DEFORMATION OF THE UNIVERSAL ENVELOPING ALGEBRA OF $\Gamma(\sigma_1, \sigma_2, \sigma_3)$

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ABSTRACT. The defining relations for the Lie superalgebra $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ as a contragredient algebra are discussed and a PBW type basis theorem is proved for the corresponding $q$-deformation.

1. Introduction. In this note, we study the $q$-analog of the universal enveloping algebra of the Lie superalgebra $G = \Gamma(\sigma_1, \sigma_2, \sigma_3)$. This Lie superalgebra is special: as a contragredient algebra, the defining matrix of $G$ over the complex number field $\mathbb{C}$ depends on a parameter, the algebra itself already admits a one-parameter deformation. To apply the idea of Drinfeld and Jimbo to define the $q$-analog of the universal enveloping algebra $U(G)$, one needs to work with a non-integer defining matrix. Hence in general, the deformation is defined over some transcendental function field extension of $\mathbb{C}$ (or just the field $\mathbb{C}$, if one takes the deformation parameter to be a suitable complex number). The deformation thus defined will actually be a two-parameter family of algebras.

We discuss the defining relations for $G$ as a contragredient algebra in Section 2. Although these defining relations are known to the experts (cf. the discussion in [8]), we are unable to find a suitable reference, so we provide a complete proof for these relations.

In Section 3, we define the deformation $\mathcal{U}$ of $U(G)$ and study its structure. As in the other cases of type II classical contragredient Lie superalgebras (see [4] for the definition of type II Lie superalgebras, see [5] for a definition of the $q$-deformation of $U(\mathfrak{osp}(m, 2n))$, the usual Drinfeld-Jimbo deformation of $U(G)$ does not contain a copy of the standard deformation of $U(G_0)$, where $G_0$ is the even part of $G$, since there are not enough group like elements in it. However, we show that in our case, one can introduce suitable elements in $\mathcal{U}$ such that a PBW type theorem (Theorem 3.3) holds for $\mathcal{U}$.

2. The defining relations for $G$. We use the notation adopted in [9]. Recall that the algebra $G$ is defined as a contragredient Lie superalgebra with three nonzero elements $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{C}$ satisfying $\sigma_1 + \sigma_2 + \sigma_3 = 0$, with generators $e_i, f_i, h_i$ ($i = 1, 2, 3$) and the defining matrix $(a_{ij})_{3 \times 3}$ given as follows:

$$
egin{pmatrix}
0 & 2\sigma_2 & 2\sigma_3 \\
-1 & 2 & 0 \\
-1 & 0 & 2
\end{pmatrix}.
$$
The grading on \( G \) is given by

\[
\deg h_i = 0, i = 1, 2, 3; \quad \deg e_i = \deg f_i = 0, i = 2, 3; \quad \deg e_1 = \deg f_1 = 1.
\]

**Proposition 2.1.** The defining relations for \( G \) as a contragredient Lie superalgebra are

1. \([h_i, h_j] = 0, \quad i, j = 1, 2, 3;\)
2. \([h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad i, j = 1, 2, 3;\)
3. \([e_i, f_j] = \delta_{ij} h_i, \quad i, j = 1, 2, 3;\)
4. \((\text{ad } e_i)^{1-a_{ij}}(e_j) = 0, \quad (\text{ad } f_i)^{1-a_{ij}}(f_j) = 0, \quad i = 2, 3, \quad j = 1, 2, 3;\)
5. \([e_1, e_1] = 0, \quad [f_1, f_1] = 0.\)

Relations (1)–(5) clearly hold in \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \), so we assume that \( G \) is defined as a contragredient Lie superalgebra by using the given generators and these relations and show that \( G \) is isomorphic to \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \). The proof will be organized in several lemmas.

Note that by the Jacobi identity, we have

\[
[e_1, [e_1, e_1]] = 0, \quad [f_1, [f_1, f_1]] = 0, \quad i = 2, 3.
\]

Let

\[
e_0 = (2\sigma_1)^{-1} [e_1, [e_3, [e_2, e_1]]] = (2\sigma_1)^{-1} [e_1, [e_3, e_2, e_1]],
\]

\[
f_0 = (2\sigma_1)^{-1} [f_1, [f_3, [f_2, f_1]]] = (2\sigma_1)^{-1} [f_1, [f_3, f_2, f_1]],
\]

\[
h_0 = [e_0, f_0].
\]

**Lemma 2.2.** The subalgebra \( \langle e_0, f_0, h_0 \rangle \) of \( G \) generated by \( e_0, f_0, h_0 \) is isomorphic to \( \text{sl}(2) \), and \( \langle e_i, f_i, h_i; i = 0, 2, 3 \rangle \cong \text{sl}(2) \oplus \text{sl}(2) \oplus \text{sl}(2) \).

**Proof.** A straightforward computation shows that

\[
h_0 = (2\sigma_1)^{-1}(2\sigma_2 h_2 + 2\sigma_3 h_3 - 2h_1).
\]

Hence \([h_0, e_0] = 2e_0, [h_0, f_0] = -2f_0, \) and \( \langle e_0, f_0, h_0 \rangle \cong \text{sl}(2). \) For the second statement, first we note that by the definitions of \( e_0 \) and \( f_0 \), we have

\[
[e_0, f_i] = 0, \quad [f_0, e_i] = 0, \quad i = 2, 3.
\]
Then we note that
\[ [e_2, e_0] = (2\sigma_1)^{-1} [e_2, [[e_1, e_3], [e_2, e_1]]] \]
\[ = (2\sigma_1)^{-1} [e_2, [e_1, e_3], [e_2, e_1]] \]
\[ = -(2\sigma_1)^{-1} [e_3, [e_2, e_1], [e_2, e_1]] \]
\[ = -(4\sigma_1)^{-1} [e_3, [e_2, e_1], [e_2, e_1]] \]
\[ = -(4\sigma_1)^{-1} [e_3, [e_2, [e_1, e_2], e_1]] \]
\[ = 0, \]

and similarly
\[ [e_3, e_0] = 0, \quad [f_2, f_0] = 0, \quad [f_3, f_0] = 0. \]

Now the lemma follows from these identities.

Let \( G_0 = (e_i, f_i, h_i; i = 0, 2, 3) \) (Lemma 2.4 below will show that \( G_0 \) is indeed the even part of \( G \) and thus justify our notation).

**LEMMA 2.3.** Let \( e_{111} = [e_3, [e_2, e_1]] \), then as a \( G_0 \)-module via the adjoint representation, the submodule \( (e_{111}) \) generated by \( e_{111} \) is isomorphic to \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \), where \( \mathbb{C}^2 \) is the two-dimensional natural representation of \( \text{sl}(2) \).

**PROOF.** By the definition of \( e_{111}, e_{111} \neq 0 \). Note that we have \( [e_i, e_{111}] = 0, i = 2, 3 \). Note also that \( [e_0, e_1] = 0 \), so since \( [e_i, e_j] = 0 \) for \( i, j \neq 1 \), we see that
\[ [e_0, e_{111}] = \left[ e_0, [e_3, [e_2, e_1]] \right] \]
\[ = \left[ e_3, [e_2, [e_0, e_1]] \right] \]
\[ = 0. \]

Thus \( e_{111} \) is a highest weight vector. Now since \( [h_i, e_{111}] = e_{111} \) and \( (ad f_i)^2(e_{111}) = 0 \) \((i = 0, 2, 3)\), by the representation theory of the semisimple Lie algebras, the lemma follows as desired.

Define the following elements of \( G \):
\[ e_{112} = [f_3, e_{111}], \quad e_{121} = [f_2, e_{111}], \]
\[ f_{212} = [f_3, f_1], \quad f_{221} = [f_2, f_1], \quad f_{222} = [f_3, [f_2, f_1]]. \]

Then \( e_1, f_1 \), together with the \( e_{ijk} \) and the \( f_{mn} \) form a basis of the \( G_0 \)-module \( (e_{111}) \).

**LEMMA 2.4.** Table I in [9] holds for the elements we defined above, where \((ijk)\) corresponds to the \( e_{ijk} \) or the \( f_{ijk} \) with \((122) \leftrightarrow e_1 \) and \((211) \leftrightarrow f_1 \).

**PROOF.** We only verify that \( e_{111}^2 = 0 \), the other relations can be verified similarly. Since \( [e_1, e_1] = 0 \), we have
\[ [e_2, [e_1, e_1]] = 2[[e_2, e_1], e_1] = 0, \]
and hence by applying $\text{ad} \, e_2$ and using (4) in Proposition 2.1, we have
\[
[e_2, e_1], [e_2, e_1] = 0.
\]

Therefore
\[
x =: [e_3, [e_2, e_1], [e_2, e_1]] = 1/2 \left[ e_3, [e_2, e_1], [e_2, e_1] \right] = 0,
\]
and thus
\[
[e_{111}, e_{111}] = [e_3, x] = 0.
\]

**Proof of Proposition 2.1.** The proposition follows from Lemma 2.2-Lemma 2.4 with $G_0$ being the even part of $\Gamma(\sigma_1, \sigma_2, \sigma_3), (e_{111})$ being the odd part of $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ (for the structure of $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, see [9, Section 2]).

3. **Deformation of $U(G)$ and a PBW type theorem.** Let $q$ be a variable over $\mathbb{C}$, and let $q_1 = q^{-1}, q_i = q^{2\pi i} (i = 2, 3)$ (the $q_i$ are well defined complex value functions as long as $q \neq 0$). Let $\mathcal{A} = \mathbb{C}[q^{\pm 1}, q_i^{\pm 1}, i = 2, 3]$, and let $\mathcal{F}$ be the quotient field of $\mathcal{A}$.

We define the algebra $\mathcal{U}$ to be the $\mathbb{Z}_2$-graded associative algebra with 1 over $\mathcal{F}$ generated by the elements $E_i, F_i, K_i^{\pm 1} (i = 1, 2, 3)$, with grading given by
\[
\deg E_i = \deg F_i = 0, \quad i = 2, 3; \quad \deg K_i^{\pm 1} = 0, \quad i = 1, 2, 3; \quad \deg E_1 = \deg F_1 = 1,
\]
and with the following generating relations:

(3.1) \quad $K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad 1 \leq i, j \leq 3,$

(3.2) \quad $K_i E_j K_i^{-1} = q_i^{a b} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a b} F_j, \quad 1 \leq i, j \leq 3,$

(3.3) \quad $E_i F_j - (-1)^{a b} F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad a = \deg E_i, \quad b = \deg F_j, \quad 1 \leq i, j \leq 3.$

(3.4) \quad $E_2 E_3 = E_3 E_2, \quad F_2 F_3 = F_3 F_2,$

(3.5) \quad $E_i^2 E_1 - (q_i + q_i^{-1}) E_i E_1 E_i + E_1 E_i^2 = 0, \quad i = 2, 3,$

$F_i^2 F_1 - (q_i + q_i^{-1}) F_i F_1 F_i + F_1 F_i^2 = 0, \quad i = 2, 3,$

(3.6) \quad $E_1^2 = F_1^2 = 0.$
The algebra \( \mathcal{U} \) is a \( \mathbb{Z}_2 \)-graded Hopf algebra with comultiplication \( \Delta \), antipode \( S \) and counit \( \varepsilon \) defined by

\[(3.7)\quad \Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta K_i = K_i \otimes K_i;\]

\[(3.8)\quad SE_i = -K_i^{-1}E_i, \quad SF_i = -F_iK_i, \quad SK_i = K_i^{-1};\]

\[(3.9)\quad \varepsilon E_i = 0, \quad \varepsilon F_i = 0, \quad \varepsilon K_i = 1.\]

There exists a \( \mathbb{C} \)-algebra anti-automorphism \( \theta \) of \( \mathcal{U} \) given by

\[(3.10)\quad \theta E_i = F_i, \quad \theta F_i = E_i, \quad \theta K_i = K_i^{-1}, \quad \theta q = q^{-1}\]

and \( \theta(uv) = \theta(v) \theta(u) \), for all \( u, v \in \mathcal{U} \).

The adjoint action of \( \mathcal{U} \) on itself is given by

\[(3.11)\quad \text{ad}_q x(y) = \sum (-1)^{\deg(b) \deg(y)} a_i y S(b_i),\]

where \( \Delta x = \sum a_i \otimes b_i \). Note that by using the adjoint action, relations (3.4) and (3.5) can be replaced by

\[(3.12)\quad (\text{ad}_q E_i)^{1-q} E_j = 0, \quad i = 2, 3, \quad 1 \leq j \leq 3.\]

Introduce the following elements of \( \mathcal{U} \):

\[(3.13)\quad E_{121} = \text{ad}_q E_3(E_1) = E_3 E_1 - q_3^{-1} E_1 E_3,\]

\( E_{112} = \text{ad}_q E_2(E_1) = E_2 E_1 - q_2^{-1} E_1 E_2,\)

\( E_{111} = \text{ad}_q E_3 \text{ad}_q E_2(E_1) = \text{ad}_q (E_3 E_2)(E_1),\)

\( E_0 = (q_2 + q_2^{-1}) E_1 E_{111} + (q_3 + q_3^{-1}) E_{111} E_1 + (q_3 q_2^{-1} - q_3^{-1} q_2) E_{121} E_{112},\)

and let

\[(3.14)\quad F_{212} = \theta E_{121}, \quad F_{221} = \theta E_{112}, \quad F_{222} = \theta E_{111}, \quad F_0 = \theta E_0.\]

**Lemma 3.1.** The following formulas hold in \( \mathcal{U} \):

(1) \( E_{ijk}^2 = 0, \quad F_{ijk}^2 = 0, \)

(2) \( E_1 E_{121} + q_3 E_{121} E_1 = 0, \quad F_1 F_{212} + q_3 F_{212} F_1 = 0, \)

(3) \( E_1 E_{112} + q_2 E_{112} E_1 = 0, \quad F_1 F_{221} + q_2 F_{221} F_1 = 0, \)

(4) \( E_{112} E_{111} + q_3 E_{111} E_{112} = 0, \quad F_{221} F_{222} + q_3 F_{222} F_{221} = 0, \)

(5) \( E_{121} E_{111} + q_2 E_{111} E_{121} = 0, \quad F_{212} F_{222} + q_2 F_{222} F_{212} = 0, \)

(6) \( E_1 E_{111} + q^{-2} E_{111} E_1 + q_2 E_{112} E_{121} + q_3 E_{121} E_{112} = 0.\)

**Proof.** We only need to prove those formulas involving \( E \), those involving \( F \) can then be obtained by applying \( \theta \). Formulas (2) and (3) are clear, (4) and (5) can be verified...
by using $E_{111} = \text{ad}_q E_2(E_{121})$ or $E_{111} = \text{ad}_q E_3(E_{112})$, (6) can be verified by using (2) and (3). To verify (1), note that formulas (3.5) and (3.6) imply that

$$E_1 E_1^2 E_1 = (q_i + q_i^{-1}) E_1 E_1 E_1 = (q_i + q_i^{-1}) E_1 E_1 E_1,$$

$i = 2, 3$.

Thus $E_{121}^2 = 0$, $E_{112}^2 = 0$. Similarly, using $E_{112}^2 = 0$ and $(\text{ad}_q E_3)^2 E_{112} = 0$ instead of (3.5) and (3.6), we get

$$E_{111}^2 = \left(\text{ad}_q E_3(E_{112})\right)^2 = 0.$$ 

The proof of the lemma is now complete.

The following lemma provides some formulas involving the element $E_0$.

**Lemma 3.2.** The following formulas hold in $\mathcal{U}$:

1. $E_0 E_2 = E_2 E_0$, $E_0 E_3 = E_3 E_0$, $E_0 F_2 = F_2 E_0$, $E_0 F_3 = F_3 E_0$,
2. $E_0 E_1 - q^{2\omega} E_1 E_0 = q_1(1 - q^{4\omega}) E_1 E_{111} E_1$,
3. $E_0 E_{121} = E_{121} E_0$, $E_{112} E_0 = E_0 E_{112}$, $E_0 E_{111} = E_{111} E_0$.

**Proof.** The proofs for those formulas involving only the $E$'s are just direct applications of Lemma 3.1. To verify the last two formulas in (1), we use the following formulas

$$F_2 E_{112} - E_{112} F_2 = E_1 K_2^{-1},$$

(3.15)

$$F_2 E_{121} = E_{121} F_2,$$

$$F_2 E_{111} - E_{111} F_2 = E_{121} K_2^{-1},$$

(3.16)

$$F_3 E_{112} = E_{112} F_3,$$

$$F_3 E_{121} - E_{121} F_3 = E_1 K_3^{-1},$$

$$F_3 E_{111} - E_{111} F_3 = E_{112} K_3^{-1}.$$

**Remark.** Compare with the corresponding formulas in $U(G)$, one would like to have a vector $E_0$ which satisfies (1) in Lemma 3.2 and has a better commutation relation with $E_1$, but this does not seem to be possible, since a search along this line will lead to the left hand side of (6) in Lemma 3.1, which is 0.

Let $\mathcal{U}_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathcal{U}$ generated by $E_i, F_i, K_i^{\pm 1}$ and

$$[K_i; 0] = K_i - K_i^{-1}, \quad i = 1, 2, 3.$$

For $\epsilon \in \mathbb{C}^\times$, let $\mathcal{U}_\epsilon = \mathcal{U}_{\mathcal{A}}/(\epsilon - 1) \mathcal{U}_{\mathcal{A}}$. Then the algebra $\mathcal{U}_1$ is an associative algebra over $\mathbb{C}$ with generators $E_i, F_i, K_i, H_i = [K_i; 0]$ ($i = 1, 2, 3$) and the defining relations (which can be verified easily):

$$K_i \text{ are central elements with } K_i^2 = 1,$$

(3.17)
(3.18) \[ [E_i, F_j] = \delta_{ij} H_i, \quad [H_i, E_j] = a_{ij} K_i E_j, \quad [H_i, F_j] = -a_{ij} K_i F_j, \]

(3.19) \[ (\text{ad} E_i)^{1-a_{ij}}(E_j) = 0, \quad (\text{ad} F_i)^{1-a_{ij}}(F_j) = 0, \quad i = 2, 3, \quad j = 1, 2, 3, \]

(3.20) \[ E_i^2 = 0, \quad F_i^2 = 0. \]

Therefore, \( \mathcal{U}_i / (K_i - 1; i = 1, 2, 3) \cong U(G) \), the universal enveloping algebra of \( G \). Note that the image of \( E_0 \) in \( U(G) \) is \( 2e_0 \), where \( e_0 \) is defined in Section 2.

Let \( \mathcal{U}^+, \mathcal{U}^-, \mathcal{U}^0 \) be the subalgebras of \( \mathcal{U} \) generated by the \( E_i \), the \( F_i \), and the \( K_i^{\pm 1} \) \((i = 1, 2, 3)\) respectively. Then just as in the Lie algebra case (see [7]), one can show that \( \mathcal{U} = \mathcal{U}^- \mathcal{U}^0 \mathcal{U}^+ \) and (use the comultiplication) that \( \mathcal{U} \cong \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+ \) as \( \mathcal{F} \)-vector spaces.

For \( \delta = (\delta_1, \delta_2, \delta_3, \delta_4), \delta_i = 0 \) or 1; \( m = (m_1, m_2, m_3), m_i \in \mathbb{Z}_+ \), let

(3.21)

\[
E^{(\delta, m)} = E_{11}^{\delta_1} E_{121}^{\delta_2} E_{112}^{\delta_3} E_{12}^{m_1} E_{2}^{m_2} E_{3}^{m_3}, \\
F^{(\delta, m)} = F_{221}^{\delta_1} F_{212}^{\delta_2} F_{22}^{\delta_3} F_{1}^{m_1} F_{2}^{m_2} F_{3}^{m_3}.
\]

For \( t = (t_1, t_2, t_3), t_i \in \mathbb{Z} \), let

(3.22) \[ K^t = K_1^{t_1} K_2^{t_2} K_3^{t_3}. \]

Then the \( K^t \) form a basis of \( \mathcal{U}^0 \), and we have the following theorem:

**Theorem 3.3.** The elements of the form \( E^{(\delta, m)} \) (resp. \( F^{(\delta, m)} \)) form a basis of \( \mathcal{U}^+ \) (resp. \( \mathcal{U}^- \)), and the elements of the form

\[ F^{(\delta, m)} K^t E^{(\delta', m')} \]

form a basis of \( \mathcal{U} \).

**Proof.** We only need to prove that the elements of the form \( E^{(\delta, m)} \) form a basis of \( \mathcal{U}^+ \), since the statement about \( \mathcal{U}^- \) will follow from symmetry and the statement about \( \mathcal{U} \) will follow from the fact that \( \mathcal{U} \cong \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+ \). We first show that these elements span \( \mathcal{U}^+ \), that is, by using the commutation relations in \( \mathcal{U}^+ \) we can express any monomial of \( \mathcal{U}^+ \) as a linear combination of these elements. In fact, Lemma 3.1 and Lemma 3.2 along with the defining relations of \( \mathcal{U} \) provide the commutation relations we need. In particular, to bring the terms \( E_{112} E_{121} \) and \( E_{1} E_{111} \) to the right order, we use formula (6) in Lemma 3.1 together with the definition of \( E_0 \). Then, we show that these elements are linearly independent over \( \mathcal{F} \). Note that by (3.13), these elements are in fact in \( \mathcal{U}_R \). So if we have a linear relation

(3.23) \[ \sum_{i=1}^r c_i E^{(\delta_i, m_i)} = 0, \]
with $0 \neq c_i \in \mathcal{F}$ $(1 \leq i \leq r)$, then by multiplying a suitable element from $\mathcal{A}$, we may assume that $c_i \in \mathcal{A}$. Now if there exists a $c_i$ such that $c_i(1) \neq 0$, then the image of the right hand side of (3.20) gives a nontrivial linear relation in $U(G)$. But by the PBW theorem of $U(G)$, the images of the $E^{(\delta, m)}$ in $U(G)$ form a basis of $U(G)$, and we have a contradiction. If $c_i(1) = 0$ for all $1 \leq i \leq r$, then by the results in [1, Ch. 3, Section 3], we may assume that the order of $1$ for $c_i$ is $n_i$, and set $n = \min\{n_i : 1 \leq i \leq r\}$. Then $\lim_{q \to 1} c_i / (q - 1)^n \neq 0$ for some $i$, hence by (3.20) we have

$$\lim_{q \to 1} \left( \frac{1}{(q - 1)^n} \sum_{\delta=1}^{r} c_i E^{(\delta, m_i)} \right) = \sum_{i=1}^{r} \left( \lim_{q \to 1} \frac{c_i}{(q - 1)^n} \right) E^{(\delta, m_i)} = 0,$$

which provides a nontrivial linear relation in $U(G)$ contradicting the PBW theorem for $U(G)$. Hence the elements of the form $E^{(\delta, m)}$ are linearly independent, and the proof of the theorem is now complete.

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