Aut}^{\mathbf{C}}(W): S\psi = \psi, S\varphi = \varphi\}.$$

These are necessarily groups of automorphisms of quadratic algebras. For background material use [**8**]. The only thing in Section 1 which is vital for the sequel is formula (1.23).

The main ideas of this paper occur in Section 2 where we show that the group $G_0(\psi) = G^{\theta}(\psi)$ of linear transformations of $V \simeq \mathbf{R}^7$ leaving a maximal 3-form ψ invariant is the compact group of type G_2. Section 3 treats the complex case and Section 4 the non-compact, doubly connected, connected real simple Lie group $G^{\gamma}(\psi)$ of type G_2. Theorems (2.9) and (4.9) classify the "generic" 3-forms on $\mathbf{R}^7$. Theorem (3.10) describes the "maximal" 3-forms on $\mathbf{C}^7$. The symmetric spaces are described concretely in Theorems (2.14) and (4.5).

In Section 5 we give very concrete descriptions of the boundaries and parabolic subgroups of $G^{\gamma}(\psi)$. We shall return to this subject later. One of our principal objects was to obtain convenient matrix descriptions of these objects with a view to analytic applications. There is some historical interest in the results since Cartan's first work [1] presents the Lie algebra $\mathfrak{g}^{\gamma}(\psi)$ as infinitesimal transformations of the minimal boundaries B_{α} and B_{β}. In Section 6 we establish the global forms of some local statements about G_2 in Cartan's thesis [**2**].

This work arose from conversations with my student, Maurice Chayet, and many of the results come from discussions with him. He deserves not only my thanks but also a substantial amount of credit. We hope to obtain similar concrete descriptions for other exceptional simple groups. An effort has been made here to avoid explicit use of the Cayley product wherever possible. In a few places, e.g. the proof of Theorem (2.13) I have not succeeded. Unfortunately, while brute force calculations for G_2 are manageable, they get out of hand for the other exceptional groups.

Notational conventions. If V is a vector space $\mathrm{END}^{\mathbf{R}}(V)$ denotes the ring of $\mathbf{R}$-linear transformations of V; similarly $\mathrm{AUT}^{\mathbf{R}}(V)$ is a group, etc. If σ is an involution of V then V^{σ} denotes the fixed subspace of σ; for a group G contained in $\mathrm{AUT}(V)$, G^{σ} is the centralizer of σ in G. When a non-degenerate quadratic form φ is fixed in the context we write # to indicate the transpose, thus

$$\varphi(X^{\#}\cdot, \cdot) = \varphi(\cdot, X\cdot)$$

defines $X^{\#}$ for $X \in \mathrm{END}(V)$. If $v \in V$ then $v^{\#}$ is the linear functional $w \mapsto \varphi(w, v)$ and $uv^{\#}$ is the linear transformation $w \mapsto \varphi(w, v)$u.

1. Vector-products. A complex conjugation γ for a complex vector space W is an element $\gamma \in \mathrm{AUT}^{\mathbf{R}}(W)$ such that $\gamma^2 = \mathrm{I}$ and

$$\gamma(cv) = \bar{c}\gamma v \quad \text{for } v \in V, c \in \mathbf{C}.$$

If V is finite-dimensional and φ is a non-degenerate symmetric bilinear form on W then (W, φ) has an hermitean complex conjugation θ, a complex conjugation such that

(1.1) $\quad <u, v> = \varphi(u, \theta v)$

defines a Hilbert space inner product on W.

Let ψ be an alternating 3-form on W. One says ψ is a non-degenerate if for each $w \in W \setminus \{0\}$ $\psi(\cdot, \cdot, w)$ is a non-trivial 2-form. If γ is a complex conjugation W then $\gamma\psi$ defined by

$$\gamma\psi(u, v, w) = \bar{\psi}(\gamma u, \gamma v, \gamma w)$$

is an alternating 3-form which is non-degenerate if and only if ψ is. The forms

$$\mathrm{Re}_\gamma\psi = \tfrac{1}{2}(\psi + \gamma\psi), \quad \mathrm{Im}_\gamma\psi = \frac{1}{2i}(\psi - \gamma\psi)$$

are invariant under γ and

$$\psi = \mathrm{Re}_\gamma\psi + i\,\mathrm{Im}_\gamma\psi.$$

Hence, if ψ is non-degenerate its restriction to W^γ is a real non-degenerate form.

(1.2) *Definition.* A *vector-product structure* for (W, φ) is an alternating 3-form ψ such that $\theta\psi = \psi$ where θ is an hermitean complex conjugation.

The vector-product structure ψ determines a monomorphism of complex vector spaces

$$W \xrightarrow{L} \mathrm{ANTI}^{\mathbf{C}}(W, \varphi), \quad u \mapsto L_u$$

defined by

(1.3) $\quad \varphi(L_u, v, w) = \psi(u, v, w).$

Here $\mathrm{ANTI}^{\mathbf{C}}(W,\varphi)$ is the sub-Lie algebra of $\mathrm{END}^{\mathbf{C}}(W)$ consisting of the elements L such that

$$\varphi(Lv, w) = -\varphi(v, Lw).$$

The vector-product is the complex-linear map

(1.4) $\quad W \otimes_{\mathbf{C}} W \to W; \quad u \otimes v \mapsto L_u v = u \times v.$

(1.5) *Definition.* The non-associative complex algebra with unity given by a vector product structure (W, φ, ψ) is

$$\mathrm{ALG}(W, \varphi, \psi) = \mathbf{C}1 + W$$

with the product

$$(a1 + u)(b1 + v) = (ab - \varphi(u, v))1 + (av + bu - u \times v).$$

The adjoint associated to θ is

$$(a1 + u)^* = \bar{a}1 - \theta u.$$

(1.6) THEOREM. *The algebras A over $\mathbf{C}$ which are "quadratic", i.e., each $x \in A$ satisfies an equation*

$$x^2 - 2\tau(x)x + \varphi(x)1 = 0 \quad \textit{with } \tau(x), \varphi(x) \in \mathbf{C},$$

satisfy the "flexible law"

$$x(yx) = (xy)x,$$

and have a "positive-definite adjoint", i.e., an $\mathbf{R}$-linear involution $x \mapsto x^*$ such that

$$(c1)^* = \bar{c}1, \ (xy)^* = y^*x^*, \textit{ and } \tau(x^*x) > 0 \quad \textit{for all } x \in A\text{-}\{0\}$$

are precisely the algebras $\mathrm{ALG}(W, \varphi, \psi)$ *defined in* (1.5) *with*

$$W = \{x \in A: \tau(x) = 0\}, \ \varphi(v) = \varphi(v, w) \quad \textit{for } v \in W, \quad \textit{and}$$
$$\psi(u, v, w) = -\tau(uvw) \quad \textit{for } u, v, w \in W.$$

Proof. It is a routine calculation that $\mathrm{ALG}(W, \varphi, \psi)$ satisfies all the conditions. Conversely, in any quadratic algebra we have

$$\tau(uv) + \tau(vu) = -2\varphi(u, v)$$

where $\varphi(u, v)$ is defined by

$$2\varphi(u, v) = \varphi((u + v)^2) - \varphi(u^2) - \varphi(v^2).$$

It follows from the flexible law that τ is a commutative and associative trace form:

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$$\tau(xy) = \tau(yx) \quad \text{and} \quad \tau((xy)z) = \tau(x(yz)).$$

The existence of the positive-definite adjoint only serves to give the existence of an hermitean complex conjugation which stabilizes ψ.

(1.7) *Definition.* For a vector product (W, φ, ψ) the group $G(\varphi, \psi)$ is the group of automorphisms of $\mathrm{ALG}(W, \varphi, \psi)$ over $\mathbf{C}$. The Lie group structure is that inherited as a closed subgroup of $\mathrm{AUT}^{\mathbf{C}}(W)$.

It is obvious that

$$G(\varphi, \psi) = \mathrm{AUT}(W, \varphi) \cap G(\psi)$$

where $\mathrm{AUT}(W, \varphi)$ is the orthogonal group for the non-degenerate symmetric bilinear form φ and $G(\psi)$ is the subgroup of $\mathrm{AUT}(W)$ preserving the 3-form ψ. These groups are not affected if one multiplies the forms by non-zero constants, but such changes alter the algebra $\mathrm{ALG}(W, \varphi, \psi)$. In particular, if one replaces φ by $c^{-1}\varphi$ then L_u is replaced by cL_u for each $u \in V$. This is but one indication that a group $G(\varphi, \psi)$ can be presented as the group of automorphisms of several different algebras.

The Lie algebra $\mathfrak{g}(\varphi, \psi)$ of $G(\varphi, \psi)$ is the algebra of derivations of $\mathrm{ALG}(W, \varphi, \psi)$. Thus, we have

$$(1.8) \qquad \mathfrak{g}(\varphi, \psi) = \{X \in \mathrm{END}^{\mathbf{C}}(V): D_X\varphi = 0, D_X\psi = 0\}$$

where

$$D_X\varphi(\cdot, \cdot) = -\varphi(X\cdot, \cdot) - \varphi(\cdot, X\cdot) \quad \text{and}$$
$$D_X\psi = -\psi(X\cdot, \cdot, \cdot) - \psi(\cdot, X\cdot, \cdot) - \psi(\cdot, \cdot, X\cdot).$$

One has

$$(1.9) \qquad X(u \times v) = (Xu) \times v + u \times (Xv) \quad \text{for } X \in \mathfrak{g}(\varphi, \psi).$$

We shall now investigate some examples of this Lie algebra.

Since φ is non-degenerate there exists $A \in \mathrm{END}^{\mathbf{C}}(W)$ such that

$$(1.10) \qquad \operatorname{trace} L_uL_v = -\varphi(Au, v).$$

Obviously, A is φ-symmetric, but the existence of an hermitean complex conjugation θ for the vector product structure (W, φ, ψ) gives more. The adjoint for the Hilbert space obtained by giving W the inner product (1.1) is $X \mapsto X^*$ where

$$X^* = \theta X^{\#}\theta, \quad X^{\#} = \varphi\text{-transposed of } X.$$

Since each L_u is anti-symmetric, i.e., $L_u^{\#} = -L_u$, we have

$$L_u^* = -\theta L_u\theta = L_{\theta u}.$$

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If $u \neq 0$ then $L_u \neq 0$ and $\operatorname{tr} L_u L_u^* > 0$. This gives

(1.11) PROPOSITION. *For a vector product structure* (W, φ, ψ) *the linear transformation* A *defined by* (1.10) *is semi-simple with strictly-positive eigenvalues. Moreover* A *commutes with the elements of* $G(\varphi, \psi)$.

This leads us to define a linear map

$$\mathrm{END}^{\mathbf{C}}(W) \xrightarrow{\Delta} \mathrm{ANTI}^{\mathbf{C}}(W, \varphi) \text{ by}$$

(1.12) $\varphi(\Delta(X)v, w) = \operatorname{tr} X[L_v, L_w]$ for $v, w \in W$.

Some direct calculations give

LEMMA. *The map* Δ *given by* (1.11) *has the properties*:

(1.13) $\Delta(X) = 0$ *if* $X = X^{\#}$,

(1.14) $\Delta(X) = -AX$ *if* $X \in \mathfrak{g}(\varphi, \psi)$,

(1.15) $\Delta(uv^{\#}) = [L_u, L_v]$,

(1.16) $\Delta(L_u) = \sum_{i=1}^{n} [L_{e_i}, L_{e_i \times u}]$

where

$e_i, \ldots, e_n$ *is an* φ*-orthonormal basis for* W.

Proof. If $Y \in \mathrm{ANT1}^{\mathbf{C}}(W, \varphi)$ and X is φ-symmetric then $\operatorname{tr} XY = 0$. This gives (1.13) with $Y = [L_u, L_v]$. For (1.14) one uses

$$\operatorname{tr} X[L_u, L_v] = \operatorname{tr} [X, L_u]L_v.$$

If $X \in \mathfrak{g}(\varphi, \psi)$ then $[X, L_u] = L_{xu}$ and

$$\operatorname{tr} L_{Xu}L_v = -\varphi(AXu, v)$$

by (1.10). (1.15) and (1.16) are straightforward computations.

(1.17) *Example.* Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. Put W for the underlying vector space of $\mathfrak{g}$,

$$\varphi(u, v) = -\mathrm{KILL}(u, v), \quad \psi(u, v, w) = -\mathrm{KILL}(u, [v, w])$$

where KILL is the Killing form. Then (W, φ, ψ) is a vector product structure since one can take θ a compact complex conjugation for $\mathfrak{g}$. One has $L_u = \operatorname{Ad} u$ and $\mathfrak{g}(\varphi, \psi) \simeq \mathfrak{g}$, $G(\varphi, \psi) = \mathrm{AUT}^{\mathbf{C}}(\mathfrak{g})$, the group of Lie-algebra automorphisms of the complex Lie algebra $\mathfrak{g}$. Here $A = I$.

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This shows that every semi-simple Lie algebra is the algebra of derivations of some $\mathrm{ALG}(W, \varphi, \psi)$ as described in Theorem (1.6). The interesting question is when semi-simple Lie algebras arise in a non-trivial way. We shall be concerned with the case in which $\mathrm{ALG}(W, \varphi, \psi)$ is the complex Cayley algebra. The corresponding vector product structure is the *Cayley product.* We shall not use the properties of the Cayley algebra directly, but rather some facts developed in Section 2 following. The link with the usual presentations is based on

(1.18) PROPOSITION. $\mathrm{ALG}(W, \varphi, \psi)$ *is alternative if and only if the vector product* (W, φ, ψ) *satisfies the Lagrange identity* (2.4) *below, i.e., for all* $u \in W$,

$$L_u^2 = -\varphi(u, u)I + uu^{\#}.$$

Proof. To say that $\mathrm{ALG}(W, \varphi, \psi)$ is alternative is to say $\Lambda_u^2 = \Lambda_{u^2}$ where Λ_v is left-multiplication by $v \in \mathrm{ALG}$ viewed as a linear transformation of $\mathbf{C}1 + W$. Reinterpretation in terms of L_u for $u \in W$ is the Lagrange identity.

This identity has some vital consequences.

(1.19) PROPOSITION. *Suppose the vector product structure* (W, φ, ψ) *satisfies the Lagrange identity. Then for the operator* A *defined by* (1.10) *one has*

(1.20) $\quad A = (n - 1)I$

and for the map Δ *of* (1.12),

(1.21) $\quad \Delta(L_u) = (2n - 8)L_u$

where $n = \dim_{\mathbf{C}} W$.

Proof. Taking the trace of the Lagrange identity gives

$$\mathrm{tr}\, L_u^2 = -(n - 1)\varphi(u, u),$$

and this is (1.20). Formula (1.16) combined with a consequence (2.8) of the Lagrange identity.

(1.22) THEOREM. *Suppose* (W, φ, ψ) *satisfies the Lagrange identity and* $\dim_{\mathbf{C}} W \neq 3$. *Then* $\dim_{\mathbf{C}} W = 7$, *and there is an orthogonal direct sum decomposition*

$$\mathrm{ANT1}^{\mathbf{C}}(W, \varphi) = \mathfrak{g}(\varphi, \psi) + L(W)$$

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where $L(W) = \{L_u: u \in W\}$. *Moreover,* $\mathfrak{g}(\varphi, \psi)$ *is generated by the elements*

(1.23) $D(u, v) = [L_u, L_v] + L_{u\times v}$.

Proof. Consider $\mathrm{ANT1}^{\mathbf{C}}(W, \varphi)$ as a Hilbert space with inner product $<X, Y> = \operatorname{tr} XY^*$. Note that for all $X \in \mathrm{ANT1}^{\mathbf{C}}(W, \varphi)$ one has

$$\begin{aligned}\operatorname{tr} \Delta(X)L_u &= \sum \varphi(\Delta(X)L_u e_i, e_i)\\ &= \sum \operatorname{tr} X[L_{u\times e_i, e_i}] \quad \text{by definition of } \Delta, (1.12)\\ &= \operatorname{tr} X\,\Delta(L_u) \quad \text{by (1.16)}\\ &= (2n-8)\operatorname{tr} X\, L_u \quad \text{by (1.21).}\end{aligned}$$

We conclude that if X is orthogonal to $L(W)$ then $\Delta(X) \perp L(W)$. One readily computes that the projection of $uv^{\#} - vu^{\#}$ on $L(W)$ is $-2(n-1)^{-1}L_{u\times v}$. Thus

$$uv^{\#} - vu^{\#} = Y - 2(n-1)^{-1}L_{u\times v} \quad \text{where } Y \perp L(W).$$

Applying the map Δ and using (1.15) and (1.21) we get

$$2[L_u, L_v] = \Delta(Y) - 2L_{u\times v};$$

so $\Delta(Y) = 2D(u, v)$ according to (1.23) as definition.

Formula (2.8) below is

$$uv^{\#} - vu^{\#} = -\tfrac{1}{3}D(u, v) - \tfrac{1}{3}L_{u\times v}.$$

Since $D(u, v) = \frac{1}{2}\Delta(Y)$ is orthogonal to $L(W)$, it follows that $\frac{1}{3} = 2(n-1)^{-1}$, i.e., $n = 7$. It also follows that the $D(u, v)$ span $L(W)^{\perp}$. Now suppose $X \in \mathfrak{g}(\varphi, \psi)$. By (1.14) and (1.20) we have

$$-6 \operatorname{tr} X\, L_u = \operatorname{tr} \Delta(X)L_u = \operatorname{tr} X\, \Delta(L_u) = 6 \operatorname{tr} X\, L_u.$$

Therefore $\mathfrak{g}(\varphi, \psi) \subset L(W)^{\perp}$. The argument at the beginning of Section 2 shows that $\dim \mathfrak{g}(\varphi, \psi) \geqq 14$; hence $\mathfrak{g}(\varphi, \psi) = L(W)^{\perp}$.

2. Cayley products. Let ψ be an alternating 3-form on a real vector space V. We say that ψ is *maximal* if for each $u \in V\setminus\{0\}$ the 2-form ψ_u defined by $\psi_u(\cdot, \cdot) = \psi(u, \cdot, \cdot)$ gives a symplectic structure to $V/\mathbf{R}u$. This can only occur for V odd-dimensional, say $\dim V = 2m + 1$. To within scalar multiples there is a unique non-trivial $(2m+1)$-form δ on V, and its contraction δ_u gives the non-trivial $2m$-form on $V/\mathbf{R}u$. Since ψ_u is a sympletic structure on $V/\mathbf{R}u$ we have

(2.1) $\psi_u \wedge \overset{m}{\cdots} \wedge \psi_u = m!\varphi(u)\delta_u$

where $\varphi: V \to \mathbf{R}$ is homogeneous of degree $m - 1$ and $\varphi(u) = 0$ only if $u = 0$. The last forces m to be odd. If we change δ to $c\delta$ with $c \in \mathbf{R} \setminus \{0\}$ then φ changes to $c^{-1}\varphi$. The orientation of δ is fixed by the condition $\varphi(u) > 0$ for $u \in V \setminus \{0\}$. If we further impose a suitable normalization condition then the constant c is completely determined. Thus for example, when $m = 1$ the normalization is $\varphi \equiv 1$, $\delta \equiv \psi$.

The interesting case is $m = 3$, for then φ is a positive definite quadratic form on V. By polarization we obtain a bilinear form also denoted φ. The normalization is such that $\delta^2(e_1, \ldots, e_7) = 1$ for any φ-orthonormal basis $e_1, \ldots, e_7$ for V, and the basic identity is equivalent to

$$(2.2) \qquad \psi \wedge \psi_u \wedge \psi_v = 6\, \varphi(u, v)\delta.$$

Put $G_0(\psi)$ for the subgroup of $\mathrm{AUT}^{\mathbf{R}}(V)$ leaving ψ fixed under the induced action on alternating 3-forms. Since the space of alternating 3-forms on V_0 has dimension 35 and $\dim \mathrm{AUT}^{\mathbf{R}}(V) = 49$ we conclude that $\dim G_0(\psi) \geqq 14$. On the other hand

(2.3) $G_0(\psi)$ is a subgroup of $\mathrm{AUT}^+(V, \varphi)$,

the latter group being a version of $S_0(7)$.

A dimension-counting argument gives the crucial

(2.4) LAGRANGE IDENTITY. $u \times (u \times v) = -\varphi(u, u)v + \varphi(u, v)u.$

Proof. Consider $u \in V \setminus \{0\}$. The orbit $G_0(\psi)u$ has dimension $\leqq 6$. From the fact that $\dim G_0(\psi) \geqq 14$ we conclude that, for the subgroup $G_0(\psi, u)$ leaving u fixed,

$$\dim G_0(\psi, u) \geqq 8.$$

Now $G_0(\psi, u)$ acts as orthogonal transformation on

$$u^\perp = \{v \in V: \varphi(u, V) = 0\}$$

while L_u is a skew-symmetric linear transformation which commutes with the elements of $G_0(\psi, u)$. Thus in some orthonormal basis for $u^\perp$, L_u has the matrix

$$\begin{bmatrix} 0 & a & & & & \\ -a & 0 & & & & \\ & & 0 & b & & \\ & & -b & 0 & & \\ & & & & 0 & c \\ & & & & -c & 0 \end{bmatrix}$$

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The basic identity (2.1) gives $a^2b^2c^2 = \varphi(u, u)$. Unless $a^2 = b^2 = c^2$ there is a 2-dimensional $G_0(\psi, u)$-invariant subspace which implies

$$\dim G_0(\psi, u) \geqq \dim O(2) \times O(4) = 7,$$

a contradiction. The conclusion is that

$$L_u^2 = -\varphi(u, u)I_{u^\perp},$$

and this is the Lagrange identity.

The polarized form of the identity is

$$(2.5) \qquad L_uL_v + L_vL_u = uv^{\#} + vu^{\#} - 2\varphi(u, v)I$$

where $\#$ designates the transposed with respect to φ. From this we obtain

$$(2.6) \qquad (u \times v) \times (u \times w) = \varphi(u, u)v \times w + 2\psi(u, v, w)u + \varphi(u, v)u \times w - \varphi(u, w)u \times v.$$

(The trick is to use (2.5) on L_zL_wu with $z = u \times v$.)

This identity gives

(2.7) PROPOSITION. *For each $u \in V \setminus \{0\}$ the group $G_0(\psi, u)$ is a version of $SU(3)$.*

Proof. We may assume $\varphi(u, u) = 1$ so that $L_u^2 = -I_{u^\perp}$. Thus L_u is a complex structure for $u^\perp$. One easily calculates that

$$\langle v, w\rangle = \varphi(v, w) - i\psi(u, v, w)$$

is a Hilbert space inner product on $u^\perp$ with the complex structure L_u and $G_0(\psi, u)$ is a subgroup of the unitary group. Now define a 3-form χ on $u^\perp$ by

$$\chi(v_1, v_2, v_3) = \psi(v_1, v_2, v_3) + i\psi(L_uv_1, L_uv_2, L_uv_3).$$

With the aid of (2.6) one verifies that χ is a complex 3-form. It is non-trivial. If S is any complex automorphism of $u^\perp$ we have

$$\chi(Sv_1, Sv_2, Sv_3) = (\det_{\mathbf{C}} S)\chi(v_1, v_2, v_3)$$

from which we conclude that $\det_{\mathbf{C}} S = 1$ for $S \in G_0(\psi, u)$. Since

$$\dim G_0(\psi, u) \geqq 8,$$

it follows that $G_0(\psi, u)$ is all of $SU(3)$.

If one replaces w by $u \times w$ in (2.6) and uses (2.4) he obtains the identity

(2.8) $$[L_u, L_v] + 2L_{u\times v} - 3(uv^{\#} - vu^{\#}) = 0$$

where $[A, B] \equiv AB - BA$. This enables us to prove

(2.9) THEOREM. *Put $\Omega_3^{\mathbf{R}}(V)$ for the 35-dimensional vector space of real alternating 3-forms on the 7-dimensional real vector space V. The maximal forms constitute a subset $\Psi(V) \cup -\Psi(V)$ which is a single orbit of* $\mathrm{AUT}^{\mathbf{R}}(V)$ *for the induced action on $\Omega_3^{\mathbf{R}}(V)$. The subset $\Psi(V)$ is open and connected. Given a positive-definite quadratic form φ on V, the set of $\psi \in \Psi(V)$ which yield φ in* (2.2) *constitute a compact, connected, 7-dimensional regular submanifold $\Psi(V, \varphi)$ which is a single orbit for the action of* AUT^+ $(V, \varphi) \simeq SO(7)$ *on $\Omega_3^{\mathbf{R}}(V)$.*

Proof. Choose an orientation for V. According to (2.2) each maximal 3-form ψ determines a non-trivial 7-form δ; put $\Psi(V)$ for the collection of those ψ such that δ is positively oriented. Since all positive-definite quadratic forms are AUT^+ (V)-equivalent, it suffices to fix φ and prove that $\Psi(V, \varphi)$ is an orbit of $\mathrm{AUT}^+(V, \varphi)$. To this effect consider ψ and ψ' in $\Psi(V, \varphi)$ and fix $u \in V$ with $\varphi(u, u) = 1$. Then L_u and L'_u are both φ-orthogonal complex structures for $u^\perp$. Hence, as operators on $u^\perp$, L'_u is $\mathrm{AUT}^+(u^\perp, \varphi)$-conjugate to $\pm L_u$, i.e., there exists $S \in \mathrm{AUT}^+(V, \varphi)$ with

$$Su = u \quad \text{and} \quad L'_u = \pm SL_uS^{-1};$$

the $+$ sign must be chosen for the orientations to be the same. Replacing ψ' by $S^{-1}\psi'$ we have $L'_u = L_u$. Now choose $v \in V$ with $\varphi(u, v) = 0$, $\varphi(v, v) = 1$. Put

$$w = u^\perp \cap v^\perp \cap (u \times v)^\perp;$$

Note that $u \times v = L_u v = L'_u v$ is unambiguously defined. By (2.4), L_v and L'_v agree on $W^\perp$.

Moreover W is an invariant subspace for L_u, L_v, and L'_v. Let M_u be the restriction of L_u to W and M the restriction of L_v or L'_v. Then we have, see (2.4) and (2.5),

$$M_u^2 = -I, \quad M^2 = -I, \quad M_uM + MM_u = 0$$

where both M_u and M are skew symmetric. Choose a basis such that

$$M_u + \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C, D 2×2 matrices. Since M is skew-symmetric we get

$$A = (\cos 2\alpha)J \quad \text{for some } \alpha \in \mathbf{R}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The addition equations yield

$$B = (\sin 2\alpha)\, J = C, \quad D = -(\cos 2\alpha)J.$$

Thus $M = R(\alpha)M_vR^{-1}(\alpha)$ where

$$\mathrm{R}(\alpha) = \begin{pmatrix} \cos \alpha I & -\sin \alpha I \\ \sin \alpha I & \cos \alpha I \end{pmatrix}, \quad M_v = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

Without loss of generality one may suppose that M_v is the restriction of L_v to W. Extending $R(\alpha)$ orthogonally as the identity on $W^{\perp}$ we have

$$L'_v = R(\alpha)L_vR^{-1}(\alpha), \quad L_u = R(\alpha)L_uR^{-1}(\alpha).$$

Hence, if we replace ψ' by $R^{-1}(\alpha)\psi'$ we have $\psi, \psi' \in \Psi(V, \varphi)$ with the same left-multiplications L_u and L_v. The identity (2.8) shows that $L'_{u\times v} = L_{u\times v}$, and the Lagrange identity allows one to complete the multiplication table given L_u, L_v and $L_{u\times v}$. Therefore $\psi' = \psi$. The rest is routine.

(2.10) COROLLARY. *The groups $G_0(\psi)$ for ψ a maximal 3-form are conjugate in* $\mathrm{AUT}^+(V)$. *If ψ and ψ' are maximal 3-forms yielding the same quadratic form φ then $G_0(\psi)$ and $G_0(\psi')$ are conjugate in $AUT^+(V, \varphi)$.*

One knows that the Cayley numbers have the form $\mathrm{ALG}^{\mathbf{R}}(V, \varphi, \psi)$, see (1.5), for some maximal 3-form ψ, but (2.9) asserts that all these algebras are isomorphic. Thus the groups $G_0(\psi)$ are versions of the exceptional simple compact group G_2. The simplicity of $G_0(\psi)$ is a standard algebraic fact we require. From what has gone before we know that $G_0(\psi)/G_0(\psi, u)$, where $u \in V\setminus\{0\}$, is a compact 6-dimensional submanifold of the sphere S^6, hence it is S^6. From (2.7) we conclude that $G_0(\psi)$ is connected and simply-connected. To summarize we record

(2.11) THEOREM. *The groups $G_0(\psi)$ for ψ a maximal 3-form are compact, connected, simply-connected, simple, 14-dimensional Lie groups.*

Put $\mathrm{STIEF}(3, C_0, \varphi)$ for the Stiefel manifold of φ-orthonormal 3-frames in V_0. The *Zorn manifold* is defined as, cf. [**9**],

$$\text{(2.12)} \qquad \Gamma_0(\psi) = \{P \in \mathrm{STIEF}(3, V_0, \varphi): \psi(P_1, P_2, P_3) = 1\}.$$

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(2.13) THEOREM. *For each* $P \in \Gamma_0(\psi)$ *the map* $S \mapsto SP$ *is a diffeomorphism of* $G_0(\psi)$ *onto* $\Gamma_0(\psi)$. *The subgroup of* $\mathrm{AUT}^{\mathbf{R}}(V_0)$ *mapping* $\Gamma_0(\psi)$ *to itself is* $\{\pm I\} \times G_0(\psi)$.

Proof. Since $\dim G_0(\psi) \geqq 14$ it follows from (2.7) that $G_0(\psi)$ acts transitively on the 6-sphere $\{u \,\varphi\, V_0 : \varphi(u, u) = 1\}$. Therefore, to show that $G_0(\psi)$ maps onto $\Gamma_0(\psi)$ we have only to show that

$$G_0(\psi, P_1)P = \{Q \in \Gamma_0(\psi) : Q_1 = P_1\}.$$

The group $G_0(\psi;P_1)$ being a version of $SU(3)$ acts transitively on the unit sphere in $P_1^{\perp}$. Thus we need only show that

$$G_0(\psi, P_1, P_2)P = \{Q \in \Gamma_0(\psi) : Q_1 = P_1, Q_2 = P_2\}.$$

Now $G_0(\psi, P_1, P_2)$ is a version of $SU(2)$ action of $P_1^{\perp} \cap P_2^{\perp} \cap (P_1 \times P_2)^{\perp}$ with complex structure L_{P_1}. The action on the unit sphere is a diffeomorphism of $SU(2)$ with S^3. Not only have we proved the map onto but also one-to-one. Finally, suppose $S \in \mathrm{AUT}^{\mathbf{R}}(V_0)$ leaves $\Gamma_0(\psi)$ invariant. Correcting by an element of $G_0(\psi)$ we may assume $SP = P$. Thus S acts as orthogonal transformations of the 4-dimensional space $P_1^{\perp} \cap P_2^{\perp} \cap P_3^{\perp}$. If $Q \in P_1^{\perp} \cap P_2^{\perp} \cap P_3^{\perp}$ and $\varphi(Q, Q) = 1$ then

$$(P_1, P_2, Q) \in \Gamma_0(\psi) \quad \text{if } Q \perp (P_1 \times P_2).$$

It follows that

$$S(P_1 \times P_2) = \epsilon P_1 \times P_2 \quad \text{where } \epsilon = \pm 1.$$

One now considers the point

$$(2^{-1/2}(P_1 + P_3), P_1 \times P_2, 2^{-1/2}(P_2 - (P_1 \times P_2) \times P_3)) \in \Gamma_0(\psi);$$

the forced conclusion is that

$$S[(P_1 \times P_2) \times P_3] = (P_1 \times P_2) \times P_3.$$

Similar arguments give

$$S(P_1 \times P_3) = \epsilon P_1 \times P_3 \quad \text{and} \quad S(P_2 \times P_3) = \epsilon\, P_2 \times P_3.$$

Thus $\det S = \epsilon^3$, so $S \in G_0(\psi)$ if $\det S > 0$.

Consider the Grassmann manifold $\mathrm{GRASS}(4, V_0)$ of 4-planes through the origin in V_0. Endowed with the metric induced by a positive-definite quadratic form φ, $\mathrm{GRASS}(4, V_0)$ is a doubly-connected Riemannian symmetric space. Each 4-plane π is the fixed point of a unique geodesic symmetry $-\sigma$ where $\sigma \in \mathrm{AUT}^+(V_0, \varphi)$ is the orthogonal involution whose -1 eigenspace $V_0^{-\sigma}$ is π and whose $+1$ eigenspace V_0^{σ} is $\pi^{\perp}$.

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(2.14) THEOREM. *Given a maximal 3-form ψ on V_0 put $\Sigma_0(\psi)$ for the subset of* GRASS(4, V_0) *constituted by the ψ-isotropic 4-planes π, i.e.,*

$$\psi(u, v, w) = 0 \quad \text{for all } u, v, w \in \pi.$$

Then $\Sigma_0(\psi)$ is a regularly imbedded 8-dimensional, simply-connected, compact submanifold of GRASS(4, V_0). *If* GRASS(4, V_0) *is given the Riemannian structure induced by φ, the positive-definite quadratic form derived from ψ, then $\Sigma_0(\psi)$ is the sub-symmetric space constituted by the 4-dimensional subspaces π of V_0 of the form $\pi = V_0^{-\sigma}$ with $\sigma \in G_0(\psi)$. The group of isometries of $\Sigma_0(\psi)$ is $G_0(\psi)$, and the subgroup $G_0^{\sigma}(\psi)$ leaving $\pi = V_0^{-\sigma} \in \Sigma_0(\psi)$ fixed is canonically isomorphic with* $\mathrm{AUT}^+(V_0^{-\sigma}, \varphi)$.

Proof. Consider the map f: $\Gamma_0(\psi) \to$ GRASS(4, V_o) given by $f(P) = \pi$ where π is the 4-dimensional subspace spanned by P_1, P_2, P_3 and $(P_1 \times P_2) \times P_3$. It is clear that

$$f(\Gamma_0(\psi)) \subset \Sigma_0(\psi).$$

On the other hand given $\pi \in \Sigma_0(\psi)$, any three orthonormal vectors P_1, P_2, $P_3 \in \pi$ determine a point $P \in \Gamma_0(\psi)$. Therefore

$$f(\Gamma_0(\psi)) = \Sigma_0(\psi).$$

By (2.13), given $P \in \Gamma_0(\psi)$ there is a unique $\sigma \in G_0(\psi)$ such that $\sigma P = -P$. Since $\sigma^2 P = P$ it follows that $\sigma^2 = I$. It is also obvious that $f(P) \subset V_0^{-\sigma}$, but one sees that dim $V_0^{-\sigma} = 4$. It follow that $f(P) = V_0^{-\sigma}$, and that $-\sigma$ is the geodesic symmetry of GRASS(4, V_0) in the φ-metric leaving $f(P)$ invariant. Since $f(SP) = Sf(P)$ for $S \in G_0(\psi)$ we have that $\Sigma_0(\psi)$ is a homogeneous space of $G_0(\psi)$. Given $P \in \Gamma_0(\psi)$, we obtain a homorphism

$$\mathrm{AUT}^+(f(P), \varphi) \to G_0(\psi)$$

by $T \mapsto S$ where S is the unique element of $G_0(\psi)$ such that

$$SP = (TP_1, TP_2, TP_3).$$

This is a monomorphism for if $TP_k = P_k$, $k = 1, 2, 3$, for 3-orthonormal vectors in a 4-dimensional orthogonal space and T is a rotation then $T = I$. The subgroup of $G_0(\psi)$ leaving $\pi \in \Sigma_0(\psi)$ fixed is clearly the centralizer $G_0^{\sigma}(\psi)$ of σ where $\pi = V_0^{-\sigma}$. The action of $G_o^{\sigma}(\psi)$ on the vector space $V_0^{-\sigma}$ given by restriction of the action on V_0 is a homomorphism

$$G_0^{\sigma}(\psi) \to \mathrm{AUT}^+(V_0^{-\sigma}, \varphi).$$

We have constructed an inverse above; then the restriction map is a canonical isomorphism. The rest is routine.

See [**6**] for material on symmetric spaces and matters used here and in section 4.

3. Complex Cayley products. Let W be a complex 7-dimensional vector space. We say that a complex alternating 3-form ψ on W is *maximal* if

$$(2.2) \qquad \psi \wedge \psi_u \wedge \psi_v = 6\varphi(u, v)\delta$$

holds with φ non-degenerate where δ is a non-trivial complex 7-form. We fix δ up to sign by the condition

$$(3.1) \qquad \varphi_{u_1} \wedge \cdots \wedge \varphi_{u_7} = \delta(u_1, \cdots u_7)\delta$$

where $\varphi_u = \varphi(\cdot, u)$. This is compatible with previous normalization in the real case.

Put $G(\psi)$ for the subgroup of $\mathrm{AUT}^{\mathbf{C}}(W)$ leaving ψ fixed for the induced action on alternating 3-forms. If $S \in G(\psi)$ then (2.2) gives

$$(3.2) \qquad \varphi(Su, Sv) = (\det S)^{-1}\varphi(u, v).$$

Combining this with (3.1) yields

$$(\det S)^9 = 1 \quad \text{for } S \in G(\psi).$$

This is in accordance with the fact that if we replace ϕ by $\epsilon^{-1}\varphi$ and δ by $\epsilon\delta$ where $\epsilon^9 = 1$ then (2.2) and (3.1) still hold. Therefore we shall consider the group

$$G_1(\psi) = \{S \in G(\psi) : \det S = 1\}$$

which is then a subgroup of the version of $SO(7, \mathbf{C})$ given by $\mathrm{AUT}_1(W, \varphi)$. Observe that $G_1(\psi)$ is precisely the group of automorphisms of ALG (W, φ, ψ) for any choice of φ satisfying (2.2) and (3.1).

We now mimic the argument of the proof of (2.4) using

(3.3) LEMMA. *Let U be a complex orthogonal space and L a skew-symmetric linear transformation. Then for the eigenspaces U_λ of L, i.e., the kernels of $(L - \lambda I)^n$, $n = \dim U$, we have*

$$\dim U_\lambda = \dim U_{-\lambda}, \quad U_\lambda \perp U_\mu \text{ if } \lambda + \mu \neq 0,$$

and, if $\lambda \neq 0$, the orthogonal structure restricted to $U_\lambda + U_{-\lambda}$ is non-degenerate.

Now fix $u \in V$ with $\varphi(u, u) \neq 0$. Put

$$G_1(\psi, u) = \{S \in G_1(\psi): Su = u\},$$

$$H = \mathrm{AUT}_1^{\mathbf{C}}(u^\perp, \varphi) \simeq SO(6, \mathbf{C}),$$

$$L = \text{restriction of } L_u \text{ to } u^\perp,$$

$$H(L) = \text{centralizer of } L \text{ in } H.$$

Since ψ is non-degenerate, L has an eigenvalue $\lambda \neq 0$ with $\dim U_\lambda = d$, $d = 1, 2$ or 3. By Lemma (3.3), $U_\lambda + U_{-\lambda}$ is an $H(L)$-invariant subspace of dimension $2d$ from which it follows that $H(L)$ is a subgroup of $O(2d, \mathbf{C}) \times O(6-2d, \mathbf{C})$. This gives $\dim_{\mathbf{C}} H(L) < 8$ unless $d = 3$. Since $G_1(\psi, u)$ may be regarded as a subgroup of $H(L)$ and $\dim_{\mathbf{C}} G_1(\psi, u) \geqq 8$ we conclude that $d = 3$. Thus

$$(L^2 - \lambda^2 I)^3 = 0,$$

but if $L^2 - \lambda^2 I \neq 0$ we again get $\dim_{\mathbf{C}} H(L) < 8$, a contradiction. Hence $L^2 = \lambda^2 I$. Using (3.1) and (3.2) we get

$$-\lambda^6 = \det L = \varphi(u, u)^3.$$

The conclusion is $L^2 = -\epsilon\varphi(u, u)I_{u^\perp}$, or

$$(3.4) \qquad u \times (u \times v) = \epsilon(-\varphi(u, u)v + \varphi(u, v)u), \quad \epsilon^3 = 1.$$

This identity is established for $\varphi(u, u) \neq 0$ and ϵ depending on u, but since $u \mapsto \epsilon$ is continuous on the connected set $\{u \in V: \varphi(u, u) \neq 0\}$ we conclude that ϵ is constant. Finally, (3.4) holds when $\varphi(u, u) = 0$ by continuity.

We have already observed that one can replace φ by $c^{-1}\varphi$ where $c^9 = 1$ and leave ψ fixed. The effect on (3.4) is to replace ϵ by $c^3\epsilon$. Thus we have

(3.5) PROPOSITION. *If ψ is a complex maximal 3-form there is a quadratic form φ, unique up to multiplication by a cube root of unity, such that* (2.2), (3.1), *and* (2.4), *the Lagrange Identity* (*namely* (3.4) *with $\epsilon = 1$*), *hold.*

From (3.2) we see that $(\det S)^3 = 1$ for $S \in G(\psi)$. On the other hand, if $\epsilon^3 = 1$ then $\epsilon I \in G(\psi)$. This gives

(3.6) PROPOSITION. $G(\psi) = G_1(\psi) \times \{\epsilon I: \epsilon^3 = 1\}$, *direct product.*

Continuing the reasoning leading to (3.4), let us suppose φ is chosen as in (3.5) and $\varphi(u, u) = -1$. From (2.6) we see that if $v \in U_1$ and $w \in U_{-1}$ then

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$$v \times w = -\varphi(v, w)u.$$

Contracting $\psi \wedge \psi_v \wedge \psi_w = 6\delta$ with respect to u, v, and w gives

$$\psi_v \wedge \psi_w = 2\delta_{u,v,w}.$$

The last two identities show that if v_1 and v_2 are linearly independent elements of U_1 then $v_1 \times v_2 \neq 0$.The identity (2.5) shows that $V_1 \times V_2 \in U_{-1}$. It follows that there exists $v_3 \in U_1$ with $\psi(v_1, v_2, v_3) \neq 0$. Hence ψ restricted to U_1 is a non-trivial 3-form.

Previous reasoning shows that the restriction map gives an isomorphism

$$H(L) \to \mathrm{AUT}^{\mathbf{C}}(U_1).$$

Since ψ is non-trivial, restriction gives a monomorphism

$$G_1(\psi, u) \to \mathrm{AUT}_1^{\mathbf{C}}(U_1) \simeq SL(3, \mathbf{C}).$$

Since $\dim_{\mathbf{C}} G_1(\psi, u) \geqq 8$ we conclude that $G_1(\psi, u) \simeq SL(3, \mathbf{C})$. It follows, since $\dim_{\mathbf{C}} G_1(\psi) \geqq 14$ that the orbit $G_1(\psi)u$ is a submanifold of $\{u \in W: \varphi(u, u) = -1\}$ of complex dimension $\geqq 6$. Putting these facts together we have

(3.7) THEOREM. *If ψ is a complex maximal 3-form then $G_1(\psi)$ is a connected complex Lie group of complex dimension 14. If $c \neq 0$ then $G_1(\psi)$ acts transitively on $\{u \in V: \varphi(u, u) = c\}$ with the stabilizer subgroups $G_1(\psi, u)$ isomorphic to $SL(3,\mathbf{C})$.*

For $\varphi(u, u) = -1$ we can choose a basis v_1, v_2, v_3 for U such that

$$\psi(v_1, v_2, v_3) = 2^{\frac{1}{2}}.$$

It follows that

$$w_1 = 2^{-\frac{1}{2}}v_2 \times v_3, \quad w_2 = 2^{-\frac{1}{2}}v_3 \times v_1, \quad w_3 = 2^{-\frac{1}{2}}v_1 \times v_2$$

is a basis for U_{-1} such that $\varphi(v_j, w_k) = \delta_{jk}$. If we write

$$z = -\varphi_u, \quad x^j = \varphi_{w_j}, \quad y^j = \varphi_{v_j}$$

then we get

$$\psi = dz \wedge (dx^1 \wedge dy^1 + dx^2 \wedge dy^2 + dx^3 \wedge dy^3) + 2^{\frac{1}{2}}dx^1 \wedge dx^2 \wedge dx^3 + 2^{\frac{1}{2}}dy^1 \wedge dy^2 \wedge dy^3, \tag{3.8}$$

$$\varphi = -(z)^2 + 2(x^1y^1 + x^2y^2 + x^3y^3). \tag{3.9}$$

These explicit formulae give

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(3.10) THEOREM. *Put $\Omega_3(W)$ for the 35-dimensional vector space over* $\mathbf{C}$ *of complex alternating 3-forms on the 7-dimensional complex vector space W. The maximal forms constitute a connected dense open subset $\Psi(W)$ which is a single orbit of* $\mathrm{AUT}^{\mathbf{C}}(W)$ *for the induced action on $\Omega_3(W)$. If $\psi, \psi' \in \Psi(W)$ and $G(\psi) = G(\psi')$ then $\psi' = c\psi$, $c \in \mathbf{C}\setminus\{0\}$. For each $\psi \in \Psi(W)$ there exists a complex conjugation θ which is hermitean for a unique bilinear form φ satisfying* (2.2) *and* (3.1) *so that one obtains a vector product structure (W, φ, ψ)*; $\mathrm{ALG}(V, \varphi, \psi)$ *is the complexified Cayley numbers. The identity component $G_1(\psi)$ is the complexification of the compact group $G^{\theta}(\psi)$, the centralizer of θ in $\mathfrak{g}(\psi)$, treated in Section* 2.

Proof. Since W has a basis over $\mathbf{C}$ in which a given $\psi \in \Psi$ has the form (3.8) we conclude that Ψ is a single orbit of $\mathrm{AUT}^{\mathbf{C}}(W)$ from which it follows that Ψ is connected and open since

$$\dim_{\mathbf{C}}\Psi = \dim_{\mathbf{C}}\mathrm{AUT}^{\mathbf{C}}(W) - \dim_{\mathbf{C}}G(\psi) = 49 - 14 = 35 = \dim_{\mathbf{C}}\Omega_3(W).$$

Let δ be a fixed non-trivial complex 7-form on W. For any $\psi \in \Omega_3(W)$ there is a unique quadratic form $\tilde{\varphi}$ defined by

$$\psi \wedge \psi_u \wedge \psi_v = 6\tilde{\varphi}(u, v)\delta.$$

We have

$$\tilde{\varphi}_{u_1} \wedge \cdots \wedge \tilde{\varphi}_{u_7} = p(\psi)\delta_0(u_1, \cdots u_7)\delta$$

where $\Omega_3(W) \xrightarrow{p} \mathbf{C}$ is a polynomial function. To say that ψ is maximal is to say $p(\psi) \neq 0$. Hence

$$\dim_{\mathbf{C}}(\Omega_3(W)\setminus\Psi) \leqq 34.$$

If in the basis given for (3.8) one defines $\theta \in \mathrm{AUT}^{\mathbf{R}}(W)$ by

$$z\circ\theta = -\bar{z}, \quad x^k\circ\theta = \bar{y}^k, \quad y^k\,\theta = \bar{x}^k$$

then $\theta\varphi = \bar{\varphi}$ and $\varphi(\cdot, \theta\cdot)$ is hermitean positive definite where φ is given by (3.9). It follows that ψ restricted to W^{θ} is a maximal alternating 3-form in the real sense. Thus $G^{\theta}(\psi)$ is as described in Section 2. Since $G_1(\psi)$ is connected, it is generated by the exponentials of the Lie algebra $\mathfrak{g}(\psi)$ which is the complexification of $\mathfrak{g}^{\theta}(\psi)$. One concludes that $G_1(\psi)$ is the complexification of $G^{\theta}(\psi)$.

Finally, if $\psi \in \Omega_3(W)$ is any 3-form and $Y \in \mathrm{END}\,(W)$ is any endomorphism we obtain a new 3-form

$$\partial(Y)\psi = -\psi(Y\cdot, \cdot, \cdot) - \psi(\cdot, Y\cdot, \cdot) - \psi(\cdot, \cdot, Y\cdot).$$

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For $\psi \in \Psi(W)$ dimension-counting shows that every element of $\Omega_3(W)$ can be written in the form $\partial(Y)\psi$ for some Y. A 3-form in $\Omega_3(W)$ is fixed under the action of $G(\psi)$ if and only if it is annihilated by all derivatives $\partial(X)$ with $X \in \mathfrak{g}(\psi)$. Since

$$\partial(X)\partial(Y)\psi = \partial([X, Y])\psi \quad \text{for } X \in \mathfrak{g}(\psi),$$

it follows that $\partial(Y)\psi$ is fixed if and only if Y normalizes $\mathfrak{g}(\psi)$ in the Lie algebra END (W). This can only occur when $Y \in \mathbf{C}J + \mathfrak{g}(\psi)$. Thus the only elements of $\Omega_3(W)$ left fixed by $G(\psi)$ are of the form $c\psi$, $c \in \mathbf{C}$.

Finally, we record

(3.11) THEOREM. *For ψ a complex maximal 3-form, the Killing form of the real Lie algebra $\mathfrak{g}(\psi)$ is*

$$\mathrm{KILL}(X, Y) = 4 \operatorname{Re} \operatorname{tr} X Y.$$

4. Split Cayley products. Let ψ be a real alternating 3-form on a 7-dimensional real vector space V. Then the identity (2.2) holds with φ a symmetric bilinear form. We say that ψ is *pseudo-maximal* if φ is non-degenerate and indefinite. The normalization (3.1) fixes φ uniquely.

Put $W = \mathbf{C} \otimes_R V$, and let γ be the complex conjugation with $W^\gamma = V$.

Extending ψ by complex linearity gives a complex maximal 3-form on W, and the results of Section 3 apply. the subgroup of $\mathrm{AUT}^{\mathbf{R}}(V)$ leaving ψ fixed is the centralizer $G^\gamma(\psi)$ of γ in $G(\psi)$.

(4.1) THEOREM. *Suppose ψ is a pseudo-maximal 3-form on V. Then $\mathfrak{g}^\gamma(\psi)$ is a normal real form of the complex simple Lie algebra $\mathfrak{g}(\psi)$ and*

$$G^\gamma(\gamma) \xrightarrow{\mathrm{ad}} \mathrm{INT}(\mathfrak{g}^\gamma(\psi))$$

is an isomorphism of $G^\gamma(\psi)$ with the adjoint group of $\mathfrak{g}^\gamma(\psi)$ where

$$\mathrm{ad}(S)X = SXS^{-1} \quad \textit{for } S \in G^\gamma(\psi),\ X \in \mathfrak{g}^\gamma(\psi).$$

The quadratic form φ is of type $(+3, -4)$, and there exists an involution $\sigma \in G^\gamma(\psi)$ such that $\varphi(v, \sigma v) > 0$ for all $v \in V \setminus \{0\}$.

Proof. Since φ is indefinite, there exists $u \in V$ with $\varphi(u, u) = -1$. Previous reasoning gives an $\mathbf{R}$-basis for V such that ψ has the form (3.8) and φ the form (3.9). This shows that φ is of type $(+3, -4)$, and if we take θ the hermitean complex conjugation for (W, φ, ψ) constructed in the proof of (3.10), then $\sigma = \gamma\theta$ meets the requirements of the statement. Given

$l_1, \lambda_2, \lambda_3 \in \mathbf{C}$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we get an element $H(\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{g}(\psi)$ defined by

$$Hu = 0, \quad Hv_j = \lambda_j v_j, \quad Hw_j = -\lambda_j w_j$$

in the notation of the proof of (3.8). These elements form a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(\psi)$. If $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{R}$ then $H(\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{g}^\gamma(\psi)$. Hence, $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{h}^\gamma$ with $\mathfrak{h}^\gamma$ diagonalizable over $\mathbf{R}$; this proves that $\mathfrak{g}^\gamma(\psi)$ is a normal real form of $\mathfrak{g}(\psi)$. In view of (3.6), $G^\gamma(\psi) = G_1^\gamma(\psi)$ and ad is a monomorphism of $G_1(\psi)$. Moreover, for $X \in \mathfrak{g}(\psi)$,

$$\text{ad} \exp X = \exp \text{ad} X,$$

from which it follows that there is a monomorphism of $\text{INT}(\mathfrak{g}^\gamma(\psi))$ into $G^\gamma(\psi)$. The isomorphism of $G^\gamma(\psi)$ and $\text{INT}(\mathfrak{g}^\gamma(\psi))$ follows from the fact that $G^\gamma(\psi)$ is connected. This is not hard to prove directly, but it is also a consequence of the fact that $\text{AUT}(\mathfrak{g}^\gamma(\psi))$, the group of all $\mathbf{R}$-Lie algebra automorphisms of $\mathfrak{g}^\gamma(\psi)$, is connected. A classical proof in the spirit of Cartan [**4**] is given below (4.8).

Reinterpretation of Theorem (3.7) and Proposition (2.7) give

(4.2) PROPOSITION. *If $u \in V$ and $\varphi(u, u) < 0$ then restriction to U_1, the $+1$-eigenspace of L_u, gives an isomorphism of the stabilizer subgroup*

$$G^\gamma(\psi, u) \to \text{AUT}^{\mathbf{R}}(U_1, \psi) \simeq SL(3, \mathbf{R}).$$

If $\varphi(u, u) > 0$ then L_u is a complex structure for $u^\perp$ and the hermitean form on $u^\perp$

$$<v, w> = \varphi(v, w) - i\psi(u, v, w)$$

is of type $(+1, -2)$. Thus, following (2.7), the restriction homomorphism to $u^\perp$ gives an isomorphism of $G^\gamma(\psi, u)$ with a version of $SU(1, 2)$.

The determination of $G^\gamma(\psi, v)$ where $v \in V \setminus \{0\}$ and $\varphi(v, v) = 0$ is given in (5.23) below.

We next find the maximal compact subgroups of $G^\gamma(\psi)$.

(4.3) TERMINOLOGY. If ψ is a pseudo-maximal 3-form on V^γ, an involution $\sigma \in G^\gamma(\psi)$ such that $\varphi(v, \sigma v) > 0$ for all $v \in V^\gamma \setminus \{0\}$ is called a *Cartan involution.*

Theorem (4.1) guarantees the existence of Cartan involutions.

(4.4) PROPOSITION. *Let σ be a Cartan involution of $G^\gamma(\psi)$. Put $G^{\gamma,\sigma}(\psi)$ for its centralizer in $G^\gamma(\psi)$ and $V^{-\sigma}$ for its -1-eigenspace in V^γ. Then restriction to $V^{-\sigma}$ gives an isomorphism*

$$G^{\gamma,\sigma}(\psi) \to \mathrm{AUT}^+(V^{-\sigma}, \varphi) \simeq SO(4).$$

Proof. The element $\theta = \gamma\sigma \in \mathrm{AUT}^{\mathbf{R}}(W)$ is an hermitean complex conjugation for (W, φ, ψ); so $G^\theta(\psi)$ is one of the groups described in Section 2. Moreover

$$W^{\theta,-\sigma} = V^{-\sigma} \quad \text{and} \quad G^{\theta,\sigma}(\psi) = G^{\gamma,\sigma}(\psi).$$

Theorem (2.14) now gives the result.

We denote by HYPER$(V, -\varphi)$ the submanifold of GRASS$(4, V)$ consisting of the 4-dimensional subspaces π of V on which $-\varphi$ is positive definite. Each such subspace π is of the form $\pi = V^{-\sigma}$ where σ is a φ-orthogonal involution of V such that $\varphi(v, \sigma v) > 0$ for all $v \in V \setminus \{0\}$. The space HYPER$(V^\gamma, -\varphi)$ is the symmetric space corresponding to the simple Lie algebra ANT1(V, φ) of φ-skew-symmetric $\mathbf{R}$-linear transformations of V^γ. The group of isometrics is the disconnected group $\mathrm{AUT}^+(V, \varphi) \simeq SO(4, 3)$.

(4.5) THEOREM. *Given a pseudo-maximal* 3-*form* ψ *on* V^γ *put* $\Sigma^\gamma(\psi)$ *for the subset of* HYPER(V, φ) *constituted by the* ψ-*isotropic* 4-*planes* π *on which* -φ *is positive-definite. Then* $\Sigma^\gamma(\psi)$ *is a regularly imbedded submanifold diffeomorphic to* $\mathbf{R}^8$. *For the Riemannian structure inherited from* HYPER$(V, -\varphi)$ *the group of isometries is* $G^\gamma(\psi)$, *and* $\Sigma^\gamma(\psi)$ *is the subsymmetric space of* HYPER$(V, -\varphi)$ *constituted by the* $\pi = V^{-\sigma}$ *where* σ *is a Cartan involution as defined in* (4.3). $G^\gamma(\psi)$ *acts transitively on* Σ

${}^\gamma(\psi)$, *and the stability subgroup of the point* $\pi = V^{-\sigma}$ *is the maximal compact subgroup* $G^{\gamma,\sigma}(\psi)$.

Proof. Suppose $\pi \in \Sigma^\gamma(\psi)$. Given $e \in \pi \setminus \{0\}$, the multiplication operator L_e is, by the Lagrange Identity, an invertible linear transformation of $e^\perp$ which has the φ-orthogonal direct sum decomposition

$$e^\perp = (\pi \cap e^\perp) + \pi^\perp.$$

To say that π is ψ-isotropic is to say that $L_e\pi \subset \pi^\perp$. Therefore L_e gives an isomorphism of $\pi \cap e^\perp$ with $\pi^\perp$.

We conclude that, for the vector product, one has

$$\pi \times \pi = \pi^\perp, \quad \pi \times \pi^\perp = \pi.$$

If $f \in \pi^\perp$ and $\varphi(f, f) = 1$ then $L_f^2 = \mathrm{Id}$ on $f^\perp$. It follows by anti-symmetry that since $L_f\pi = \pi$ then $L_f\pi^\perp = \pi^\perp \cap f^\perp$. Therefore, we also have

$$\pi^\perp \times \pi^\perp = \pi^\perp.$$

Let σ be the φ-orthogonal involution such that $\pi = V^{-\sigma}$ and $\pi^{\perp} = V^{\sigma}$; we see that σ is an automorphism of the vector product structure. Therefore σ is a Cartan involution in the sense of (4.3). On the other hand, since $\sigma \in G^{\gamma}(\psi)$, $X \mapsto \sigma X \sigma$ is a Lie algebra involution of $\mathfrak{g}^{\gamma}(\psi)$. From (3.11) we have for the Killing form

$$\mathrm{KILL}(X, \sigma X \sigma) = 4 \operatorname{tr} X\sigma X\sigma = -4 \operatorname{tr} XX^*$$

where X^* is the transposed of X with respect to the positive-definite quadratic form $\varphi(\cdot, \sigma\cdot)$; for all $X \in \mathrm{END}^{\mathbf{R}}(V)$ one has $X^* = \sigma X^{\#} \sigma$, and $X^{\#} = -X$ for $X \in \mathfrak{g}^{\gamma}(\psi)$. Thus σ is a Cartan involution in the ordinary sense for the Lie algebra $\mathfrak{g}^{\gamma}(\psi)$. Since all Cartan involutions of a non-compact, real, simple Lie algebra are conjugate under the adjoint group, it follows that (4.3) describes all Cartan involutions of $\mathfrak{g}^{\gamma}(\psi)$. The rest of the theorem is a matter of identifying the symmetric space of a Lie algebra with the Cartan involutions; see [7] for the Riemannian structures (one uses the fact that the restriction of the Killing form of $\mathrm{ANT1}(V, \varphi)$ to $\mathfrak{g}^{\gamma}(\psi)$ is a multiple of the Killing form of $\mathfrak{g}^{\gamma}(\psi)$). For the full group of isometries see (4.8) below.

It is useful to write the isomorphism of Proposition (4.4) explicitly.

(4.6) PROPOSITION. *The inverse of the restriction isomorphism*

$$G^{\gamma,\sigma}(\psi) \to \mathrm{AUT}^{+}(V^{-\sigma}, \varphi)$$

of Proposition (4.4) *is*

$$\begin{aligned} &\mathrm{AUT}^{+}(V^{-\sigma}, \varphi) \xrightarrow{R} G^{\gamma,\sigma}(\psi) \quad \textit{where} \\ &R(S)u = Su \quad \textit{if } u \in V^{-\sigma}, \\ &R(S)v = Se \times \bar{S}(e \times v) \quad \textit{if } v \in V^{\sigma}, \end{aligned}$$

e being any element of $V^{-\sigma}$ *such that* $\varphi(e, e) = -1$.

From this one obtains Cartan's results [4, p. 494].

(4.7) PROPOSITION. *Under the adjoint action of* $G^{\gamma}(\psi)$ *on* $\mathfrak{g}^{\gamma}(\psi)$, *the maximal compact subgroup* $G^{\gamma,\sigma}(\psi)$ *acts on* $\mathfrak{g}^{\gamma,\sigma}(\psi)$ *according to the adjoint action of* $SO(4)$ *and on* $\mathfrak{g}^{\gamma,-\sigma}(\psi)$ *according to an irreducible* 8*-dimensional representation* ρ *of* $SO(4)$.

Proof. The first statement is obvious. For the second we note that $\mathfrak{g}^{\gamma,-\sigma}(\psi)$ is spanned by the $D(u, v)$ with $u \in V^{-\sigma}$, $v \in V^{\sigma}$ and the action of $\mathrm{AUT}^{+}(V^{-\sigma}, \varphi)$ is

$$\rho(S)D(u, v) = R(S)D(u, v)R^{-1}(S) = D(R(S)u, R(S)v)$$

which is an 8-dimensional irreducible representation.

(4.8) COROLLARY. $\mathrm{AUT}(\mathfrak{g}^{\gamma}(\psi)) = \mathrm{INT}(\mathfrak{g}^{\gamma}(\psi))$.

Proof. Each class of AUT/INT is represented by an automorphism leaving $\mathfrak{g}^{\gamma,\sigma}(\psi)$ and $\mathfrak{g}^{\gamma,-\sigma}(\psi)$ invariant. Let τ be such an automorphism and suppose it is the outer automorphism class of $\mathfrak{g}^{\gamma,\sigma}(\psi)$. Then τ corresponds to the automorphism $\tau(S) = TST^{-1}$ of $\mathrm{AUT}^{+}(V^{-\sigma}, \varphi)$ given by an element $T \in \mathrm{AUT}(V^{-\sigma}, \varphi)$ of determinant -1. One calculates that $\rho \circ \tau$ and ρ are inequivalent which is a contradiction. In terms of the Lie algebra $\mathfrak{so}(4)$ with primitive roots μ and ν, τ corresponds to an interchange of μ and ν. The highest weight of the adjoint representation is $\mu + \nu$; that of ρ is $\frac{3}{2}\mu + \frac{1}{2}\nu$.

There is an almost exact analogue of (2.9).

(4.9) THEOREM. *The pseudo-maximal forms on the real* 7*-dimensional vector space* V *constitute a subset* $\Psi'(V) \cup -\Psi'(V)$ of $\Omega_3^{\mathbf{R}}(V)$ *which is a single orbit of* $\mathrm{AUT}^{\mathbf{R}}(V)$. *The subset* $\Psi'(V)$ *is open and connected. Given a quadratic form* φ *of type* $(+3, -4)$ *on* V, *the set of* $\psi \in \Psi'(V)$ *which yield* φ *in* (2.2) *form a disconnected set* $\Psi'(V, \varphi)$ *which is a single orbit for the action of* $\mathrm{AUT}^{+}(V, \varphi) \simeq SO(3, 4)$. *Each of the two components of* $\Psi'(V, \varphi)$ *is a* 7*-dimensional regular submanifold of* $\Psi'(V)$.

Proof. The fact that V has a basis in which ψ has the form (3.8) is equivalent to the statement that the pseudo-maximal 3-forms give a single orbit for $\mathrm{AUT}^{+}(\mathbf{R})$. The rest follows routinely from (3.10).

(4.10) COROLLARY. *The groups* $G^{\gamma}(\psi)$ *for* ψ *a pseudo-maximal* 3*-form are conjugate in* $\mathrm{AUT}_1^{\mathbf{R}}(V) \simeq SL(7, \mathbf{R})$. *The groups* $G^{\gamma}(\psi)$ *with* $\psi \in \Psi'(V, \varphi)$ *form* 2 *distinct conjugacy classes in the identity component of* $\mathrm{AUT}^{+}(V, \varphi)$.

5. The boundaries of $G^{\gamma}(\psi)$. Let ψ be a pseudo-maximal 3-form on a 7-dimensional real vector space V. The quadratic form φ and the group $G^{\gamma}(\psi)$ are as in Section 2. The boundaries of the group $G^{\gamma}(\psi)$, the adjoint group of the simple Lie algebra $\mathfrak{g}^{\gamma}(\psi)$, are written B_{β}, B_{α}, and B.

(5.1) *Notation.* The differentiable manifold B_{β} is the manifold of 1-dimensional φ-isotropic subspaces l of V.

(5.2) *Remark.* B_{β} has a simply-connected double-covering $\widetilde{B}_{\beta}$ given by the manifold of oriented φ-isotropic lines l^{+} through the origin of V. There is a diffeomorphism $\widetilde{B}_{\beta} \approx S^3 \times S^2$, so for the fundamental group one has

$\pi_1(B_\beta) \approx \mathbf{Z}_2$.

(5.3) *Remark.* B_β is defined independently of the split Cayley product in V. It is also a boundary for the group $\mathrm{AUT}_0(V, \varphi)$, the connected component of the identity in $\mathrm{AUT}(V, \varphi)$ which is a version of $SO_0(3, 4)$. It is the unique minimal (5-dimensional) boundary of $\mathrm{AUT}_0(V, \varphi)$.

When one considers only the orthogonal structure (V, φ) one obtains a flag for each $l \in B_\beta$ given by

$$V_1(l) = l, \quad V_6(l) = l^\perp.$$

The vector-product structure provides a refinement:

$$V_3(l) = \{v \in V: v \times l = 0\}$$
$$V_4(l) = V_3(l)^\perp.$$

One has

(5.4) PROPOSITION. *The manifold B_β is a compact 5-dimensional homogeneous space of $G^\gamma(\psi)$. The stability subgroup of a point $l \in B_\beta$ is the maximal parabolic subgroup*

(5.5) $\quad P(l) = \{S \in G^\gamma(\psi): SV_k(l) \subset V_k(l), k = 1, 3, 4, 6\}.$

For each Cartan involution σ, the maximal compact subgroup $G^{\gamma,\sigma}(\psi)$ operates transitively on B_β and $\widetilde{B}_\beta$.

The statement will be proved in the sequel.

(5.6) *Notation.* The differentiable manifold B_α is the manifold of 2-dimensional ψ-isotropic subspaces p of V. Thus $p \in B_\alpha$ if $v \times w = 0$ whenever $v, w \in p$.

(5.7) *Remark.* B_α has a simply-connected double covering $\widetilde{B}_\alpha$ given by the manifold of oriented ψ-isotropic planes p^+ through the origin in V. B_α is diffeomorphic to B_β. (See Theorem (6.4) below.)

(5.8) *Remark.* A ψ-isotropic 2-dimensional subspace p of V is also φ-isotropic. The φ-isotropic planes through the origin of V constitute a 7-dimensional boundary of the group $\mathrm{AUT}_0(V, \varphi)$ in which B_α is a regularly imbedded 5-dimensional sub-manifold.

The flag associated with $p \in B_\alpha$ is

$$V_2(p) = p, \quad V_5(p) = p^\perp.$$

(5.9) PROPOSITION. *The manifold B_α is a compact 5-dimensional homogeneous space of $G^\gamma(\psi)$. The stability group of a point $p \in B_\alpha$ is the maximal parabolic subgroup*

(5.10) $P(p) = \{S \in G^\gamma(\psi): SV_k(p) \subset V_k(p): k = 2, 5\}.$

For each Cartan involution σ, *the maximal compact subgroup* $G^{\gamma,\sigma}(\psi)$ *operates transitively on* B_α *and* $\widetilde{B}_\alpha$.

Again the proof is contained below.
The maximal boundary occurs next.

(5.11) *Notation.* The differentiable manifold B is the manifold of ordered pairs $b = (l, p)$ where p is a 2-dimensional ψ-isotropic subspace of V and l is a 1-dimensional subspace with $l \subset p$.

(5.12) *Remark.* B has a 4-group covering space B^+ whose points are $b^+ = (l^+, p^+)$ determined by the respective oriented subspaces. The space B^+ is doubly connected, and

$$\pi_1(B) = \text{quaternion group}.$$

(5.13) *Remark.* The pairs (l, p) of φ-isotropic subspaces of V with $l \subset p$, $\dim l = 1$, $\dim p = 2$, constitute an 8-dimensional boundary of the group $\mathrm{AUT}_0(V, \varphi)$ in which B is a regularly imbedded 6-dimensional submanifold.

The stability subgroup for the action of $G^\gamma(\psi)$ on B is clearly

(5.14) $P(b) = P(l) \cap P(p), \quad b = (l, p).$

There is a very concrete description of B^+ (and hence of B).

(5.15) THEOREM. *Let* σ *be a Cartan involution of* $G^\gamma(\psi)$. *Put* $W = \mathbf{C} \otimes_{\mathbf{R}} V$ *and let* θ *be the complex conjugation* $\theta = \gamma\sigma$. *Put* $\Gamma^\theta(\psi)$ *for the Zorn manifold, see* (2.12), *associated with the maximal* 3-*form* ψ *on* W^θ. *Define*

$$K(\sigma) = \{P \in \Gamma^\theta(\psi): \sigma P = -P\}.$$

Then for each $P \in K(\sigma)$ *the map* $S \mapsto SP$ *is a diffeomorphism of* $G^{\gamma,\sigma}(\psi)$ *with* $K(\sigma)$, *and there exists a* $G^{\gamma,\sigma}(\psi)$-*equivariant diffeomorphism* $B^+ \to K(\sigma)$.

Proof. Fix the Cartan involution σ. Given $b^+ = (l^+, p^+) \in B^+$ there are uniquely determined vectors $v \in l^+$, $w \in p^+$ such that

$$\varphi(v, \sigma v) = 1, \quad \varphi(w, \sigma w) = 1, \quad \varphi(v, \sigma w) = 0.$$

Define points $P_1, P_2, P_3 \in W$ by

$$\begin{aligned} P_1 &= -i2^{-\frac{1}{2}}(v - \sigma v), \\ P_2 &= i2^{-\frac{1}{2}}(w - \sigma w), \\ P_3 &= -iv \times \sigma v. \end{aligned}$$

Then $P = (P_1, P_2, P_3) \in W^\theta$ and P_1, P_2, P_3 is an orthonormal triad. Since $\sigma P = -P$ and $\sigma \in G(\psi)$ we have $\psi(P_1, P_2, P_3) = 0$. Hence $P \in K(\sigma)$. The map $b^+ \mapsto P$ is invertible because

$$v = 2^{-\frac{1}{2}}(iP_1 + P_3 \times P_1),$$
$$w = 2^{-\frac{1}{2}}(-iP_2 + P_3 \times P_2).$$

The map is $G^{\gamma,\sigma}(\psi) = G^{\theta,\sigma}(\psi)$-equivariant since it is defined in terms of ψ, φ, and σ. In view of Theorem (2.13), the map $S \mapsto SP$ is a diffeomorphism of $G^{\theta,\sigma}(\psi)$ with $K(\sigma)$.

(5.16) COROLLARY. *The manifold B^+ is diffeomorphic to $SO(4)$, and the manifold B is diffeomorphic to $S^3 \times S^3$ modulo the action of the quaternion group where S^3 is identified with the unit quaternions and the action of the quaternion group is (left) multiplication of each factor.*

Note that the transitivity of the action of $G^{\gamma,\sigma}(\psi)$ on B, B_β, and B_α result from (5.15).

It remains to give the explicit determination of the parabolic subgroups. For their description we shall make extensive use of the elements of $\mathfrak{g}^\gamma(\psi)$ given by (1.23), namely

$$D(u, v) = [L_u, L_v] + L_{u \times v}.$$

(5.17) PROPOSITION. *Given $l \in B_\beta$ there is a unique vector-space monomorphism*

$$\mathrm{HOM}^{\mathbf{R}}(V/V_6(l), V_4(l)/V_1(l)) \xrightarrow{D_l} \mathfrak{g}^\gamma(\psi)$$

such that

$$D_l(h)v + V_1 = h(v + V_6).$$

The image, $\mathfrak{N}_0(l)$, is an abelian Lie algebra. Moreover

$$\mathfrak{n}_0(l)V_k(l) \subset V_{k-3}(l) \quad \textit{for } k = 3, 4, 6, 7.$$

Proof. The defining property gives that the elements of $\mathfrak{n}_0(l)$ map $V_7 \to V_4$, and so, by orthogonality, $V_3 \to 0$. Let w_1 be a generator for $V/V_6(l)$. Then there is a unique $v_1 \in V_1(l)$ such that $\varphi(v_1, w_1) = 1$. The element $D_l(h)w_1$ is uniquely specified by the defining condition and the orthogonality condition

$$\varphi(D_l(h)w_1, w_1) = 0.$$

Since

$$V = \mathbf{R}w_1 + V_3 + w_1 \times V_3,$$

$D_h(h)$ is completely determined as a derivation of the vector product (V, φ, ψ). Existence is given by the explicit formula

$$D_l(h) = D(v_1, hw_1)$$

which is well-defined, since $D(v_1, v_1) = 0$, and independent of the choice of w_1.

(5.18) THEOREM. *Given $b = (l, p) \in B$ there is a complete flag for V given by*

$$V_k(b) = V_k(l) \quad \textit{for } k = 1, 3, 4, 6; \quad V_k(b) = V_k(p) \quad \textit{for } k = 2, 5$$

with $V_k \subset V_{k+1}$, $\dim V_k = k$. Put $\mathfrak{n}(b)$ for the subalgebra of $\mathfrak{g}^\gamma(\psi)$ determined by the conditions

$$\mathfrak{n}(b)V_k(b) \subset V_{k-1}(b).$$

Let

$$\mathfrak{n}(b) \xrightarrow{\tau} \mathrm{END}(V_4(l)/V_1(l))$$

be the restriction homomorphism. Then τ is an epimorphism onto $\mathfrak{ut}(b)$, the Lie algebra of upper triangular matrices for the induced flag on V_4/V_1; the kernel of τ is $\mathfrak{n}_0(l)$ described in (5.17) *and there is a natural isomorphism*

$$\mathfrak{ut}(b) \times \mathrm{HOM}^{\mathbf{R}}(V/V_6(l), V_4(l)/V_1(l)) \rightarrow \mathfrak{n}(b).$$

Thus $\mathfrak{n}(b)$ is a 6-dimensional unipotent subalgebra of $\mathfrak{g}^\gamma(\psi)$ whose centralizer $P(b)$ in $G^\gamma(\psi)$ is the subgroup determined by $P(b)V_k(b) \subset V_k(b)$. The image of the homomorphism

$$P(b) \rightarrow \prod_{k=1}^{7} \mathrm{AUT}(V_k/V_{k-1})$$

is

$$\{(\chi_1, \chi_2^{-1}, \chi_3^{-1}, 1, \chi_3, \chi_2, \chi_1^{-1}): \chi_1\chi_2\chi_3 = 1\}.$$

Hence $P(b)$ is a minimal parabolic subgroup with Langlands decomposition $\mathrm{MAN}(b)$; *M a 4-group, $A \simeq \mathbf{R}^2$, $N(b) = \mathrm{Exp}\,\mathfrak{n}(b)$.*

(5.19) *Remark.* For A one has $\chi_j = \exp \lambda_j$, $\lambda_j \in \mathbf{R}$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$. The ordered system of positive roots associated with $\mathfrak{n}(b)$ is

$$(5.20) \qquad \alpha = -\lambda_3,\ \beta = \lambda_3 - \lambda_2,\ \alpha + \beta = -\lambda_2; \\ 2\alpha + \beta = +\lambda_1,\ 3\alpha + \beta = \lambda_1 - \lambda_3,\ 3\alpha + 2\beta = \lambda_1 - \lambda_2.$$

The first three associated with $\mathfrak{ut}(b)$ and the last three with $\mathfrak{n}_0(l)$.

(5.21) THEOREM. *Given $p \in B_\alpha$ the corresponding maximal parabolic subgroup is the semi-direct product*

$$P(p) = \mathrm{AUT}(p) \rtimes \mathrm{Exp}(\mathfrak{n}(p))$$

where $\mathfrak{n}(p)$ is the 5-dimensional Heisenberg Lie algebra composed of the elements

$$D(u, v),\ u \in p^\perp,\ v \in p$$

with centre made up of those elements with $u \in p$ and the Lie bracket given by

$$[D(u, v), D(u', v')] = (4/3)\varphi(u, u')D(v, v').$$

The epimorphism $P(p) \to \mathrm{AUT}(p)$ is the restriction map while the action on $p^\perp/p$ is given by an epimorphism $\mathrm{AUT}(p) \to \mathrm{AUT}(p^\perp/p, \varphi)$ which is a version of the adjoint epimorphism $GL(2, \mathbf{R}) \to SO(1, 2)$.

(5.22) THEOREM. *Given $l \in B_\beta$ the corresponding maximal parabolic subgroup is the semi-direct product*

$$P(l) = \mathrm{AUT}(V_3(l)/V_1(l)) \rtimes Exp(\mathfrak{n}(l))$$

where $\mathfrak{n}(l)$ is the 5-dimensional nilpotent Lie algebra composed of the elements

$$D(v, w),\ v \in l,\ w \in l^\perp$$

with centre $\mathfrak{z}(l)$ consisting of those elements such that $v \times w = 0$ and Lie bracket given by

$$[D(v, w), D(v, w')] = D(v, v \times (w \times w')).$$

The quotient $\mathfrak{n}(l)/\mathfrak{z}(l)$ is the 3-dimensional Heisenberg Lie algebra with centre $\mathfrak{n}_0(l)/\mathfrak{z}(l)$. The restriction homomorphism

$$P(l) \xrightarrow{\rho} \mathrm{AUT}(V_3U)/V_1(l))$$

is an epimorphism, and for $S \in P(l)$, $v \in l$ we have

$$Sv = (\det \rho(S))v.$$

(5.23) COROLLARY. *If $v \in V\setminus\{0\}$ and $\varphi(v, v) = 0$ then the stability subgroup is isomorphic to a semi-direct product,*

$$G^\gamma(\psi, v) \simeq SL(2, \mathbf{R}) \rtimes N$$

where $N = \mathrm{Exp}\ \mathfrak{n}(l)$, $l = \mathbf{R}v$, is a 5-dimensional unipotent group.

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6. Historical remarks. In the late 19th century, the theory of Lie groups is essentially the theory of Lie algebras, and the theorems about groups are proved for local Lie groups. The first presentation of the complex simple Lie algebra of type G_2, $\mathfrak{g}(\psi)$ in our notation, were given by Engel [**5**] and Cartan [**1**] in 1893. A fuller description can be found on pp. 146-151 of Cartan's thesis [**2**]. With suitable interpretations these results all have global versions.

The quadric B_β defined by (5.1) has tangent bundle $T(B_\beta)$ with a canonical identification

$$T_l(B_\beta) \simeq V_6(l)/V_1(l),$$

and the restriction of the quadratic form φ to this space gives a pseudo-riemannian structure to B_β. In each $T_l(B_\beta)$ we may consider the manifold of isotropic 2-dimensional subspaces. This gives rise to a bundle

(6.1) *Notation.* Π is the bundle of φ-isotropic tangent 2-planes to the quadric B_β.

Observe that Π is an 8-dimensional boundary of the group $\mathrm{AUT}_0(V, \varphi)$ whose points may be written as ordered pairs (l, t) where t is a 3-dimensional φ-isotropic subspace of V and l is a 1-dimensional subspace $l \subset t$. The pseudo-maximal 3-form gives rise to a section of $\Pi \rightarrow B_\beta$ which we call a *Cartan-Engel* section. This is the section s defined by

$$s(l) = (l, t), \quad t = V_3(l), \quad \text{i.e., } l \times t = 0.$$

Engel and Cartan gave a definition of the complex Lie algebra $\mathfrak{g}(\psi)$ in terms of such a section. In modern language, with a description of the group rather than the Lie algebra, one has

(6.2) THEOREM. *The group $G^\gamma(\psi)$ is the group of φ-orthogonal transformations of V acting on B_β in such a way that the induced action on the bundle Π preserves the Cartan-Engel section determined by ψ.*

(In Cartan's notation the group of such transformations of B_β is denoted G_b.)

The description of the group in terms of B_α is more complicated. The result is (see (5.7))

(6.3) THEOREM. *Given a Cartan involution σ there is a 1-form λ_σ on $\widetilde{B}_\alpha$ such that $(\widetilde{B}_\alpha, \lambda_\sigma)$ is an exact contact manifold and $G^\gamma(\psi)$ is a group of contact transformations. The subgroup of $G^\gamma(\psi)$ formed by the special contact transformations is $G^{\gamma,\sigma}(\psi)$.*

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Proof. Given $p^+ \in \widetilde{B}_\alpha$ choose a $\varphi(\cdot, \sigma\cdot)$-orthonormal oriented frame (v, w) and put $Z_\sigma(p^+) = D(v, w)$. This element of $\mathfrak{g}^\gamma(\psi)$ depends only on p^+ and σ. Each $X \in \mathfrak{g}^\gamma(\psi)$ gives rise to an infinitesimal transformation of $\widetilde{B}_\alpha$ denoted A_X. Then

$$\lambda_\sigma(A_X) = \text{trace } XZ_\sigma(p^+)$$

defines a 1-form on $\widetilde{B}_\alpha$. One verifies that λ_σ gives a contact structure. Given $S \in G^\gamma(\psi)$ let $T(S)$ be the induced map of the tangent bundle $T(\widetilde{B}_\alpha)$. Then

$$\lambda_\sigma \circ T(S)|_{p^+} = c(S, p^+)\lambda_\sigma|_{Sp^+}$$

where $c(S)$ is determined by

$$SZ_\sigma(p^+)S^{-1} = c(S, p^+)Z_\sigma(Sp^+);$$

and so $c(S, p^+) = 1$ for all $p^+ \in \widetilde{B}_\alpha$ if $S \in G^{\gamma,\sigma}(\psi)$.

The above is the first form of $G^\gamma(\psi)$ given by Engel and Cartan (the group G_a in Cartan's notation). Since $(\widetilde{B}_\sigma, \lambda_\sigma)$ is an exact contact manifold, the Lie algebra $\mathfrak{g}^\gamma(\psi)$ can be described in terms of a vector space of functions on $\widetilde{B}_\alpha$ and Poisson brackets.

The manifolds $\widetilde{B}_\alpha$ and $\widetilde{B}_\beta$ are distinct as $G^\gamma(\psi)$-manifolds because the stability groups are not isomorphic. They are also distinct $G^{\gamma,\sigma}(\psi)$-manifolds for in $\widetilde{B}_\beta$ the stabilizer subgroups have the appearance

$$\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}, \quad R \in SO(2)$$

while for $\widetilde{B}_\alpha$ they are of the form

$$\begin{pmatrix} R & 0 \\ 0 & R^2 \end{pmatrix}$$

and these subgroups are not conjugate via an automorphism of $SO(4)$. Nevertheless, $\widetilde{B}_\alpha$ and $\widetilde{B}_\alpha$ are diffeomorphic as are B_α and B_β with the first a covering diffeomorphism of the second. To see this let Q_1 be the group of unit quaternions and $\mathbf{T}$ the subgroup consisting of the elements $\cos x + i \sin x$. We can view $\widetilde{B}_\beta$ as a homogeneous space of $Q_1 \times Q_1$ with stability group of some point given by

$$\widetilde{K}_\beta = \{(z, z): z \in \mathbf{T}\}.$$

Similarly $\widetilde{B}_\alpha \simeq (Q_1 \times Q_1)/\widetilde{K}_\alpha$ where

$$\widetilde{K}_\alpha = \{(z^3, z): z \in \mathbf{T}\}.$$

The group $SO(4)$ is identified with $Q_1 \times Q_1$ modulo the centre which is generated by $(-1, -1)$. Denote by K_α and K_β the respective images in $SO(4)$. Both have unique coset representatives of the form

$$\pm (x, y),\ x \in Q_1,\ y = a + bj + ck \quad \text{with } a^2 + b^2 + c^2 = 1.$$

Thus $\widetilde{B}_\alpha$ and $\widetilde{B}_\beta$ have coincident sections in $G^{\gamma,\sigma}(\psi) \simeq SO(4)$. Moreover the covering transformations over B_α and B_β are diffeomorphic. In summary, we have

(6.4) THEOREM. *B is a regular submanifold of $B_\alpha \times B_\beta$ in such a way that the projections $B \to B_\alpha$, $B \to B_\beta$ give circle bundles. There are smooth sections $B_\alpha \xrightarrow{a} B$ and $B_\beta \xrightarrow{b} B$ such that $a(B_\alpha) = b(B_\beta)$.*

I take this theorem to be a global interpretation of the statement at the top of page 151 in [**2**].

Added in proof. Much of the essential content of Theorem 3.10 was previously found by J. A. Schouten, in *Klassifizierung der alternierenden Grössen dritten Grades in* 7 *Dimenionen*, Rendiconti del Circolo Matematico di Palermo *55* (1931), 137-156.

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