ATOMLESS LATTICE-ORDERED GROUPS

In Memoriam—C. S. Milloy

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ABSTRACT. We show the existence of atomless lattice-ordered groups which have doubly transitive representations. In so doing, we answer a question of M. Giraudet from 1981 [4].

Let $G$ be a lattice-ordered group with identity $e$. A strictly positive element of $G$ is called an atom if it cannot be written as the join of two disjoint strictly positive elements of $G$. Note that every strictly positive element of any linearly ordered group is an atom.

If $(\Omega, \leq)$ is an infinite chain (linearly ordered set), we write $A(\Omega)$ for $\text{Aut}(\Omega, \leq)$. $A(\Omega)$ is a group under composition and a lattice under the pointwise ordering. An important sublattice subgroup of this lattice-ordered group is $B(\Omega)$, the subset of all elements of $A(\Omega)$ whose support is bounded both above and below ($\text{supp}(g) = \{ \alpha \in \Omega : \alpha g \neq \alpha \}$).

A subgroup $G$ of $A(\Omega)$ is said to be doubly transitive on $\Omega$ if for all $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ in $(\Omega, \leq)$, there is a $g \in G$ such that $\alpha_j g = \beta_j (j = 1, 2)$. (Sublattice subgroups of $A(\Omega)$ that are doubly transitive are $m$-transitive for all positive integers $m$—see [2, Lemma 1.10.1]).

In 1981, M. Giraudet [4] asked (Problem 10.16) if for some infinite chain $(\Omega, \leq)$, there is an atomless doubly transitive sublattice subgroup $H$ of $B(\Omega)$; the other question of [4], Problem 10.15, was referred to and partially answered in [1]. The purpose of this short note is to observe that a construction due to Keith R. Pierce (see [5] or [2, Chapter 10]) provides a positive answer. Indeed

THEOREM. For every lattice-ordered group $G$, there is a lattice-ordered group $H$ containing $G$ as a sublattice subgroup and such that every pair of strictly positive elements of $H$ are conjugate in $H$. Moreover, we can find such an $H$ that is a doubly transitive sublattice subgroup of $B(\Omega)$ for some infinite chain $(\Omega, \leq)$.

The proof we give assumes the Generalized Continuum Hypothesis; this dependence can be avoided, see [2, p. 205].

PROOF. All but the last sentence of the theorem is established in Theorem 10.8 of [2]. Indeed, by [2, Corollary 2L], it suffices to prove the theorem for $G \subseteq B(T)$, $G$ doubly transitive on $T$ and $|T| = |G|$, a regular uncountable cardinal. Now the proof of...
[2, Lemma 10.9] shows that if $F$ is a sublattice subgroup of $B(T)$, then $F \psi \subseteq B(T_\psi)$. Similarly, the proof of [2, Lemma 10.10] establishes that $F \subseteq B(T_\psi)$ for such $F$. Hence, as noted in [2], for the chain $(\Omega_1, \leq)$ obtained on p. 203 of [2], $G \subseteq B(\Omega_1)$. Moreover, any two strictly positive elements of $G$ are conjugate in $B(\Omega_1)$. The construction ensures that $|\Omega_1| = |G|$ and that $B(\Omega_1)$ is doubly transitive (since $\Omega_1$ is an $\alpha$-set—see [2, pp. 203 and 187]). For each pair of strictly positive elements of the image of $G$, choose an element of $B(\Omega_1)$ conjugating the first to the second. Also for each pair of strictly increasing pairs of elements of $\Omega_1$, choose an element of $B(\Omega_1)$ mapping the former to the latter. Let $G^\dagger$ be the sublattice subgroup of $B(\Omega_1)$ generated by the image of $G$ and these $|G| + |G|$ elements of $B(\Omega_1)$. Then $|G^\dagger| = |G|$, $G^\dagger \subseteq B(\Omega_1)$ and $G^\dagger$ is doubly transitive on $\Omega_1$.

We can therefore proceed by induction: $G_0 = G$, $G_{m+1} = (G_m)^\dagger$, for each natural number $m$, to obtain $G_{m+1} \subseteq B(\Omega_{m+1})$, a sublattice subgroup acting doubly transitively on $\Omega_{m+1}$, $|G_{m+1}| = |G|$ and $G_{m+1}$ containing an image of $G_m$. Consequently, $H = \bigcup_{m=0}^\infty G_m$ acts doubly transitively on $\Omega = T \cup \bigcup_{m=1}^\infty \Omega_m$ and satisfies the conclusion of the theorem. 

**COROLLARY 1.** Every lattice-ordered group $G$ can be embedded in an atomless lattice-ordered group $H$. Moreover, $H$ can be chosen to be a doubly transitive sublattice subgroup of $B(\Omega)$ for some suitable infinite chain $(\Omega, \leq)$.

**PROOF.** Let $H$ be as in the theorem. Let $h_1 \in H$ be strictly positive. Let $\alpha, \beta \in \Omega$ with $\alpha < \text{supp}(h_1) < \beta$. Since $H$ is transitive on $\Omega$ (indeed, doubly transitive), there is $b \in H$ such that $\alpha b = \beta$. Let $h_2 = b^{-1}h_1b \in H$. Then $h_1 \wedge h_2 = e$ and $h_1, h_2 \neq e$. Hence $h = h_1 \vee h_2$ is not an atom. If $f \in H$ is strictly positive, then for some $a \in H$, $f = a^{-1}ha$. Now $f$ is the join of the disjoint strictly positive elements $a^{-1}h_1a$ and $a^{-1}h_2a$, and so is not an atom.

As noted in [2, Theorem 12E], some algebraically closed lattice-ordered groups are doubly transitive. By [3, Proposition 0.2.3], they are not completely distributive; so any doubly transitive algebraically closed lattice-ordered group is not contained in the set of elements of bounded support of that chain [2, Theorems 8D and 8.2.1]. However, by Corollary 1, we immediately obtain another rich non-trivial class of atomless lattice-ordered groups.

**COROLLARY 2.** Every algebraically closed lattice-ordered group is atomless.

**REFERENCES**


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