ON D. J. LEWIS'S EQUATION \( x^3 + 117y^3 = 5 \)

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In a recent publication [2], D. J. Lewis stated that the Diophantine equation \( x^3 + 117y^3 = 5 \) has at most 18 integer solutions, but the exact number is unknown. In this paper we shall solve this problem by proving the following

**THEOREM.** The equation \( x^3 + 117y^3 = 5 \) has no integer solutions.

**Proof.** Let \( \theta^3 = 117 \), \( \theta \) real. From Selmer [3], we obtain the following properties of the field \( \mathbb{Q}(\theta) \):

1. An integral basis of \( \mathbb{Q}(\theta) \) is \( (1, \theta, \theta^2/3) \).
2. \( [5] = [5, \theta - 3][5, \theta^2 + 3\theta + 4] \).

Since \( [5, \theta - 3] = [8 - \theta^2/3] \), we get

\[
5 = (8 - \theta^2/3)(64 + 13\theta + 8\theta^2/3),
\]

where the factors of 5 are primes in \( \mathbb{Q}(\theta) \), \( N(8 - \theta^2/3) = 5 \) and \( N(64 + 13\theta + 8\theta^2/3) = 25 \). By Voronoi's algorithm [1, Ch. 4], we get that the fundamental unit of \( \mathbb{Q}(\theta) \) is \( \varepsilon_0 = 412 - 50\theta - 7\theta^2 \).

We want all the integers of \( \mathbb{Q}(\theta) \) of norm 5 of the form \( a + b\theta \). Setting

\[
a + b\theta = (8 - \theta^2/3)\varepsilon_0^n, \quad n \in \mathbb{Z},
\]

we have an impossible situation, since

\[
e_0 \equiv 1 + \theta - \theta^2, \quad e_0^2 \equiv 1 - \theta - \theta^2, \quad e_0^3 \equiv 1 \pmod{3}.
\]

Thus \( x^3 + 117y^3 = 5 \) has no integer solutions.

**BIBLIOGRAPHY**


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