RIGID AND FINITELY $V$-DETERMINED GERMS OF $C^\infty$-MAPPINGS

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1. The result. Let $\mathcal{E}$ (respectively $\mathcal{E}_{[\mu]}$, $0 \leq \mu \leq \infty$) denote the ring of germs at $0 \in \mathbb{R}^n$ of all $C^\infty$ functions (respectively $C^\mu$ functions) from $\mathbb{R}^n$ to $\mathbb{R}$. For a given $\varphi = (\varphi_1, \ldots, \varphi_p) \in \mathcal{E}_p$, $p \leq n$, where $\mathcal{E}_p$ is the space of all germs of $C^\infty$ mappings $\mathbb{R}^n \to \mathbb{R}^p$, let $J(\varphi)$ denote the ideal in $\mathcal{E}$ generated by $\varphi_1, \ldots, \varphi_p$ and the Jacobian determinants

$$\frac{D(\varphi_1, \ldots, \varphi_p)}{D(x_{i_1}, \ldots, x_{i_p})},$$

where $1 \leq i_1 < \ldots < i_p \leq n$. Let

$$\mathcal{M}^\infty = \{ \varphi \in \mathcal{E} : D^\alpha \varphi(0) = 0, |\alpha| = 0, 1, \ldots \}.$$

Clearly, $\mathcal{M}^\infty$ is an ideal in $\mathcal{E}$ and $\mathcal{M}^\infty = \cap_{s=1}^{\infty} \mathcal{M}^s$, where $\mathcal{M}$ is the (unique) maximal ideal of $\mathcal{E}$. For $\varphi \in \mathcal{E}_p$ and $s \leq \infty$ denote by

$$j^s(\varphi) = \left( \sum_{|\alpha| = 0} \frac{1}{\alpha!} D^\alpha \varphi_1(0)x^\alpha, \ldots, \sum_{|\alpha| = 0} \frac{1}{\alpha!} D^\alpha \varphi_p(0)x^\alpha \right)$$

the Taylor’s expansion of $\varphi$ at 0 up the order $s$, called the $s$-jet of $\varphi$, and for $\varphi = (\varphi_1, \ldots, \varphi_p) \in \mathcal{E}_{[\mu]}$ let $\mathcal{E}_{[\mu]}(\varphi)$ denote the ideal in $\mathcal{E}_{[\mu]}$ generated by $\varphi_1, \ldots, \varphi_p$.

Definition. We call a given germ $\varphi \in \mathcal{E}_p$ finitely $V$-determined (respectively $C^\mu$-rigid) if there exists a positive integer $s$ for which the following holds: for any $\psi \in \mathcal{E}_p$ with the same $s$-jet $j^s(\varphi) = j^s(\psi)$, the germs of the varieties $\varphi^{-1}(0)$ and $\psi^{-1}(0)$ are homeomorphic (respectively, one can find a local $C^\mu$ diffeomorphism $\tau : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\mathcal{E}_{[\mu]}(\varphi \circ \tau) = \mathcal{E}_{[\mu]}(\psi)$).

For $\varphi \in \mathcal{E}_p$ write

$$Z_\varphi(x) = \sum_{i=1}^p \varphi_i^2(x) + \sum \left[ \frac{D(\varphi_1, \ldots, \varphi_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x) \right]^2,$$

the second summation being taken for $1 \leq i_1 < \ldots < i_p \leq n$.

Theorem 1. For $\varphi \in \mathcal{E}_p$, the following conditions are equivalent:

(a) For each $\mu \in \mathbb{N}$, $\varphi$ is $C^\mu$-rigid;
(b) $\varphi$ is finitely $V$-determined;
(c) $Z_\varphi(x) \geq c|x|^\alpha$ in a neighborhood of 0, where $c$ and $\alpha$ are positive constants;
(d) $\mathcal{M}^\infty \subset J(\varphi)$.

Received February 29, 1972.
We are merely interested in those \( \varphi \) with \( \varphi(0) = 0 \); Theorem 1 reduces to triviality if \( \varphi(0) \neq 0 \).

Observe that if \( \varphi \in \mathcal{E}^p \) is finitely \( V \)-determined then, by definition, the germ of \( \varphi^{-1}(0) \) is homeomorphic to the germ of an algebraic variety; if \( \varphi \) is \( C^0 \)-rigid then the variety \( \varphi^{-1}(0) \) can be transformed under a local \( C^0 \) diffeomorphism of \( \mathbb{R}^n \) onto an algebraic variety. For general \( \varphi \), however, there is no criterion for (the germ of) \( \varphi^{-1}(0) \) to be homeomorphic with an algebraic variety. Thom has conjectured that if \( \varphi \) is analytic then this is the case.

The problems concerning sufficiency of jets and finitely determined mappings have been considered by several authors [1; 2; 3; 4; 5; 6; 7; 8; 9; 12; 13]. We recall the definition. Denote by \( J^r(n, p) \) the space of \( r \)-jets of mappings from \( \mathbb{R}^n \) to \( \mathbb{R}^p \) (this space can be identified with the space of all \( p \)-tuples \( w = (w_1, \ldots, w_p) \) of polynomials \( w_i \) of degree \( \leq r \) in \( n \) variables). A jet \( w \in J^r(n, p) \) is called \( V \)-sufficient (respectively \( C^0 \)-sufficient, \( r \leq \mu \leq \infty \)) in \( \mathcal{E}^p \) if for any \( \varphi \in \mathcal{E}^p \) with \( j^r(\varphi) = w \), the germs of varieties \( w^{-1}(0) \) and \( \varphi^{-1}(0) \) are homeomorphic (respectively, there exists a local \( C^0 \) diffeomorphism \( \tau \), such that \( \varphi \circ \tau = w \)). A germ \( \varphi \in \mathcal{E}^p \) is called \textit{finitely \( C^0 \)-determined} if there exists a positive integer \( r \) such that \( j^r(\varphi) \) is \( C^0 \)-sufficient in \( \mathcal{E}^p \).

Many problems concerning \( V \)- and \( C^0 \)-sufficiency in \( J^r(n, 1) \) (\( p = 1 \)) have been solved. In particular, for \( \varphi \in \mathcal{E} \), it has been proved in [3] (compare also [1] and [13]) the equivalence of the following four conditions:

- \( (a) \) For each \( \mu \in \mathbb{N} \), \( \varphi \) is finitely \( C^0 \)-determined;
- \( (b) \) \( \varphi \) is finitely \( V \)-determined;
- \( (c) \) \( |\text{grad } \varphi(x)| \geq c|x|^\alpha \) for \( |x| < \delta \), where \( c, \alpha, \delta \) are positive constants;
- \( (d) \) \( \mathcal{M}^\infty \subset \mathcal{P}(\varphi) \), where \( \mathcal{P}(\varphi) \) is the ideal in \( \mathcal{E} \) generated by the partial derivatives \( \partial \varphi / \partial x_1, \ldots, \partial \varphi / \partial x_n \).

Observe that for \( \varphi \in \mathcal{E}^p \), \( p > 1 \), conditions \( (a) \) and \( (b) \) are not equivalent. The fact that \( \varphi \) is \( V \)-determined does not even imply that \( \varphi \) is finitely \( C^0 \)-determined.

Counterexample (Mather). Let \( \varphi(x, y) = (x, y^3) \). It is easy to see that \( j^3(\varphi) \) is \( V \)-sufficient. But \( \varphi \) is not finitely \( C^0 \)-determined. For any \( s \in \mathbb{N} \), \( \psi_s = (x, y^3 + xy^{s+1}) \) is a realization of \( j^3(\varphi) \). But \( \varphi \) and \( \psi_s \) have different topological types, since \( \varphi \) is a homeomorphism, while \( \psi_s \) is not.

Remarks 1. One can prove that \( \varphi \in \mathcal{E}^p \) is \( C^0 \)-rigid if and only if \( J(\varphi) \) is the ideal of definition of \( \mathcal{E} \), i.e. there exists a positive integer \( s \) such that \( \mathcal{M}^s \subset J(\varphi) \) [4, Theorem 4(b); 13].

2. Observe that if \( \varphi \in \mathcal{E}^p \) is an analytic mapping then each of the conditions \( (a) \), \( (b) \), \( (c) \) and \( (d) \) in Theorem 1 is equivalent (by Lojasiewicz inequality) to the condition.

- \( (e) \) \textit{In a neighborhood of zero, } \( Z_\varphi(x) = 0 \text{ implies } x = 0 \).

This generalizes Theorem 5 in [8].
2. Proof of Theorem 1. We shall assume that \( p \geq 2 \). The case \( p = 1 \) was explained above. In fact, our proof would not work in the case \( p = 1 \).

(a) \( \Rightarrow \) (b). This is trivial.

(b) \( \Rightarrow \) (c). Observe firstly that if \( \varphi \in \mathcal{E}^p \) is finitely \( V \)-determined then the (germ of) \( \varphi^{-1}(0) \setminus \{0\} \) is either empty or a topological submanifold of codimension \( p \) in \( \mathbb{R}^n \). Indeed, suppose the \( s \)-jet \( w = j^s(\varphi) \) is \( V \)-sufficient, choose a system of \( p \) homogeneous polynomials \( h = (h_1, \ldots, h_p) \) of degree \( s + 1 \) in such a way that \( Z_{w+k}(x) \neq 0 \) for all \( x \) in a neighborhood of \( 0 \in \mathbb{R}^n \), \( x \neq 0 \) (this is possible, for example by [4, Proposition 1(b)]). By assumption, the germs of \( \varphi^{-1}(0) \) and \( (w + h)^{-1}(0) \) are homeomorphic, but it is clear that \( (w + h)^{-1}(0) \setminus \{0\} \) is either empty or a smooth submanifold of \( \mathbb{R}^n \) of codimension \( p \).

Now assume that (c) is false, we shall derive a contradiction. We shall find an application \( \varphi \in \mathcal{E}^p \) such that \( j^s(\varphi) = j^s(\varphi) \) and the germ of \( \varphi^{-1}(0) \setminus \{0\} \) is not a topological manifold (and is not empty). The idea is similar to that in [1] and [8] and is due to S. Lojasiewicz.

Since (c) is false, we can find a sequence \( \{a_i\}_{i \in \mathbb{N}}, a_i \in \mathbb{R}^n, a_i \neq 0, a_i \to 0 \), such that for each \( s \in \mathbb{N} \),

\[
Z_p(a_i) = o(|a_i|).
\]

For \( p \) vectors \( u_1, \ldots, u_p \) in \( \mathbb{R}^n \) write \( d(u_1, \ldots, u_p) = \min \{|\alpha_1|, \ldots, |\alpha_p|\} \), where \( \alpha_k \) denotes the distance from \( u_k \) to the linear subspace of \( \mathbb{R}^n \) spanned by the vectors \( u_{ij}, j \neq k \), and let \( \text{Vol}(u_1, \ldots, u_p) \) denote the \( p \)-dimensional volume of the parallelotope with edges \( u_1, \ldots, u_p \). Then \( \text{Vol}(u_1, \ldots, u_p) \geq (d(u_1, \ldots, u_p))^p \). Moreover if we write \( u_k = (u_{ki}, \ldots, u_{kp}) \) then

\[
\sum_{1 \leq i_1 < \ldots < i_p \leq n} \left| \begin{array}{c}
\vdots \\
|u_{1i_1}, \ldots, u_{1i_p}|
\vdots \\
|u_{pi_1}, \ldots, u_{pi_p}|
\end{array} \right|^2 = (\text{Vol}(u_1, \ldots, u_p))^2,
\]

where \( 1 \leq i_1 < \ldots < i_p \leq n \). The above formula can be verified by checking the axioms for a volume (see for example [11]).

Now consider \( u_k = \text{grad} \varphi_k \) at \( a_i \); without loss of generality, we may assume that for all \( i \in \mathbb{N} \)

\[
d(\text{grad} \varphi_1(a_i), \ldots, \text{grad} \varphi_p(a_i)) = \delta_i,
\]

where \( \delta_i \) is the distance from \( \text{grad} \varphi_p(a_i) \) to the subspace spanned by \( \text{grad} \varphi_1(a_i), \ldots, \text{grad} \varphi_{p-1}(a_i) \). Since

\[
Z_p(a_i) = \sum_{k=1}^p \varphi_k^2(a_i) + (\text{Vol}(\text{grad} \varphi_1(a_i), \ldots, \text{grad} \varphi_p(a_i)))^2 \geq \sum_{k=1}^p \varphi_k^2(a_i) + \delta_i^{2p},
\]

condition (*) implies that

(1) \( |\varphi_k(a_i)| = o(|a_i|^s) \), for all \( s \in \mathbb{N} \) and \( 1 \leq k \leq p \);

(2) \( \delta_i = o(|a_i|^s) \), for all \( s \in \mathbb{N} \).

To complete the proof we need the following
**Lemma.** Let \( \{u_1^{(p)}, \ldots, u_r^{(p)}\} \in \mathbb{N} \) be a sequence of \( p \)-tuples of vectors in \( \mathbb{R}^n \). Suppose there is a sequence of positive numbers \( \alpha_1, \alpha_2, \ldots \rightarrow 0 \) such that for all \( s \in \mathbb{N} \)

\[
\delta_s = o(\alpha_s^p),
\]

where \( \delta_s \) is the distance from \( u_s^{(p)} \) to the linear subspace spanned by \( u_1^{(1)}, \ldots, u_r^{(p-1)} \).

Then we can find a sequence \( \{\lambda_1^{(2)}, \ldots, \lambda_r^{(p)}\} \in \mathbb{N} \) of \((p-1)\)-tuples of vectors in \( \mathbb{R}^n \), satisfying the following three conditions:

(i) For all \( s \in \mathbb{N} \), \( |\lambda_s^{(k)}| = o(\alpha_s^k) \), \( 2 \leq k \leq p \);

(ii) For each \( i \in \mathbb{N} \), \( u_i^{(2)} + \lambda_i^{(2)}, \ldots, u_i^{(p)} + \lambda_i^{(p)} \) are linearly independent;

(iii) For each \( i \in \mathbb{N} \), \( u_i \) belongs to the subspace spanned by \( u_i^{(2)} \) and \( \{\lambda_1^{(2)}, \ldots, \lambda_r^{(p)}\} \).

**Proof of the Lemma.** Let \( v_i^{(p)} \) denote the orthogonal projection of \( u_i^{(p)} \) to the subspace spanned by \( u_1^{(1)}, \ldots, u_r^{(p-1)} \), and let \( v_i^{(k)} = u_i^{(k)} \), \( k \leq p - 1 \). Then \( |v_i^{(k)} - u_i^{(k)}| = o(\alpha_i^k) \), \( 1 \leq k \leq p \). For each \( i \), the vectors \( v_i^{(1)}, \ldots, v_i^{(p)} \) are linearly dependent; the subspace \( L_i \) spanned by them has dimension \( \leq p - 1 \). Now we can choose \( w_i^{(1)}, \ldots, w_i^{(p)} \), where \( w_i^{(1)} = v_i^{(1)}, \ldots, w_i^{(k)} - v_i^{(k)} = o(\alpha_i^k) \), such that a subset of linearly independent vectors \( \{w_i^{(2)}, \ldots, w_i^{(p)}\} \) is a basis of \( L_i \). Consequently, \( w_i^{(1)} = u_i^{(1)} \) is a linear combination of \( w_i^{(2)}, \ldots, w_i^{(p)} \). Now put \( \lambda_i^{(k)} = w_i^{(k)} - u_i^{(k)} \), \( 2 \leq k \leq p \).

With this Lemma, we now complete the proof that \( (b) \Rightarrow (c) \). With \( u_i^{(k)} = \text{grad} \varphi_k(a_i) \) and \( a_i = |a_i| \), choose \( \lambda_i^{(k)} \) as in the above Lemma. We may assume \( |a_{i+1}| < \frac{3}{2} |a_i| \). Let \( \psi : \mathbb{R}^n \to [0, 1] \) be \( C^\infty \), \( \psi(x) = 1 \) in a neighborhood of \( 0 \in \mathbb{R}^n \) and \( \psi(x) = 0 \) for \( |x| \geq \frac{1}{4} \). Put

\[
\eta_1(x) = \sum_{i=1}^\infty \psi \left( \frac{x - a_i}{|a_i|} \right) \left( \varphi_1(a_i) + \epsilon_i |x - a_i|^2 \right), \quad \epsilon_i > 0,
\]

\[
\eta_k(x) = \sum_{i=1}^\infty \psi \left( \frac{x - a_i}{|a_i|} \right) \left( \varphi_k(a_i) - \lambda_i^{(k)} (x - a_i) \right), \quad k = 2, \ldots, p.
\]

Observe that

(\( \alpha \)) If we choose \( \epsilon_i > 0 \) such that for each \( s \in \mathbb{N} \), \( \epsilon_i = o(|a_i|^s) \) then \( \eta = (\eta_1, \ldots, \eta_p) \) is of class \( C^\infty \);

(\( \beta \)) \( \eta \) is (infinitely) flat at the origin;

(\( \gamma \)) For each \( i \in \mathbb{N} \), \( (\varphi - \eta)(a_i) = 0 \).

Now put \( \varphi = \varphi - \eta \). We shall show that \( \epsilon_i \), \( \epsilon_i = o(|a_i|^s) \), can be chosen in such a way that near each \( a_i \), \( \varphi^{-1}(0) \) is not a topological manifold.

By condition (iii) in the Lemma, \( \text{grad} \varphi_k(a_i) \) is a linear combination of \( \text{grad} \varphi_k(a_i) + \lambda_i^{(k)} \), \( k \geq 2 \), say

\[
\text{grad} \varphi_k(a_i) = \sum_{k=2}^p \xi_k \left( \text{grad} \varphi_k(a_i) + \lambda_i^{(k)} \right), \quad \xi_k \in \mathbb{R}.
\]
Choose $\epsilon_i = o(|a_i|)$ such that each $a_i$ is a non-degenerate critical point of 

$$
\rho(x) = \varphi_1(x) - \eta_1(x) + \sum_{k=2}^{p} \xi_k(\eta_k(x) - \varphi_k(x)).
$$

In a neighborhood of any $a_i$, for fixed $i$,

$$
\varphi^{-1}(0) = \{ x \in \mathbb{R}^n : \varphi_1(x) - \eta_1(x) = \ldots = \varphi_p(x) - \eta_p(x) = 0 \} = \{ x \in \mathbb{R}^n : \rho(x) = \varphi_2(x) - \eta_2(x) = \ldots = \varphi_p(x) - \eta_p(x) = 0 \}.
$$

Hence, near $a_i$, $\varphi^{-1}(0)$ is homeomorphic to the intersection of the locus of a non-degenerate quadratic form $\rho^{-1}(0)$ (Morse Lemma) with the $(p - 1)$-codimensional differentiable submanifold of $\mathbb{R}^n$, defined by $\varphi_2(x) - \eta_2(x) = \ldots = \varphi_p(x) - \eta_p(x) = 0$; thus $\varphi^{-1}(0)$ is not a manifold near $a_i$.

(c) $\iff$ (d). This follows easily from the following theorem.

**Theorem 2 (Tougeron-Merrien).** An ideal $I$ of $C^\infty$ is elliptic if and only if $I$ is finitely generated and $M^\infty \subset I$.

Recall (compare [13]) that $I$ is called **elliptic** if it contains an element $f$ having the property that $|f(x)| \geq c|x|^\alpha$ in a neighborhood of $0 \in \mathbb{R}^n$, where $\alpha$ and $c$ are positive constants. Such a function $f$ is also called elliptic.

It is easy to see that if $I$ is elliptic and generated by $f_1, \ldots, f_q$ then the element $f_1^2 + \ldots + f_q^2$ is elliptic. Hence (c) $\iff$ (d) follows from Theorem 2, because $Z_\varphi$ is the sum of squares of generators of $J(\varphi)$.

We now prove Theorem 2. Let $f$ be an elliptic element of $I$. Let $\psi \in M^\infty$ be any element; then $\eta(x) = \psi(x)/f(x)$, $\eta(0) = 0$, is a germ of a $C^\infty$ function. Hence $\psi = \eta f \in I$ and $M^\infty \subset I$. To show that $I$ is finitely generated choose $C^\infty$ functions $\varphi_1, \ldots, \varphi_k$ so that their formal Taylor's expansions $f^\omega(\varphi_1), \ldots, f^\omega(\varphi_k)$ generate the ideal $f^\omega(I)$ in the ring of formal power series. Here $f^\omega(I)$ consists of all formal Taylor's expansions of elements of $I$. Then $\{ \varphi_1, \ldots, \varphi_k, f \}$ is clearly a set of generators of $I$.

Conversely, suppose $I$ is generated by $f_1, \ldots, f_q$ and $M^\infty \subset I$. Choose an open neighborhood $W$ of 0 at $\mathbb{R}^n$ and representations $\tilde{f}_i \in C^\infty(W)$ of $f_i$ such that for any $h \in C^\infty(W)$ with $D^\alpha h(0)$, $|\alpha| = 0, 1, 2, \ldots$, there exist $g_1, \ldots, g_q \in C^\infty(W)$ for which $h = \sum_{i=1}^{q} \tilde{f}_i g_i$.

This choice is possible. We can certainly choose $\tilde{f}_i$, defined in a neighborhood $W$ of 0, such that $\cap_{i=1}^{q} \tilde{f}_i^{-1}(0) = \{0\}$. Then by applying a partition of unity, it is easy to fulfill the above requirement.

Hence, by construction, the ideal $\tilde{I}$ of $C^\infty(W)$ generated by $\tilde{f}_i$, $1 \leq i \leq q$, of which the zero set reduces to $\{0\}$, contains all functions in $C^\infty(W)$ which are infinitely flat at 0. Now by [10, Proposition 1],

$$
\sum_{i=1}^{q} (\tilde{f}_i(x))^2 \geq c|x|^\alpha
$$

near 0 for some $c, \alpha$ positive.
Remark. Theorem 2 has been communicated to the first author by J. Mather who has also given a slightly different proof.

(d) ⇒ (a). This has been proved by Tougeron [13, p. 220]. For any subset \( I \) in \( \mathcal{E} \) and \( \mu \in \mathbb{N} \), let \( \mathcal{E}_{[\mu]}(I) \) denote the ideal generated by \( I \) in \( \mathcal{E}_{[\mu]} \). Now \( \mathcal{M}^{\infty} \subset J(\varphi) \) implies that for any \( \mu \in \mathbb{N} \), there exists \( s \in \mathbb{N} \) such that \( \mathcal{M}^s \subset \mathcal{E}_{[\mu+1]}(J(\varphi)) \). Indeed, since \( Z_\varphi \) is elliptic, \( x^\alpha/Z_\varphi(x) \) is of class \( C^{s+1} \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is any multi-index with \( |\alpha| = s \), \( s \) large. Hence \( x^\alpha \in \mathcal{E}_{[\mu+1]}(J(\varphi)) \) and \( \mathcal{M}^s \subset \mathcal{E}_{[\mu+1]}(J(\varphi)) \).

We now show that \( j^{s2}(\varphi) \) is \( C^p \)-rigid in \( \mathcal{E}^p \). Let \( \psi \in \mathcal{E}^p \) be any element with \( j^{s2}(\psi) = j^{s2}(\varphi) \). Then \( \mathcal{E}(\varphi - \psi) \subset \mathcal{M}^{s+1} \), hence

\[
\mathcal{E}(\varphi - \psi) \subset \left[ \mathcal{E}_{[\mu+1]}(\mathcal{M}) \right] \left[ \mathcal{E}_{[\mu+1]}(J(\varphi)) \right]^2.
\]

Now applying Tougeron’s theorem [4, Theorem 1(b)], the proof is complete.

References