ON THE NUMBER OF SIDES OF A PETRIE POLYGON

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Let \( \{p, q, r\} \) be the regular 4-dimensional polytope for which each face is a \( \{p, q\} \) and each vertex figure is a \( \{q, r\} \), where \( \{p, q\} \), for example, is the regular polyhedron with \( p \)-gonal faces, \( q \) at each vertex. A Petrie polygon of \( \{p, q\} \) is a skew polygon made up of edges of \( \{p, q\} \) such that every two consecutive sides belong to the same face, but no three consecutive sides do. Then a Petrie polygon of \( \{p, q, r\} \) is defined by the property that every three consecutive sides belong to a Petrie polygon of a bounding \( \{p, q\} \), but no four do. Let \( h_{p,q,r} \) be the number of sides of such a polygon, and \( g_{p,q,r} \) the order of the group of symmetries of \( \{p, q, r\} \). Our purpose here is to prove the following formula:

\[
\frac{h_{p,q,r}}{g_{p,q,r}} = \frac{1}{64} \left( 12 - p - 2q - r + \frac{4}{p} + \frac{4}{r} \right).
\]

We use the following result of Coxeter (1, p. 232; 2):

\[
\frac{h_{p,q,r}}{g_{p,q,r}} = \frac{1}{16} \left( \frac{6}{h_{p,q} + 2} + \frac{6}{h_{q,r} + 2} + \frac{1}{p} + \frac{1}{r} - 2 \right),
\]

where \( h_{p,q} \), for example, denotes the number of sides of a Petrie polygon of \( \{p, q\} \). Both proofs referred to depend on the fact that the number of hyperplanes of symmetry of \( \{p, q, r\} \) is \( 2h_{p,q,r} \). This is proved in a more general form in (3). Clearly (1) is a consequence of (2) and the following result:

If \( h \) is the number of sides of a Petrie polygon of the polyhedron \( \{p, q\} \), then

\[
h + 2 = \frac{24}{10 - p - q}.
\]

Proof of (3). The planes of symmetry of \( \{p, q\} \) divide a concentric sphere into congruent spherical triangles each of which is a fundamental region for the group \( \mathbb{S} \) of symmetries of \( \{p, q\} \) (1, p. 81). The number of triangles is thus \( g \), the order of \( \mathbb{S} \). The vertices of one of these triangles can be labelled \( P, Q, R \) so that the corresponding angles are \( \pi/p, \pi/q, \pi/2 \). There are \( g/2p \) images of \( P \) under \( \mathbb{S} \), since the subgroup leaving \( P \) fixed has order \( 2p \). At each of these points there are \( p(p - 1)/2 \) intersections of pairs of circles of symmetry. Counting intersections at the images of \( Q \) and \( R \) in a similar fashion, one gets for the total number of intersections of pairs of circles of symmetry the number

Received October 21, 1957.
$g(p + q - 1)/4$. However, the number of such circles is $3h/2$ (1, p. 68), and every two intersect in two points. Hence

$$g\left(\frac{p + q - 1}{4}\right) = \frac{3h}{2} \left(\frac{3h}{2} - 1\right).$$

Dividing (4) by the relation $g = h(h + 2)$ of Coxeter (1, p. 91), and solving for $h$, one obtains (3).

References


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