Canad. Math. Bull. Vol. **57** (2), 2014 pp. 270–276
http://dx.doi.org/10.4153/CMB-2013-001-9

Derivations on Toeplitz Algebras

Michael Didas and Jörg Eschmeier

Abstract. Let $H^2(\Omega)$ be the Hardy space on a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, and let $A \subset L^\infty(\partial\Omega)$ denote the subalgebra of all L^∞-functions f with compact Hankel operator H_f. Given any closed subalgebra $B \subset A$ containing $C(\partial\Omega)$, we describe the first Hochschild cohomology group of the corresponding Toeplitz algebra $\mathcal{T}(B) \subset B(H^2(\Omega))$. In particular, we show that every derivation on $\mathcal{T}(A)$ is inner. These results are new even for $n = 1$, where it follows that every derivation on $\mathcal{T}(H^\infty + C)$ is inner, while there are non-inner derivations on $\mathcal{T}(H^\infty + C(\partial\mathbb{B}_n))$ over the unit ball $\mathbb{B}_n$ in dimension $n > 1$.

1 Introduction

A recent result of Cao [2, Theorem 3] describes the first Hochschild cohomology group of the Toeplitz C^*-algebra generated by all Toeplitz operators with continuous symbol on the Hardy space over a strictly pseudoconvex domain in $\mathbb{C}^n$. Using a modification of Cao's arguments and a result of Davidson from 1977 [4, Corollary 4], the first author showed in [6] that every continuous derivation of the Toeplitz algebra $\mathcal{T}(H^\infty + C)$ on the Hardy space of the unit disc $\mathbb{D}$ is inner. It seems natural to ask if the first Hochschild cohomology group vanishes in this case, that is, if the latter result remains true without the continuity assumption, and, secondly, if a generalization to higher dimensions is possible. In this note we answer both questions in the affirmative by establishing a description of the first Hochschild cohomology group for a variety of Hardy-space Toeplitz algebras on strictly pseudoconvex domains, including the cases mentioned above.

Throughout this paper, we fix a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^∞-boundary. The Hardy space $H^2(\sigma)$ with respect to the normalized surface measure σ on $\partial\Omega$ can be defined as the norm closure of the set

$$A(\partial\Omega) = \left\{ f|\partial\Omega : f \in C(\overline{\Omega}), f|\Omega \text{ holomorphic} \right\}$$

in $L^2(\sigma)$. As usual, the Toeplitz operator $T_f \in B(H^2(\sigma))$ with symbol $f \in L^\infty(\sigma)$ is given by the formula

$$T_f = PM_f|H^2(\sigma),$$

where $P\colon L^2(\sigma) \to H^2(\sigma)$ denotes the orthogonal projection and $M_f\colon L^2(\sigma) \to L^2(\sigma), g \mapsto fg$, is the operator of multiplication with f. A natural question to ask is to what extent the membership of a function f to some special symbol class $S \subset L^\infty(\sigma)$ determines the behaviour of the corresponding Toeplitz operator T_f.

Received by the editors November 27, 2012.
Published electronically February 8, 2013.
AMS subject classification: **47B47**, 47B35, 47L80.
Keywords: derivations, Toeplitz algebras, strictly pseudoconvex domains.

Besides this single-operator point of view, one may ask for the properties of the so-called Toeplitz algebra

$$\mathcal{T}(S) = \overline{\mathrm{alg}}\{T_f : f \in S\} \subset B(H^2(\sigma))$$

associated with the symbol class S. Among the most important choices for S are the bounded holomorphic functions on Ω (more precisely, their non-tangential boundary values), which will be denoted by $H^\infty(\sigma)$ in the sequel, and the continuous functions $C(\partial\Omega)$, which give rise to the algebra of all analytic Toeplitz operators $\mathcal{T}(H^\infty(\sigma))$ and the Toeplitz C^*-algebra $\mathcal{T}(C(\partial\Omega))$, respectively. Another natural symbol class arising intrinsically in the theory of Toeplitz operators can best be expressed in terms of the corresponding Hankel operators

$$H_f \colon H^2(\sigma) \to L^2(\sigma), \quad h \mapsto (1-P)(fh).$$

From the work of Davidson [4] for the unit disc, Ding and Sun [8] for the unit ball, and Didas et al. [7] for the strictly pseudoconvex case, it is known that an operator $S \in B(H^2(\sigma))$ commutes modulo the compact operators with all analytic Toeplitz operators if and only if $S = T_f + K$, where K is compact and f belongs to the class

$$A = \{f \in L^\infty(\sigma) : H_f \text{ is compact}\}.$$

The identity $H_{fg} = H_f T_g + (1-P)M_f H_g$ valid for $f, g \in L^\infty(\sigma)$ shows that A is a closed subalgebra of $L^\infty(\sigma)$. Moreover, since Hankel operators with continuous symbol are compact in our setting ([13, Theorem 4.2.17]), A always contains the space $H^\infty(\sigma)+C(\partial\Omega)$. According to [1] (see also [10, Theorem 20]), the latter space is also a closed subalgebra of $L^\infty(\sigma)$. By a classical result of Hartman [9], the equality $A = H^\infty + C$ holds on the open unit disc in $\mathbb{C}$, while the inclusion $H^\infty(\sigma)+C(\partial\mathbb{B}_n) \subset A$ is known to be strict in the case of the open unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ for $n > 1$ [5].

Given any closed subalgebra $B \subset L^\infty(\sigma)$ with $C(\partial\Omega) \subset B \subset A$, our main result characterizes the first Hochschild cohomology group of the Toeplitz algebra $\mathcal{T}(B)$. We briefly recall the definition of the first Hochschild cohomology. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{E}$ be a Banach-$\mathcal{A}$-bimodule. A derivation from $\mathcal{A}$ into $\mathcal{E}$ is a (not necessarily continuous) linear map $D \colon \mathcal{A} \to \mathcal{E}$ satisfying the identity

$$D(AB) = D(A)B + AD(B) \qquad (A, B \in \mathcal{A}).$$

For a given element $S \in \mathcal{E}$, the commutator with S,

$$D \colon \mathcal{A} \to \mathcal{E}, \quad D(X) = [X, S] = XS - SX$$

defines a derivation from $\mathcal{A}$ into $\mathcal{E}$. Derivations arising in this way are called inner. Writing $Z^1(\mathcal{A}, \mathcal{E})$ for the space of all derivations from $\mathcal{A}$ into $\mathcal{E}$ and $N^1(\mathcal{A}, \mathcal{E}) \subset Z^1(\mathcal{A}, \mathcal{E})$ for the subspace consisting of all inner derivations, the first Hochschild cohomology group can be defined as the quotient

$$H^1(\mathcal{A}, \mathcal{E}) = Z^1(\mathcal{A}, \mathcal{E}) / N^1(\mathcal{A}, \mathcal{E}).$$

In particular, $H^1(\mathcal{A}, \mathcal{E})$ vanishes if and only if every derivation from $\mathcal{A}$ into $\mathcal{E}$ is inner.

Let us finally mention that, for a given Hilbert space H, we write $\mathcal{K}(H)$ for the ideal of all compact operators on H and that, for a subset $\mathcal{S} \subset B(H)$, we denote its essential commutant by

$$\mathcal{S}^{ec} = \left\{ X \in B(H) : [X, S] \in \mathcal{K}(H) \text{ for all } S \in \mathcal{S} \right\}.$$

We have now gathered all the notation required for an adequate formulation of our main result.

2 A Description of H^1 for Toeplitz Algebras

The following theorem can be thought of as a Banach-algebra version of Cao's result [2, Theorem 3] on the Toeplitz C^*-algebra.

Theorem 2.1 *Let $B \subset L^\infty(\sigma)$ be a closed subalgebra with $C(\partial\Omega) \subset B \subset A$. Then every derivation $D\colon \mathcal{T}(B) \to B(H^2(\sigma))$ is inner and the map*

$$\delta\colon H^1(\mathcal{T}(B), \mathcal{T}(B)) \longrightarrow \mathcal{T}(B)^{ec}/\mathcal{T}(B), \quad \delta([D]) = [S] \quad \textit{if} \quad D = [\,\cdot\,, S],$$

is a well-defined isomorphism of linear spaces.

We postpone the proof of this theorem for a moment in order to demonstrate some of its consequences. Let us first remark that, as A contains $C(\partial\Omega)$, the algebra $\mathcal{T}(A)$ contains the Toeplitz C^*-algebra $\mathcal{T}(C(\partial\Omega))$ and hence all compact operators on $H^2(\sigma)$ ([13, Theorem 4.2.24]). Together with the description of $\mathcal{T}(H^\infty(\sigma))^{ec}$ established in [7, Corollary 4.8], we obtain the chain of inclusions

$$\mathcal{T}(A)^{ec} \subset \mathcal{T}(H^\infty(\sigma))^{ec} = \left\{ T_f + K : f \in A, K \in \mathcal{K}\big(H^2(\sigma)\big) \right\} \subset \mathcal{T}(A).$$

The identity $T_{fg} - T_f T_g = PM_f H_g$ for $f, g \in L^\infty(\sigma)$ shows that $\mathcal{T}(A)$ is essentially commutative. Hence we can complete the above chain with the inclusion $\mathcal{T}(A) \subset \mathcal{T}(A)^{ec}$, which shows that in fact equality holds at each stage. In particular, we have

$$\mathcal{T}(A)^{ec} = \mathcal{T}\big(H^\infty(\sigma)\big)^{ec} = \mathcal{T}(A).$$

As a consequence we obtain the following special case of Theorem 2.1, which applies, for example, to the algebra $B = H^\infty(\sigma) + C(\partial\Omega)$.

Corollary 2.2 *If the algebra $B \subset A$ from Theorem 2.1 contains $H^\infty(\sigma)$, then we have $H^1(\mathcal{T}(B), \mathcal{T}(B)) \cong \mathcal{T}(A)/\mathcal{T}(B) \cong A/B$ as linear spaces.*

Proof For the first identification, it suffices to observe that

$$\mathcal{T}(A) = \mathcal{T}(A)^{ec} \subset \mathcal{T}(B)^{ec} \subset \mathcal{T}\big(H^\infty(\sigma)\big)^{ec} = \mathcal{T}(A)$$

holds and to apply Theorem 2.1. The second identification is given by the map $A/B \to \mathcal{T}(A)/\mathcal{T}(B)$, $[f] \mapsto [T_f]$, which is easily seen to be a vector-space isomorphism. For the details, see the remarks following Lemma 3.4. ∎

https://doi.org/10.4153/CMB-2013-001-9 Published online by Cambridge University Press

For $B = A = \{f \in L^\infty(\sigma) : H_f \text{ compact}\}$ the assertion of Corollary 2.2 deserves to be stated separately.

Corollary 2.3 *The first Hochschild cohomology group $H^1(\mathcal{T}(A), \mathcal{T}(A))$ vanishes on every bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^∞-boundary. In particular, every derivation on $\mathcal{T}(H^\infty(\sigma) + C(\partial\mathbb{D}))$ is inner on the unit disc $\mathbb{D}$.*

As mentioned before, it was observed by Davie and Jewell in [5] that the inclusion $H^\infty(\sigma)+C(\partial\mathbb{B}_n) \subset A$ is strict for every $n > 1$. Thus, in contrast to the case $n = 1$, by Corollary 2.2, there exist non-inner derivations on the Toeplitz algebra $\mathcal{T}(H^\infty(\sigma) + C(\partial\mathbb{B}_n))$ for every $n > 1$.

Finally, we obtain the result of Cao mentioned at the beginning, which was the starting point of our considerations. Note that $\mathcal{T}(C(\partial\Omega))^{ec} = \{T_{z_1}, \dots, T_{z_n}\}^{ec}$ (see, *e.g.*, [7, Lemma 4.1]).

Corollary 2.4 (Cao) *For the Toeplitz C^*-algebra on a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^∞-boundary, we have the isomorphism*

$$H^1\Big(\mathcal{T}\big(C(\partial\Omega)\big), \mathcal{T}\big(C(\partial\Omega)\big)\Big) \cong \{T_{z_1}, \dots, T_{z_n}\}^{ec}/\mathcal{T}\big(C(\partial\Omega)\big).$$

3 Proof of Theorem 2.1

Since in the setting of Theorem 2.1 the Toeplitz algebra $\mathcal{T}(B)$ contains the ideal of compact operators, a well-known standard argument implies that every derivation $D\colon \mathcal{T}(B) \to B(H^2(\sigma))$ is inner. For completeness sake we indicate the main ideas.

Proposition 3.1 *Let H be a Hilbert space. If $\mathcal{B}$ is a closed subalgebra of $B(H)$ containing the compact operators $\mathcal{K}(H)$, then every derivation from $\mathcal{B}$ into $B(H)$ is inner, that is, $D = [\,\cdot\,, S]$ for some operator $S \in B(H)$.*

Proof Since the ideal $\mathcal{K}(H) \subset B(H)$ has the property that $\mathcal{K}(H) = \mathcal{K}(H)^2$, where the right-hand side consists of all finite sums of products of two compact operators, the identity

$$D(XY) = D(X)Y + XD(Y)$$

shows that D maps $\mathcal{K}(H)$ into itself. Hence the restriction of D onto $\mathcal{K}(H)$ can be written as the commutator with some fixed operator $S \in B(H)$ ([11, Corollary 4.1.7]). More explicitly, there exists an operator $S \in B(H)$ such that

$$D(K) = KS - SK \quad \big(K \in \mathcal{K}(H)\big).$$

Since the identity

$$\begin{aligned} D(A)K + AD(K) = D(AK) &= AKS - SAK = (AKS - ASK) + (ASK - SAK) \\ &= AD(K) + (AS - SA)K \end{aligned}$$

holds for every $A \in \mathcal{B}$ and $K \in \mathcal{K}(H)$, it follows that $D = [\,\cdot\,, S]$. This observation completes the proof. ∎

https://doi.org/10.4153/CMB-2013-001-9 Published online by Cambridge University Press

Corollary 3.2 *If in the setting of the last proposition the quotient algebra $\mathcal{B}/\mathcal{K}(H)$ is commutative and semi-simple, then every derivation $D\colon \mathcal{B} \to \mathcal{B}$ has the form $D(X) = XS - SX$ $(X \in \mathcal{B})$ for some fixed operator $S \in \mathcal{B}^{ec}$ in the essential commutant of $\mathcal{B}$.*

Proof By the preceding proposition, there is an operator $S \in B(H)$ with $D = [\,\cdot\,, S]$. In particular, D induces a continuous derivation

$$\widehat{D}\colon \mathcal{B}/\mathcal{K}(H) \to \mathcal{B}/\mathcal{K}(H), \quad [X] \mapsto [D(X)].$$

Since $\mathcal{B}/\mathcal{K}(H)$ is supposed to be commutative and semi-simple, the Singer–Wermer theorem ([12, Theorem 1] or [3, Corollary 2.7.20]) implies that $\widehat{D} = 0$. Hence $D(\mathcal{B}) \subset \mathcal{K}(H)$ and $S \in \mathcal{B}^{ec}$. ■

Corollary 3.3 *Let $\mathcal{B} \subset B(H)$ be a unital closed subalgebra containing the compact operators $\mathcal{K}(H)$ such that the quotient algebra $\mathcal{B}/\mathcal{K}(H)$ is commutative and semi-simple. Then the mapping*

$$\delta\colon H^1(\mathcal{B}, \mathcal{B}) \to \mathcal{B}^{ec}/\mathcal{B}, \quad [D] \mapsto [S], \text{ where } D = [\,\cdot\,, S]$$

is a well-defined vector-space isomorphism.

Proof Let $D\colon \mathcal{B} \to \mathcal{B}$ be a given derivation. By Corollary 3.2 there is an operator $S \in \mathcal{B}^{ec}$ in the essential commutant of $\mathcal{B}$ such that $D = [\,\cdot\,, S]$. If $T \in B(H)$ is another operator with $D = [\,\cdot\,, T]$, then $T - S \in \mathcal{B}^c \subset \mathcal{K}(H)^c = \mathbb{C}1_H \subset \mathcal{B}$, and hence the equivalence classes of T and S in $\mathcal{B}^{ec}/\mathcal{B}$ coincide. If D is inner, then it follows that the operator S chosen above belongs to $\mathcal{B}$. Thus the map δ is well defined. Obviously, it is linear and injective. To complete the proof, observe that every operator $S \in \mathcal{B}^{ec}$ in the essential commutant of $\mathcal{B}$ induces a well-defined derivation $D\colon \mathcal{B} \to \mathcal{B}$, $A \mapsto [A, S]$. Hence δ is also surjective. ■

In the setting of Corollary 3.3, the first Hochschild cohomology group $H^1(\mathcal{B}, \mathcal{B})$ of $\mathcal{B}$ vanishes if and only if $\mathcal{B}$ is equal to its essential commutant $\mathcal{B}^{ec}$ in $B(H)$. It is elementary to check that this happens if and only if the quotient algebra $\mathcal{B}/\mathcal{K}(H)$ is a maximal abelian subalgebra of the Calkin algebra $\mathcal{C}(H) = B(H)/\mathcal{K}(H)$. This remark shows in particular that the quotient $\mathcal{T}(A)/\mathcal{K}(H^2(\sigma))$ is a maximal abelian subalgebra of the Calkin algebra $\mathcal{C}(H^2(\sigma))$.

Moreover, the proof of the main theorem can be completed by showing that the Toeplitz algebra $\mathcal{T}(B)$ induced by the symbol class $B \subset L^\infty(\sigma)$ occurring in the statement of Theorem 2.1 satisfies the requirements of Corollary 3.3. This will be done in the following lemma.

Lemma 3.4 *Let $B \subset L^\infty(\sigma)$ be a closed subalgebra with $C(\partial\Omega) \subset B \subset A$. Then the mapping $\tau\colon B \to \mathcal{T}(B)/\mathcal{K}(H^2(\sigma))$ defined by*

$$\tau(f) = T_f + \mathcal{K}\big(H^2(\sigma)\big)$$

is an isometric isomorphism between commutative semi-simple Banach algebras.

Proof Obviously the map τ is linear. By [7, Corollary 3.6] the equality of norms

$$\|f\|_{L^\infty(\sigma)} = \big\| T_f + \mathcal{K}\big(H^2(\sigma)\big) \big\|$$

holds for every function $f \in L^\infty(\sigma)$. Hence τ is isometric. Since $B \subset A$, the formula

$$T_f T_g - T_{fg} = -PM_f H_g \quad \big(f, g \in L^\infty(\sigma)\big)$$

shows that τ is an algebra homomorphism. The identity

$$T_{f_1} \cdots T_{f_r} - T_{f_1 \cdots f_r} = T_{f_1}(T_{f_2} \cdots T_{f_r} - T_{f_2 \cdots f_r}) + (T_{f_1} T_{f_2 \cdots f_r} - T_{f_1 \cdots f_r})$$

together with an elementary induction implies that

$$T_{f_1} \cdots T_{f_r} + \mathcal{K}\big(H^2(\sigma)\big) = T_{f_1 \cdots f_r} + \mathcal{K}\big(H^2(\sigma)\big)$$

belongs to the range of τ for all $f_1, \dots f_r \in B$. Since the range of τ is closed, this argument yields the surjectivity of τ. As a unital closed subalgebra of the commutative C^*-algebra $L^\infty(\sigma)$, the Banach algebra B is semi-simple. This observation completes the proof. ■

Let $B \subset L^\infty(\sigma)$ be a closed subalgebra as in Lemma 3.4. If $f \in L^\infty(\sigma)$ is a function with $T_f + \mathcal{K}(H^2(\sigma)) \in \mathcal{T}(B)/\mathcal{K}(H^2(\sigma))$, then there is a function $g \in B$ with $T_{f-g} = T_f - T_g \in \mathcal{K}(H^2(\sigma))$ and hence $f = g \in B$. Therefore in the setting of Corollary 2.2, the mapping

$$A/B \to \mathcal{T}(A)/\mathcal{T}(B), \quad f + B \mapsto T_f + \mathcal{T}(B)$$

is a vector-space isomorphism as we claimed.

References

[1] A. Aytuna and A. M. Chollet, *Une extension d'un résultat de W. Rudin*. Bull. Soc. Math. France **104**(1976), no. 4, 383–388.

[2] G. Cao, *Toeplitz algebras on strongly pseudoconvex domains*. Nagoya Math. J. **185**(2007), 171–186.

[3] H. G. Dales, *Banach algebras and automatic continuity*. London Mathematical Society Monographs, New Series, 24, The Clarendon Press, Oxford University Press, New York, 2000.

[4] K. R. Davidson, *On operators commuting with Toeplitz operators modulo the compact operators*. J. Functional Analalysis **24**(1977), no. 3, 291–302. http://dx.doi.org/10.1016/0022-1236(77)90060-X

[5] A. M. Davie and N. P. Jewell, *Toeplitz operators in several complex variables*. J. Functional Analysis 26(1977), no. 4, 356–368. http://dx.doi.org/10.1016/0022-1236(77)90020-9

[6] M. Didas, *Every continuous derivation of* $\mathcal{T}(H^\infty + C(\mathbb{T}))$ *is inner*. Preprint Nr. 317, Department of Mathematics, Saarland University. http://www.math.uni-sb.de/service/preprints/preprint317.pdf

[7] M. Didas, J. Eschmeier, and K. Everard, *On the essential commutant of analytic Toeplitz operators associated with spherical isometries*. J. Functional Analysis **261**(2011), no. 5, 1361–1383. http://dx.doi.org/10.1016/j.jfa.2011.05.005

[8] X. Ding and S. Sun, *Essential commutant of analytic Toeplitz operators*. Chinese Sci. Bull. **42**(1997), no. 7, 548–552. http://dx.doi.org/10.1007/BF03182613

[9] P. Hartman, *On completely continuous Hankel operators*. Proc. Amer. Math. Soc. **9**(1958), 862–866. http://dx.doi.org/10.1090/S0002-9939-1958-0108684-8

[10] N. P. Jewell and S. G. Krantz, *Toeplitz operators and related function algebras on certain pseudoconvex domains*. Trans. Amer. Math. Soc. **252**(1979), 297–312. http://dx.doi.org/10.1090/S0002-9947-1979-0534123-7

[11] S. Sakai, *C^*-algebras and W^*-algebras*. Springer, Berlin, 1971.
[12] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*. Math. Ann. **129**(1955), 260–264. http://dx.doi.org/10.1007/BF01362370
[13] H. Upmeier, *Toeplitz operators and index theory in several complex variables*. Operator Theory: Advances and Applications, 81, Birkhäuser Verlag, Basel, 1996.

Fachrichtung Mathematik, Universität des Saarlandes, Postfach 15 11 50, D-66041 Saarbrücken, Germany
e-mail: didas@math.uni-sb.de eschmei@math.uni-sb.de