ORLICZ SPACES WITHOUT STRONGLY EXTREME POINTS AND WITHOUT $H$-POINTS

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ABSTRACT. W. Kurc [5] has proved that in the unit sphere of Orlicz space $L^\Phi(\mu)$ generated by an Orlicz function $\Phi$ satisfying the suitable $\Delta_2$-condition and equipped with the Luxemburg norm every extreme point is strongly extreme. In this paper it is proved in the case of a nonatomic measure $\mu$ that the unit sphere of the Orlicz space $L^\Phi(\mu)$ generated by an Orlicz function $\Phi$ which does not satisfy the suitable $\Delta_2$-condition and equipped with the Luxemburg norm has no strongly extreme point and no $H$-point.

0. Introduction. In the sequel $\mathbb{R}$ denotes the reals, $\mathbb{R}_+$ denotes the nonnegative reals and $\Phi$ denotes an arbitrary Orlicz function, i.e. $\Phi: \mathbb{R} \to \mathbb{R}_+, \Phi(0) = 0$, and $\Phi$ is even and convex. $(T, \Sigma, \mu)$ denotes a positive nonatomic (finite or infinite) measure space. $L^0(\mu)$ stands for the space of (equivalence classes of) all $\Sigma$-measurable real functions defined on $T$.

Given an Orlicz function $\Phi$ we define on $L^0(\mu)$ a convex functional $I_\Phi$ by

$$I_\Phi(x) = \int_T \Phi\left(x(t)\right) d\mu \quad (\forall x \in L^0(\mu)).$$

This functional is a convex modular on $L^0(\mu)$ (see [7]), i.e. $I_\Phi(0) = 0$, $I_\Phi$ is convex and even and $x = 0$ whenever $I_\Phi(\lambda x) = 0$ for any $\lambda > 0$. The Orlicz space $L^\Phi(\mu)$ generated by an Orlicz function $\Phi$ is defined to be the set of all $x \in L^0(\mu)$ such that $I_\Phi(\lambda x) < \infty$ for some $\lambda > 0$ depending on $x$. This space can be endowed with the norm

$$\|x\|_\Phi = \inf\{\lambda < 0 : I_\Phi(x/\lambda) \leq 1\} \quad (\forall x \in L^\Phi(\mu)),$$

called the Luxemburg norm. The couple $(L^\Phi(\mu), \| \cdot \|_\Phi)$ is a Banach space (see [4], [6] and [7]).

Recall that an Orlicz function $\Phi$ satisfies the $\Delta_2$-condition for all $u \in \mathbb{R}$ (at infinity) if there are positive constants $K$ and $c$ such that $0 < \Phi(c) < +\infty$ and $\Phi(2u) \leq K\Phi(u)$ for any $u \in \mathbb{R}$ (for $u \in \mathbb{R}$ satisfying $|u| \geq c$).

We say that an Orlicz function $\Phi$ satisfies the suitable $\Delta_2$-condition if $\Phi$ satisfies the $\Delta_2$-condition for all $u \in \mathbb{R}$ whenever $\mu$ is infinite and $\Phi$ satisfies the $\Delta_2$-condition at infinity whenever $\mu$ is finite.

For an arbitrary Banach space $X$, $S(X)$ denotes its unit sphere.
A point $x \in S(X)$ is said to be strongly extreme (see [3] and [8]) if for any sequence $(x_n)$ in $X$ the conditions $\|x + x_n\| \to 1$ and $\|x - x_n\| \to 1$ imply that $\|x_n\| \to 0$.

A point $x \in S(X)$ is said to be an $H$-point if for any sequence $(x_n)$ in $X$ such that $\|x_n\| \to \|x\|$ and $x_n$ tends weakly to $x$, we have $\|x - x_n\| \to 0$.

1. Results. We start with the following:

**Theorem 1.** Let $\Phi$ be an Orlicz function which does not satisfy the suitable $\Delta_2$-condition. Let us assume additionally in the case when $\mu$ is infinite that $\Phi$ vanishes only at zero. Then $S(L^\Phi)$ has no strongly extreme point.

**Proof.** It is known that every strongly extreme point is extreme and that under the assumptions concerning $\Phi$, if $x \in S(L^\Phi)$ is extreme then it must be $I_{\Phi}(x) = 1$ (see [2]). Therefore, it suffices to consider only these points of $S(L^\Phi)$ for which $I_{\Phi}(x) = 1$.

Assume that $I_{\Phi}(x) = 1$, $\mu$ is finite (for infinite $\mu$ the proof is analogous) and $\Phi$ does not satisfy the $\Delta_2$-condition at infinity. Then there exists a sequence $(u_n)$ of positive reals with $u_n \to \infty$ as $n \to \infty$ and such that

$$\Phi \left( 1 + \frac{1}{n} u_n \right) > 2^{-n} \Phi(u_n).$$

Let $b > 0$ be large enough, so that the set $B = \{ t \in T : b^{-1} \leq |x(t)| \leq b \}$ has positive measure. Let $A_n \subset B, A_n \in \Sigma (n = 1, 2, \ldots)$ be such that $\Phi(u_n) \mu(A_n) = 2^{-n}$ (if necessary we can pass to a subsequence). Of course, $\mu(A_n) \to 0$ as $n \to \infty$. Define

$$x_n = \frac{1}{2} u_n \chi_{A_n} \text{ sgn } x \quad (n = 1, 2, \ldots).$$

We have

$$I_{\Phi}(x + x_n) = I_{\Phi}(x \chi_{T \setminus A_n}) + I_{\Phi}(x \chi_{A_n} + x_n)$$

$$\leq I_{\Phi}(x \chi_{T \setminus A_n}) + \frac{1}{2} \left( \Phi(2b) \mu(A_n) + \Phi(u_n) \mu(A_n) \right)$$

$$\to I_{\Phi}(x) = 1.$$

Moreover, $I_{\Phi}(x + x_n) \geq I_{\Phi}(x \chi_{T \setminus A_n}) \to 1$. Thus, $I_{\Phi}(x + x_n) \to 1$, whence it follows that $\|x + x_n\|_{\Phi} \to 1$. We have also

$$I_{\Phi}(x - x_n) \leq I_{\Phi}(\|x\| + |x_n|) \to 1$$

and

$$I_{\Phi}(x - x_n) \geq I_{\Phi}(x \chi_{T \setminus A_n}) \to 1,$$

whence it follows that $I_{\Phi}(x - x_n) \to 1$, i.e. $\|x - x_n\|_{\Phi} \to 1$. On the other hand

$$I_{\Phi} \left( 2 \left( 1 + \frac{1}{n} \right) x_n \right) = \Phi \left( 1 + \frac{1}{n} \right) u_n \mu(A_n) > 2^n \Phi(u_n) \mu(A_n) = 1.$$

Therefore, $\|x_n\|_{\Phi} \geq \frac{1}{2} \left( 1 + \frac{1}{n} \right)^{-1} \geq \frac{1}{4}$. This means that $x$ is not a strongly extreme point. The proof is finished.
THEOREM 2. Let $\Phi$ be an Orlicz function which does not satisfy the suitable $A_2$-condition. Let us assume additionally in the case when $\mu$ is infinite that $\Phi$ vanishes only at zero. Then $S(L^\Phi)$ has no $H$-point.

PROOF. We will restrict ourselves only to finite measure. In the case of an infinite measure the proof is analogous. Take $d > 0$ large enough, so that defining $A = \{t \in T : d^{-1} \leq |x(t)| \leq d\}$ we have $I_\Phi(x\chi_A) \geq \frac{1}{2}I_\Phi(x)$. Next, take $C_\infty \subset A, C_n \in \Sigma$ such that

$$I_\Phi(x\chi_{C_n}) = 2^{-n}I_\Phi(x) \quad (n = 1, 2, \ldots).$$

Since $\Phi$ has only finite values by the definition and $\mu$ is finite, the condition $\|x\|_\Phi = 1$ yields $I_\Phi(x) > 0$ (note that for infinite $\mu$ it can be $I_\Phi(x) = 0$ even if $\|x\|_\Phi = 1$ whenever $\Phi$ vanishes outside zero). In fact, defining $\alpha = \sup\{u \geq 0 : \Phi(u) = 0\}$ the condition $I_\Phi(x) = 0$ yields $|x(t)| \leq \alpha$ for $\mu$-a.e. $t \in T$. Next by the finiteness of $\mu$ we have $I_\Phi(\lambda x) < \infty$ for any $\lambda > 0$. This is a contradiction, because in the case when $I_\Phi(x) = 0$ the quality $\|x\|_\Phi = 1$ implies that $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$ (see [2]).

In view of the assumption that $\Phi$ does not satisfy the $A_2$-condition at infinity, there exists a sequence $(u_n)$ of positive reals such that $u_n \to \infty$ as $n \to \infty$ and

$$\Phi\left(\left(1 + \frac{1}{n}\right)u_n\right) > 2^n\Phi(u_n) \quad (n = 1, 2, \ldots).$$

Passing to subsequences of $(u_n)$ and $(C_n)$ if necessary, we can find a sequence $(D_n)$ of measurable subsets of $C_n$ such that

$$\Phi(u_n)\mu(D_n) = I_\Phi(x\chi_{C_n}) \quad (n = 1, 2, \ldots).$$

Define

$$x_n = x\chi_{\overline{T \setminus C_n}} - u_n(\text{sgn}x)\chi_{D_n}.$$

We have

$$I_\Phi(x_n) = I_\Phi(x\chi_{\overline{T \setminus C_n}}) + \Phi(u_n)\mu(D_n).$$

If $I_\Phi(x) = 1$ then $I_\Phi(x_n) = 1$ for any $n \in \mathbb{N}$, whence $\|x_n\|_\Phi = 1$. If $I_\Phi(x) < 1$, then the equality $\|x\|_\Phi = 1$ yields $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$. Since $x$ is uniformly bounded on the sets $C_n$, we get $I_\Phi(\lambda x_n) = \infty$ for any $\lambda > 1$ and $n \in \mathbb{N}$. Hence it follows that $\|x_n\|_\Phi = 1$ for any $n \in \mathbb{N}$. We have

$$x - x_n = x\chi_{C_n} + u_n\chi_{D_n} \text{sgn}x.$$

The sequence $(x\chi_{C_n})$ is norm convergent to zero, because the function $x$ is uniformly bounded on $C_n, n = 1, 2, \ldots$, and $\mu(C_n) \to 0$ as $n \to \infty$. Therefore, in order to prove that the sequence $(x_n)$ is weakly convergent to $x$ it suffices to prove that the sequence $(y_n)$, where $y_n = u_n\chi_{D_n} \text{sgn}x$, is weakly convergent to zero. We have $y_n \in E^\Phi(\mu)$ for any $n \in \mathbb{N}$, where $E^\Phi(\mu) = \{x \in L^0(\mu) : I_\Phi(\lambda x) < \infty \text{ for any } \lambda > 0\}$. Hence it follows that any linear continuous singular functional over $L^\Phi(\mu)$ vanishes at $y_n, n = 1, 2, \ldots$ (for the description of the dual space of $L^\Phi(\mu)$ see [1]). Thus, in order to prove that $y_n \to$...
0 weakly, it suffices to consider only regular (i.e. order continuous) linear continuous functionals over $L^\Phi(\mu)$. Take an arbitrary linear continuous regular functional $\xi_f$ over $L^\Phi(\mu)$ generated by a function $f \in L^\Phi(\mu)$, where $\Phi^*$ is the function complementary to $\Phi$ in the sense of Young. Let $\lambda > 0$ be such that $I_{\Phi^*}(\lambda f) < \infty$. We have

$$|\xi_f(y_n)| = \left| \int f(t)y_n(t) \, d\mu \right| \leq \frac{1}{\lambda} \left\{ I_{\Phi^*}(\lambda f\chi_{D_n}) + I_{\Phi}(y_n) \right\}$$

as $n \to \infty$ because $\mu(D_n) \to 0$ as $n \to \infty$. Moreover,

$$I_{\Phi^*}\left(1 + \frac{1}{n}\right) = I_{\Phi}\left(1 + \frac{1}{n}\right) \mu(D_n) \geq 2^{2n}\Phi(u_n) \mu(D_n) = 2^n I_{\Phi}(x) \geq 1$$

for $n \in \mathbb{N}$ large enough. Therefore

$$\|x - x_n\|_{\Phi} \geq 1/\left(1 + \frac{1}{n}\right) \geq 1/2$$

for sufficiently large $n \in \mathbb{N}$, which means that the sequence $(x_n)$ is not norm convergent to $x$, i.e. $x$ is not an $H$-point. Since $x \in S(L^\Phi)$ was arbitrary, the proof is finished.

Recall that a point $x \in S(X)$ is said to be strongly exposed if there exists a functional $x^* \in S(X^*)$ such that $x^*(x) = 1$, and for any sequence $(x_n)$ in $X$ the condition $x^*(x - x_n) \to 0$ implies that $\|x - x_n\| \to 0$.

Since any strongly exposed point is strongly extreme, we obtain from Theorem 1 the following:

**Corollary.** If $\Phi$ is an Orlicz function which does not satisfy the suitable $\Delta_2$-condition and if additionally in the case when $\mu$ is infinite $\Phi$ vanishes only at zero, then $S(L^\Phi)$ has no strongly exposed point.

**References**


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