1. Introduction. One very interesting and important problem in ring theory is the determination of the position of the singular ideal of a ring with respect to the various radicals (Jacobson, prime, Wedderburn, etc.) of the ring. A summary of the known results can be found in Faith [3, p. 47 ff.] and Lambek [5, p. 102 ff.]. Here we use a new technique to obtain extensions of these results as well as some new ones.

Throughout we adopt the Bourbaki [2] conventions for rings and modules: all rings have 1, all modules are unital, and all ring homomorphisms preserve the 1.

2. The main result. Let $A_M B$ be a bimodule. For $b \in B$ define $l(b) = (m \in M \mid mb = 0)$, an $A$-submodule of $M$. And for $m \in M$ define $l(m) = (a \in A \mid am = 0)$, a left ideal of $A$.

Now define $Z(B) = Z_M(B) = (b \in B \mid l(b) \triangledown M)$ where $\triangledown$ denotes essential extension (=large submodule), and $Z(M) = Z_A(M) = (m \in M \mid l(m) \triangledown A)$. It is easy to verify that $Z(B)$ is a two-sided ideal of $B$ and that $Z(M)$ is an $A-B$ submodule of $M$. In fact $Z_A( )$ defined in the category of $A-B$ bimodules is a subfunctor of the identity functor, usually called the singular submodule of Johnson [4].

Note also that $Z(B)$ is invariant with respect to every ring homomorphism $\psi: B \to C$ such that $A_M C$ is also a bimodule; i.e. $\psi Z(B) \subseteq Z(C)$. Hence $\psi Z(B) \subseteq Z(\psi B) \subseteq Z(C)$.

The proofs of the following two lemmas are straightforward and hence omitted. (Lemma 1 is needed for Lemma 2.)

**Lemma 1.** For any $m \in M$ and $b \in B$, $Am \cap l(b) \simeq l(mb)/l(m)$ as $A$-modules.

**Lemma 2.** With the same notation consider the following three conditions:

(i) $b \in Z(B)$
(ii) $l(m) \neq l(mb)$
(iii) $m \neq 0$.

Then any two conditions imply the third, and hence in the presence of any one the other two are equivalent.

**Main Theorem.** Suppose $A$ has maximum condition on left annihilator ideals and let $x_1, x_2, \ldots$ be a sequence of elements of $Z(B)$. Define $b_n = x_1 x_2 \ldots x_n$. Then:

(i) $M = U l(b_n)$

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(ii) If \( M \) has maximum condition on annihilator \( A \)-submodules then there exists an integer \( N \) with \( M = l(b_N) \)

(iii) If \( M \) is also \( B \)-faithful (e.g. \( B = \text{End}_A M \) or \( M \) is \( B \)-free) then \( b_N = 0 \).

**Proof.** (i) For \( m \in M, l(mb_n) \) is an ascending chain of left annihilator ideals which becomes stationary with \( l(mb_n) = l(mb_n x_{n+1}) \) say. By Lemma 2 \( mb_n = 0 \) since \( x_{n+1} \in Z(B) \). Hence \( m \in l(b_n) \).

(ii) and (iii) are now clear.

**Corollary.** Under all of the above conditions \( Z(B) \) is \( T \)-nilpotent in the sense of Bass (1), and hence \( Z(B) \subseteq \text{rad } B = \text{ prime radical of } B \). Therefore \( B \) semiprime \( \Rightarrow B \) neat in the sense of Bourbaki [2].

**Proof.** It is easy to verify that every \( T \)-nilpotent ideal is contained in the prime radical, using the equivalent definition given by Lambek [5, p. 55].

**Corollary.** If the maximum length of chains of left annihilator ideals of \( A \) is \( N \) (e.g. if \( A \) is an artinian ring of length \( N \)) then \( M = l(b_N) \) and \( b_N = 0 \) if \( M \) is \( B \)-faithful. In this case \( (Z(B))^N = 0 \), i.e. \( Z(B) \) is nilpotent and hence contained in the Wedderburn radical (= sum of all nilpotent ideals).

3. **Applications.**
   (1) Let \( A \neq M \) be a quasi-injective module and \( B = \text{End}_A M \). Then \( Z(B) = \text{Rad } B \) (= the Jacobson radical). Hence if \( A \) has maximum condition on left annihilator ideals and \( M \) has maximum condition on annihilator \( A \)-submodules then \( Z(B) = \text{Rad } B = \text{rad } B \).

   (2) If \( M = B \) then condition (iii) of the theorem holds always. Thus if \( M = B = AG \), the group ring over a finite group \( G \), and \( A \) has maximum condition on left annihilator ideals then \( Z_{AG}(AG) \subseteq \text{rad } AG \).

   (3) If \( A = M = B \) then \( Z(B) \) is the (left) singular ideal. Thus if \( B \) has maximum condition on left annihilator ideals then \( Z(B) \) is \( T \)-nilpotent and hence contained in \( \text{rad } B \).

**References**


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