PLURIHARMONIC SYMBOLS
OF COMMUTING TOEPLITZ TYPE OPERATORS
ON THE WEIGHTED BERGMAN SPACES

YOUNG JOO LEE

Abstract. A class of Toeplitz type operators acting on the weighted Bergman spaces of the unit ball in the \( n \)-dimensional complex space is considered and two pluriharmonic symbols of commuting Toeplitz type operators are completely characterized.

1. Introduction and result. Let \( B \) be the unit ball of the \( n \)-dimensional complex space \( \mathbb{C}^n \) and \( V \) denote the Lebesgue volume measure on \( B \) normalized to have total mass 1. For \( \alpha > -1 \), define a measure \( dV_\alpha \) on \( B \) by \( dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dV(z) \) where the constant \( c_\alpha \) is a normalizing constant so that \( dV_\alpha \) is a probability measure. The weighted Bergman space \( A^p_\alpha (1 \leq p < \infty) \) is the closed subspace of the usual Lebesgue space \( L^p(B, V_\alpha) \) consisting of holomorphic functions on \( B \).

Corresponding to each \( \alpha \geq 1 \), we define an integral operator

\[
P_\alpha(\psi)(z) = \lambda_\alpha \int_B \frac{(1 - |w|^2)^\alpha}{(1 - \langle z, w \rangle)^{\alpha + 1}} \psi(w) dV(w) \quad (z \in B)
\]

where \( 1/\lambda_\alpha = \int_B (1 - |w|^2)^\alpha dV(w) \) and \( \langle z, w \rangle \) is the ordinary Hermitian inner product for points \( z, w \in \mathbb{C}^n \). It is known [2, Theorem 1] that for \( p \geq 1 \) and \( \alpha, \sigma > -1 \), \( P_\sigma \) is a bounded operator from \( L^p_\alpha \) onto \( A^p_\alpha \) if and only if \( p(1 + \sigma) > 1 + \alpha \). Moreover, if \( p(1 + \sigma) > 1 + \alpha \), then \( P_\sigma \) has the following reproducing properties:

\[
(1) \quad P_\sigma f = f \quad \text{and} \quad P_\sigma \bar{f} = \bar{f}(0)
\]

for every functions \( f \in A^p_\alpha \). See [2] for more informations on \( P_\sigma \) and related facts.

Let \( p \geq 1 \) and \( \alpha, \sigma > -1 \) be such that \( p(1 + \sigma) > 1 + \alpha \). For \( u \in L^p_\alpha \), the Toeplitz type operator \( T^\sigma_u \) with symbol \( u \) is the linear operator acting on \( A^p_\alpha \) defined by

\[
T^\sigma_u(f) = P_\sigma(uf)
\]

for functions \( f \in A^p_\alpha \). Then, \( T^\sigma_u \) is a bounded operator on \( A^p_\alpha \).

In this paper, we study the commuting problem of Toeplitz type operators, acting on the weighted Bergman spaces, with pluriharmonic symbols. A function \( u \in C^2(B) \) is said
to be \emph{pluriharmonic} if its restriction to an arbitrary complex line that intersects the ball is harmonic as a function of single complex variable. As is well known \cite[Theorem 4.4.9]{6}, every pluriharmonic function on $B$ can be expressed, uniquely up to an additive constant, as the sum of a holomorphic function and an antiholomorphic function.

On the unweighted $L^2$-Bergman space, a characterization problem for pluriharmonic symbols of commuting Toeplitz operators induced by the Bergman projection was first solved by Axler and Čučković \cite{1} on the disk. Later, on the setting of the ball, some partial results on this problem were obtained by Choe and Lee \cite{3}. In \cite{9}, Zheng solved the problem completely by using the $\mathcal{M}$-harmonic function theory. Also, the same problem was considered for certain Toeplitz type operator. In a recent paper, the author \cite{5} solved the problem for the certain Toeplitz type operator acting on the unweighted $L^1$-Bergman space of the ball.

In the present paper, we consider the same characterization problem for a class of Toeplitz type operators acting on the weighted Bergman spaces and completely characterize two pluriharmonic symbols for which the associate Toeplitz type operators commute. The following is the main theorem of the paper.

\textbf{Theorem 1.} Let $p \geq 1$ and $\alpha, \sigma > -1$ be such that $p(1+\sigma) > 1 + \alpha$ and $p(n+1+\sigma) = 2(n+1+\alpha)$. Suppose that $u$ and $v$ are two bounded pluriharmonic symbols on $B$. Then $T_u^* T_v^* = T_v^* T_u^*$ on $A_p^\alpha$ if and only if one of the following properties holds:

(a) $u$ and $v$ are both holomorphic on $B$.

(b) $u$ and $v$ are both antiholomorphic on $B$.

(c) There exist constants $c$ and $d$, not both 0, such that $cu + dv$ is constant on $B$.

In the unweighted case of $p = 2$, $\alpha = 0$ and $\sigma = 0$, the above theorem was obtained in \cite{1} on the disk and then, on the ball, in \cite{3} partially and \cite{9} completely. Recently, a certain weighted case was considered. In \cite{5}, the author obtained Theorem 1 in the weighted case $p \geq 1$, $\alpha \geq 0$ and $\sigma \geq n+1$.

In Section 2, we collect some basic facts and preliminary results on $\mathcal{M}$-harmonic and pluriharmonic functions needed in the proof. In Section 3, we prove Theorem 1 and give an application (see Corollary 9).

\section{Preliminaries}

For $z, w \in B$, $z \neq 0$, define

$$\varphi_z(w) = \frac{z - |z|^{-2} \langle w, z \rangle z - \sqrt{1 - |z|^2} (w - |z|^{-2} \langle w, z \rangle z)}{1 - \langle w, z \rangle}$$

and $\varphi_0(w) = -w$. Then $\varphi_z \in \mathcal{A}$, the group of all automorphisms (= biholomorphic self-maps) of $B$ and $\varphi_z$ is an involution. That is, $\varphi_z \circ \varphi_z$ is the identity on $B$. Furthermore, each $\varphi \in \mathcal{A}$ has a unique representation $\varphi = \varphi_z \circ U$ for some $z \in B$ and $U \in \mathcal{U}$, the group of all unitary operators on $\mathbb{C}^n$. For $u \in C^2(B)$ and $z \in B$, we define

$$(\tilde{\Delta} u)(z) = \Delta(u \circ \varphi_z)(0)$$

where $\Delta$ denotes the ordinary Laplacian. The operator $\tilde{\Delta}$ is called the \emph{invariant Laplacian} because it commutes with automorphisms of $B$ in the sense that $\tilde{\Delta}(u \circ \varphi) = (\tilde{\Delta} u) \circ \varphi$ for
\( \varphi \in \mathcal{A} \). We say that a function \( u \in C^2(B) \) is \( \mathcal{M} \)-harmonic on \( B \) if it is annihilated on \( B \) by the invariant Laplacian \( \Delta_\mathcal{M} \). \( \mathcal{M} \)-harmonic functions are characterized by a certain mean value property. To be more precise, for a continuous function \( F \) on \( B \), let \( R(F) \) denote the radialization of \( F \) defined by the formula

\[
R(F)(z) = \int_{\mathcal{U}} F(Uz) \, dU \quad (z \in B)
\]

where \( dU \) denotes the Haar measure on \( \mathcal{U} \).

It is not hard to see that if \( F \) is \( \mathcal{M} \)-harmonic on \( B \) and \( \varphi \in \mathcal{A} \), then the radialization \( R(F \circ \varphi) \) is constant on \( B \) and hence extends to a continuous function on \( \bar{B} \). The following proposition shows that this property, together with a certain mean value property, conversely characterizes the \( \mathcal{M} \)-harmonicity. The following characterization of \( \mathcal{M} \)-harmonicity given by the weighted area version of invariant mean value property will be useful in the proof of Theorem 1.

**Proposition 2.** Let \( t > -1 \) and suppose \( F \in L^1_t \) is a continuous function on \( B \). Then \( F \) is \( \mathcal{M} \)-harmonic on \( B \) if and only if

\[
(F \circ \varphi)(0) = \lambda_t \int_B (F \circ \varphi)(w)(1 - |w|^2)^t \, dV(w)
\]

and \( R(F \circ \varphi) \) has a continuous extension on \( \bar{B} \) for every \( \varphi \in \mathcal{A} \).

**Proof.** See Proposition 3 of [5].

To characterize the symbols, we also need a recent result of D. Zheng [9] on \( \mathcal{M} \)-harmonic products (the original statement in [9, Theorem 2] is in a slightly different form).

**Lemma 3.** Let \( u = f + \bar{g} \) and \( v = h + \bar{k} \) be two bounded pluriharmonic functions on \( B \). If \( f \bar{k} - h \bar{g} \) is \( \mathcal{M} \)-harmonic on \( B \), then \( u \) and \( v \) are all holomorphic or antiholomorphic or there exist constants \( c \) and \( d \), not both 0, such that \( cu + dv \) is constant on \( B \).

Let \( 1 \leq s < \infty \). The Hardy space \( H^s \) is the space of all functions \( f \) holomorphic on \( B \) for which

\[
\|f\|_s = \left( \sup_{0 < r < 1} \int_0^r |f(r\zeta)|^s \, d\sigma(\zeta) \right)^{1/s} < \infty.
\]

Here the measure \( \sigma \) denotes the rotation invariant probability measure on \( S \), the boundary of \( B \). It is well known (see [6, Theorem 5.6.4]) that if \( f \in H^s \), then \( f^*(\zeta) = \lim_{n \to \infty} f(r^n\zeta) \) exists for \( \sigma \) a.e. \( \zeta \in S \). The space \( \text{BMOA} \) consists of all \( f \in H^2 \) whose boundary functions \( f^* \) are functions of bounded mean oscillations with respect to the nonisotropic metric on \( S \) that corresponds to the Korányi approach regions. See [4] for details. Note that \( \text{BMOA} \) functions are closed under composition with automorphisms and \( \text{BMOA} \subseteq H^s \) for all \( s \). Also, it turns out (see, for example, [7]) that \( f \in \text{BMOA} \) if and only if

\[
\sup_{a \in B} \|f \circ \varphi_a - f(a)\|_2 < \infty.
\]

Before turning to the proof, we prove several simple results which will be used in the proof of Theorem 1. The following proposition was obtained in [8, Proposition 3] on the disk.
**Proposition 4.** Let $f + \tilde{g}$ be a bounded pluriharmonic function on $B$. Then $f$ and $g$ are all in $\text{BMOA}$.

**Proof.** Put $u = f + \tilde{g}$ for notational simplicity. By the mean value property for holomorphic functions, we see that

$$\int_{S} (f(r\zeta) - f(0)) (g(r\zeta) - g(0)) \, d\sigma(\zeta) = 0$$

for all $0 < r < 1$. It follows from a simple manipulation that

$$\int_{S} |f(r\zeta) - f(0)|^2 \, d\sigma(\zeta) + \int_{S} |g(r\zeta) - f(0)|^2 \, d\sigma(\zeta) = \int_{S} |u(r\zeta) - u(0)|^2 \, d\sigma(\zeta)$$

$$\leq 4 \|u\|_{\infty}^2 \quad (0 < r < 1)$$

and hence $\|f - f(0)\|_2^2 + \|g - g(0)\|_2^2 \leq 4 \|u\|_{\infty}^2$. Now, for $a \in B$, replace $u$ by $u \circ \varphi_a$ to get

$$\|f \circ \varphi_a - f(a)\|_2^2 + \|g \circ \varphi_a - g(a)\|_2^2 \leq 4 \|u\|_{\infty}^2.$$  

Therefore, taking the supremum over all $a \in B$, we have $f, g \in \text{BMOA}$, as desired. The proof is complete.

It is not hard to see that by an integration in polar coordinates, $H^s \subset A_t^s$ for all $s \geq 1$ and $t > -1$. Hence, the following corollary is an immediate consequence of Proposition 4 because $\text{BMOA} \subset H^s$ for all $s < \infty$.

**Corollary 5.** Let $f + \tilde{g}$ be a bounded pluriharmonic function on $B$. Then $f$ and $g$ are all in $A_t^s$ for all $s \geq 1$ and $t > -1$.

It is shown in [3] that if $F, G \in H^2$, then the radialization $\mathcal{R}(F \tilde{G})$ has a continuous extension on $\tilde{B}$. Since $\text{BMOA}$ is invariant under composition with automorphisms, we have the following.

**Lemma 6.** Let $F, G \in \text{BMOA}$. Then $\mathcal{R}(F \circ \varphi G \circ \varphi)$ extends to a continuous function on $\tilde{B}$ for every $\varphi \in \mathcal{A}$.

3. **Proof.** In this section we will prove Theorem 1 and give a simple application. We first need some elementary properties of canonical automorphisms. The real Jacobian $J_R \varphi_z$ of $\varphi_z$ is given by

$$J_R \varphi_z(w) = \left( \frac{1 - |z|^2}{1 - \langle w, z \rangle^2} \right)^{n+1} \quad (w \in B)$$

and the identity

$$1 - \langle \varphi_z(a), \varphi_z(b) \rangle = \frac{(1 - |z|^2)(1 - \langle a, b \rangle)}{(1 - \langle a, z \rangle)(1 - \langle z, b \rangle)}$$

holds for every $a, b \in B$. See [6, Chapter 2] for details. For $\alpha > -1$ and $a \in B$, we let

$$k_\alpha^a(z) = \left( \frac{\sqrt{1 - |a|^2}}{1 - \langle z, a \rangle} \right)^{n+1+\alpha} \quad (z \in B)$$
for notational simplicity. Then, by (2) and (3), we have a useful change-of-variable formula:

\[
\int_B F \, dV_\alpha = \int_B F \circ \varphi_a |k_\alpha^a|^2 \, dV_\alpha \quad (a \in B)
\]

for all measurable \( F \) on \( B \) whenever the integrals make sense.

Let \( p \geq 1 \) and \( \alpha > -1 \). For \( a \in B \), let \( U_a \) denote the linear operator on \( L^p_\alpha \) defined by

\[
U_a f = (f \circ \varphi_a)(k_\alpha^a)^{2/p}.
\]

Then, by the change-of-variable formula (4), one can easily see that

\[
\int_B |U_a f|^p \, dV_\alpha = \int_B |f \circ \varphi_a|^p |k_\alpha^a|^2 \, dV_\alpha = \int_B |f|^p \, dV_\alpha
\]

for every \( f \in L^p_\alpha \). Hence \( U_a \) is an isometry of \( L^p_\alpha \) into \( L^p_\alpha \). Using (3), one can easily show that \( k_\alpha^a(\varphi_a)k_\alpha^a = 1 \) on \( B \). It follows that \( U_a U_a \) is the identity operator on \( L^p_\alpha \) and hence \( U_a \) takes \( A^p_\alpha \) onto \( A^p_\alpha \).

Before proving Theorem 1, we have a couple of lemmas. The following lemma says that \( P_\sigma \) and \( U_a \) commute on \( L^p_\alpha \) in certain cases.

**Lemma 7.** Let \( p \geq 1 \) and \( \alpha, \sigma > -1 \) be such that \( p(1 + \sigma) > 1 + \alpha \) and \( p(n + 1 + \sigma) = 2(n + 1 + \alpha) \). For \( a \in B \), we have \( P_\sigma U_a = U_a P_\sigma \) on \( L^p_\alpha \).

**Proof.** Let \( f \in L^p_\alpha \) and \( z \in B \). By the change-of-variable formula (4) and simple manipulations using (3), one can see

\[
P_\sigma U_a f(z) = \lambda_\sigma \int_B \frac{(1 - |w|^2) f(\varphi_a(w)) (k_\alpha^a(w))^{2/p}}{(1 - \langle z, w \rangle)^{n+1+\sigma}} \, dV(w)
\]

\[
= \lambda_\sigma \int_B \frac{(1 - |\varphi_a(w)|^2) f(w) (k_\alpha^a(\varphi_a(w)))^{2/p}}{(1 - \langle z, \varphi_a(w) \rangle)^{n+1+\sigma}} \left( \frac{1 - |a|^2}{|1 - \langle w, a \rangle|^2} \right)^{n+1} \, dV(w)
\]

\[
= \frac{(1 - |a|^2)^{n+1+\sigma} - \frac{2n+1}{p}}{(1 - \langle z, a \rangle)^{n+1+\sigma}}
\]

\[
\times \lambda_\sigma \int_B \frac{(1 - |w|^2)^p f(w)(1 - \langle w, a \rangle)^{2n+1+\sigma}}{(1 - \langle \varphi_a(w), z \rangle)^{n+1+\sigma}} \, dV(w).
\]

On the other hand, since \( p(n + 1 + \sigma) = 2(n + 1 + \alpha) \) by the assumption, the last expression of the above turns into

\[
\left( \frac{1 - |a|^2}{1 - \langle z, a \rangle} \right)^{2n+1+\sigma} \lambda_\sigma \int_B \frac{(1 - |w|^2)^p}{(1 - \langle \varphi_a(z), w \rangle)} f(w) \, dV(w) = (k_\alpha^a(z))^{2/p} P_\sigma f(\varphi_a(z)),
\]

which is exactly \( U_a P_\sigma f(z) \). Hence \( P_\sigma U_a = U_a P_\sigma \) on \( L^p_\alpha \), as desired. The proof is complete. \( \Box \)
LEMMA 8. Let $p \geq 1$ and $\alpha, \sigma > -1$ be such that $p(1+\sigma) > 1+\alpha$ and $p(n+1+\sigma) = 2(n+1+\alpha)$. For $a \in B$ and $u \in L^\infty$, we have $U_a T^\sigma_{w,i_n} = T^\sigma_{w,i_n}$ on $A^\sigma_a$.

**Proof.** Let $f \in A^\sigma_a$. By Lemma 7, one obtains
\[
T^\sigma_{w,i_n} U_a f = T^\sigma_{w,i_n} [(f \circ \varphi_o)(k^\sigma_o)^{\frac{1}{p}}]
\]
\[
= P_\sigma [(u \circ \varphi_o)(f \circ \varphi_o)(k^\sigma_o)^{\frac{1}{p}}]
\]
\[
= P_\sigma U_a (uf)
\]
\[
= U_a T^\sigma_{w,i_n} f.
\]
Thus $T^\sigma_{w,i_n} U_a = U_a T^\sigma_{w,i_n}$ on $A^\sigma_a$. Now, use the fact $U_a U_a$ is the identity operator on $A^\sigma_a$ to get the desired result. This completes the proof.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** We begin with easy direction. First suppose that (a) holds, so that $u$ and $v$ are holomorphic on $B$. Then, by (1), we can see that $T^\sigma_u$ and $T^\sigma_v$ are, respectively, the operators on $A^\sigma_a$ of multiplication by $u$ and $v$. Thus $T^\sigma_u T^\sigma_v = T^\sigma_v T^\sigma_u$ on $A^\sigma_a$. Now assume (b), so that $\bar{u}$ and $\bar{v}$ are holomorphic on $B$. Using the explicit formula for $P_\sigma$, an application of Fubini’s theorem and (1), one can see that $T^\sigma_u T^\sigma_v = P_\sigma(uf)$ for every bounded functions $f$ in $A^\sigma_a$. Note that the set of all bounded functions in $A^\sigma_a$ forms a dense subset of $A^\sigma_a$. It follows from the continuity that $T^\sigma_u$ and $T^\sigma_v$ commute, as desired.

Finally, suppose (c) holds and assume $c \neq 0$ (the other case is in a similar fashion). Then $u = c_1 v + c_2$ for some constants $c_1$ and $c_2$, which implies $T^\sigma_u = c_1 T^\sigma_v + c_2$, so that $T^\sigma_v T^\sigma_u = c_1 T^\sigma_u T^\sigma_v + c_2 T^\sigma_u = T^\sigma_v T^\sigma_u$ on $A^\sigma_a$. Hence $T^\sigma_u$ and $T^\sigma_v$ commute on $A^\sigma_a$.

Now we prove the converse implication. Write $u = f + \bar{g}$ and $v = h + \bar{k}$ for some holomorphic $f$, $g$, $h$ and $k$. Then, by Corollary 5, the functions $f$, $g$, $h$ and $k$ are all in $A^p_a$. Moreover, $fh$, $h\bar{g}$ and $f\bar{k}$ are all in $L^1_a$ by Corollary 5 again. Let 1 denote the constant function 1 on $B$. Then, by the reproducing properties (1), we have
\[
T^\sigma_u T^\sigma_v 1 = T^\sigma_u (P_\sigma v)
\]
\[
= T^\sigma_u (h + \bar{k}(0))
\]
\[
= P_\sigma (fh + \bar{k}(0)f + h\bar{g} + g\bar{k}(0))
\]
\[
= fh + \bar{k}(0)f + P_\sigma (h\bar{g}) + g\bar{k}(0)
\]

Note that $f_B F dV_a = F(0)$ for holomorphic functions $F \in L^1_a$. It follows that
\[
\int_B (T^\sigma_u T^\sigma_v 1) dV_a = (T^\sigma_u T^\sigma_v 1)(0)
\]
\[
= f(0)h(0) + f(0)\bar{k}(0) + \bar{g}(0)\bar{k}(0) + P_\sigma (h\bar{g})(0)
\]
\[
= f(0)h(0) + f(0)\bar{k}(0) + \bar{g}(0)\bar{k}(0) + \lambda_\sigma \int_B h(w)\bar{g}(w)(1 + |w|^2)^{\sigma} dV(w).
\]

Similarly,
\[
\int_B (T^\sigma_v T^\sigma_u 1) dV_a = f(0)h(0) + h(0)\bar{g}(0) + \bar{g}(0)\bar{k}(0) + \lambda_\sigma \int_B f(w)\bar{k}(w)(1 + |w|^2)^{\sigma} dV(w).
\]
Since $T_u^\sigma T_v^\sigma = T_v^\sigma T_u^\sigma$ by the assumption, letting $\Lambda = f k - h g$, we have by (5) and (6) that

$$
\lambda_\sigma \int_B \Lambda(w)(1 - |w|^2)^\sigma \, dV(w) = \Lambda(0).
$$

Let $a \in B$. Multiplying both sides of the equation $T_u^\sigma T_v^\sigma = T_v^\sigma T_u^\sigma$ by $U_a$ on the left and by $U_a$ on the right, we obtain since $U_a U_a$ is the identity operator

$$
U_a T_v^\sigma U_a T_u^\sigma U_a = U_a T_u^\sigma U_a U_a T_v^\sigma U_a
$$

and therefore by Lemma 8

$$
T_{\varphi \varphi_u}^\sigma T_{\varphi \varphi_v}^\sigma = T_{\varphi \varphi_v}^\sigma T_{\varphi \varphi_u}^\sigma.
$$

Equation (7) was derived under the assumption that $T_u^\sigma T_v^\sigma = T_v^\sigma T_u^\sigma$. Thus (8) says that (7) remains valid with $\Lambda \circ \varphi_u$ in place of $\Lambda$. That is,

$$
\lambda_\sigma \int_B (\Lambda \circ \varphi_u)(w)(1 - |w|^2)^\sigma \, dV(w) = \Lambda(\varphi_u(a))
$$

for any $a \in B$. Let $\varphi \in \mathcal{A}$ and suppose $\varphi$ has the representation $\varphi = \varphi_u \circ U$ for some $a \in B$ and $U \in \mathcal{U}$. Then, by the rotation-invariance of the measure $dV$ and (9), we have the weighted area version of the invariant mean value property for $\Lambda$:

$$
\lambda_\sigma \int_B (\Lambda \circ \varphi)(w)(1 - |w|^2)^\sigma \, dV(w) = \lambda_\sigma \int_B (\Lambda \circ \varphi_u)(w)(1 - |w|^2)^\sigma \, dV(w)
$$

$$
= \Lambda(\varphi_u(a))
$$

$$
= (\Lambda \circ \varphi)(0)
$$

because $\varphi(0) = a$. On the other hand, since $f, g, h$ and $k$ are all in BMOA by Proposition 4, the function $\mathcal{R}(\Lambda \circ \varphi)$ has a continuous extension on $\hat{B}$ for any $\varphi \in \mathcal{A}$ by Lemma 6. Note from Corollary 5 that $\Lambda \in L^1_a$. It follows from Proposition 2 with $t = \sigma$ that $\Lambda = f k - h g$ is $M$-harmonic on $B$. Now, the characterization follows from Lemma 3. This completes the proof.

We conclude this paper with a simple application. We note that pluriharmonic functions are closed under complex conjugation.

**Corollary 9.** Let $p \geq 1$ and $\alpha, \sigma > -1$ be such that $p(1 + \sigma) > 1 + \alpha$ and $p(n + 1 + \sigma) = 2(n + 1 + \alpha)$. Suppose $u$ is a bounded pluriharmonic symbol on $B$. Then $T_u^\sigma T_u^\sigma = T_u^\sigma T_u^\sigma$ if and only if the image of $B$ under $u$ lies on some line in $\mathbb{C}$.

**Proof.** If $u(B)$ lies on some line in $\mathbb{C}$, a rotation and a translation show that there exist constants $c$ ($|c| = 1$) and $d$ such that $cu + d$ is real valued on $B$. It follows that

$$
T_u^\sigma = \frac{1}{c}(T_{cu + d}^\sigma - d) = \frac{1}{c}(T_{cu + d}^\sigma - d).
$$

Now, since $T_u^\sigma = \frac{1}{c}(T_{cu + d}^\sigma - d)$, we see that $T_u^\sigma$ and $T_u^\sigma$ commute. Conversely, assume $T_u^\sigma T_u^\sigma = T_u^\sigma T_u^\sigma$ on $A_u^\sigma$. Then, by Theorem 1, $u$ and $\bar{u}$ are holomorphic on $B$ or a nontrivial linear combination of $u$ and $\bar{u}$ is constant on $B$. The first case implies $u$ is constant on $B$, so we are done. Also, a simple manipulation shows that the latter case implies $u(B)$ lies on some line in $\mathbb{C}$. This completes the proof.
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Department of Mathematics
Mokpo National University
Chonnam 534-729
Korea
e-mail: yjlee@chungkye.mokpo.ac.kr