ON PRIME RINGS WITH ASCENDING CHAIN CONDITION ON ANNIHILATOR RIGHT IDEALS AND NONZERO INJECTIVE RIGHT IDEALS

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If $I$ is a right ideal of a ring $R$, $I$ is said to be an annihilator right ideal provided that there is a subset $S$ in $R$ such that

$I = \{ r \in R \mid sr = 0, \quad \forall s \in S \}$.

$I$ is said to be injective if it is injective as a submodule of the right regular $R$-module $R_R$. The purpose of this note is to prove that a prime ring $R$ (not necessarily with 1) which satisfies the ascending chain condition on annihilator right ideals is a simple ring with descending chain condition on one sided ideals if $R$ contains a nonzero right ideal which is injective.

**Lemma 1.** Let $M$ and $T$ be right $R$-modules such that $M$ is injective and $T$ has zero singular submodule [4] and no nonzero injective submodule. Then $\text{Hom}_R(M, T) = \{0\}$.

**Proof.** Suppose $f \in \text{Hom}_R(M, T)$ such that $f \neq 0$. Let $K$ be the kernel of $f$. Then $K$ is a proper submodule of $M$ and there exists $m \in M$ such that $f(m) \neq 0$. Let $(K:m) = \{ r \in R \mid mr \in K \}$. Since the singular submodule of $T$ is zero and $f(m)(K:m) = \{0\}$ the right ideal $(K:m)$ has zero intersection with some nonzero right ideal $J$ in $R$. Then $mJ \neq \{0\}$ and $K \cap mJ = \{0\}$. Let $m\hat{J}$ be the injective hull of $mJ$. Since $M$ is injective, $m\hat{J}$ is a submodule of $M$. $m\hat{J} \cap K = \{0\}$ since $mJ$ has nonzero intersection with each submodule which has nonzero intersection with $m\hat{J}$ (See [4, p. 712]). Hence $f$ restricted to $m\hat{J}$ is a monomorphism and $f(m\hat{J})$ is an injective submodule of $T$. This is a contradiction.

The following lemma is a consequence of [4, Theorem 1.1].

**Lemma 2.** Let $R$ be a prime ring with zero (right) singular ideal. Then there is a prime ring $R_u$ with 1 in which $R$ is a two-sided ideal such that $R_u$ is a prime ring with zero singular ideal and every nonzero submodule of $R_u$, as (right) $R$-module, has nonzero intersection with $R$. Furthermore, if $I$ is a nonzero right ideal of $R$ such that $I$ is injective, then $I$ is an annihilator right ideal of $R$.

**Proof.** In view of [4, Theorem 1.1], it needs only to be shown that $R_u$ is a prime ring and $I$ is an annihilator right ideal of $R$. Let $S_1$, $S_2$ be right ideals of $R_u$ such that $S_1S_2 = \{0\}$. If $S_i \neq \{0\}$, $i = 1, 2$, then $S_i \cap R \neq \{0\}$ for all $i = 1, 2$. Since $S_i \cap R$ is a nonzero right ideal in $R$ for each $i = 1, 2$, and $R$ is a prime ring, it must be true that either $S_1 = \{0\}$ or $S_2 = \{0\}$. It is easy to show that if $I$ is an injective right ideal of $R$ then $I$ is an injective right ideal of $R_u$. Thus there exists a right ideal $K$ in $R_u$.
such that $R_u = I \oplus K$ by [1, Theorem 1]. Since $1 \in R_u$, there must exist an idempotent $e \in I$ such that $I = eI = eR$. Let $L = R(1-e)$. Since $R$ is a two-sided ideal in $R_u$, $L \subseteq R$. Let $t \in R$ such that $Lt = \{0\}$. Then $(1-e)t = 0$ since $R_u$ is a prime ring and $R$ is a two-sided ideal in $R_u$. Thus $t = et$ and $I = \{r \in R \mid tr = 0, \forall t \in L\}$.

**Theorem.** The following two statements are equivalent:

(a) $R$ is a simple ring with descending chain condition on right ideals.

(b) $R$ is a prime ring with ascending chain condition on annihilator right ideals and $R$ contains a nonzero right ideal which is injective.

**Proof.** (a) $\Rightarrow$ (b). $R$ is certainly a prime ring and $R$ satisfies the ascending chain condition on right ideals by [3, p. 48, Theorem 15]. Furthermore, $R$ is injective by [2, p. 11, Theorem 4.2].

(b) $\Rightarrow$ (a). Let $I_0$ be a nonzero right ideal of $R$ such that $I_0$ is injective. By [5, Lemma 2.1], the singular ideal of $R$ is zero. If $I_0 = R$ then $R$ is an injective $R_u$-module where $R_u$ is the ring given in Lemma 2. Hence there must exist a $R_u$-module $T$ in $R_u$ such that $R \oplus T = R_u$ by [1, Theorem 1]. $T$ is also an $R$-module. Hence by Lemma 2, if $T$ were not zero then $T \cap R \neq \{0\}$. Thus $R = R_u$. If $I_0 \neq R$, then there must exist a nonzero right ideal $K$ in $R_u$ such that $R = I_0 \oplus K$. Since, for each $k \in K$, the left multiplication by $k$ is an $R_u$-homomorphism of $I_0$ into $K$ and $KI_0 \neq 0$, by Lemma 1 it must be true that $K$ contains a nonzero right ideal $K$ which is injective. Let $I_1 = I_0 \oplus K_1$. Then $I_1$ is an injective right ideal of $R$. Inductively we construct the sequences of injective right ideals $\{I_i\}$ and $\{K_{i+1}\}$ such that $I_{i+1} = I_i \oplus K_{i+1}$ for all $i = 0, 1, 2, \ldots$. By Lemma 2, $I_i$ is an annihilator right ideal of $R$ for all $i = 0, 1, 2, \ldots$. Since $I_i \subseteq I_{i+1}$ for $i = 0, 1, 2, \ldots$ and $R$ satisfies the ascending chain condition on annihilator right ideals, there must exist a positive integer $n$ such that $R = I_n \oplus K_{n+1}$ and $K_{n+1}$ does not contain any nonzero injective right ideal of $R$. Since in this case $\text{Hom}_{R_u}(I_n, K_{n+1}) = \{0\}$ by Lemma 1, and each element of $K_{n+1}$ determines a homomorphism of $I_n$ into $K_{n+1}$, $K_{n+1}I_n = \{0\}$. Since $R_u$ is a prime ring, this implies $K_{n+1} = \{0\}$ and $I_n = R = R_u$. Now by [5, Theorem 1] (a) is true.

**References**


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