A NOTE ON NORMAL ATTRACTION TO A STABLE LAW

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Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed random variables, with the common distribution function \( F(x) \). The sequence is said to be normally attracted to a stable law \( V \) with characteristic exponent \( \alpha \), if for some \( a_n S_n/n^{1/\alpha} \to V \) (converges in distribution to \( V \)). Necessary and sufficient conditions for normal attraction are known (cf [1, p. 181]). We prove a theorem that relates the limiting behaviour of the distribution of \( S_{k_n}/k_n^{1/\alpha} \) to that of \( S_n/n^{1/\alpha} \).

Distributions are assumed throughout to be nondegenerate.

THEOREM. Let \( k_n \) be a sequence of positive integers converging to \( \infty \), and such that \( k_{n+1}/k_n \) is bounded. Let \( r \) be a real nonzero number. In order that \( S_{k_n}/k_n^{1/\alpha} \) converge in distribution to a stable law with characteristic exponent \( \alpha \), it is necessary that \( r = \alpha \). Convergence to the normal law can take place iff \{X\} is normally attracted to the normal law. If \( k_n/k_{n+1} \to 1 \), \( S_{k_n}/k_n^{1/\alpha} \) can converge in distribution only to a stable law, and this convergence takes place iff \{X\} is normally attracted.

Proof. We assume, without any loss of generality, that the \( X_i \)'s are symmetric. For each \( x > 0 \), let \( G(x) = P(|X_1| > x) \).

Now, \( S_{k_n}/k_n^{1/\alpha} \) converges in distribution iff there exists \( \sigma^2 \geq 0 \), and a function \( L(x) \), such that (1) and (2) given below, hold [1, p. 124, Theorem 4]:

(1) \[ k_n G(k_n^{1/\alpha}x) \to L(x), \quad x > 0 \]

(2) \[ \lim_{n \to \infty} \lim_{\epsilon \to 0} \int_{|y| < k_n^{1/\alpha}\epsilon} y^2 dF(y) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{|y| < k_n^{1/\alpha}\epsilon} y^2 dF(y) = \sigma^2. \]

Hence, because the limiting distribution is assumed to be nondegenerate, it follows that

(3) \[ 0 < r \leq 2. \]

Should the limit law be stable we would have [1, p. 164, p. 128] for some \( c > 0 \).

(4) \[ \begin{align*}
L(x) &= c/x^\alpha, \quad \alpha < 2 \\
&= 0, \quad \alpha = 2
\end{align*} \]

and

(5) \[ \begin{align*}
\lim_{n \to \infty} k_n^{(r-2)/r} \int_{|y| < k_n^{1/\alpha}\epsilon} y^2 dF(y) &= \sigma^2 > 0, \quad \text{if} \quad \alpha = 2.
\end{align*} \]
Next, for any $y > k$, there exists $n$ such that $k_n \leq y \leq k_{n+1}$, and hence, for any $x > 0$,

$$(6) \quad k_n G(k_{n+1}^{1/r}x) \leq yG(y^{1/r}x) \leq k_{n+1} G(k_n^{1/r}x)$$

Let $S_n/k_n^{1/r} \overset{D}{\to} V$, where $V$ is a stable law with characteristic exponent $\alpha$. Assume, at first, that $\alpha < 2$. Since $k_n/k_{n+1}$ is bounded, by hypothesis, we have from (1), (4), and (6), that $y^{r}G(y)$ is bounded. But, by (1) and (4), $k_n x^r G(k_n^{1/r}x) \to cx^r$. This shows that $y^{r}G(y)$ can be bounded only if $r = \alpha$.

Assume, next $\alpha = 2$. It follows from (1), (4), and (6) that, for all $x > 0$,

$$(7) \quad yG(y^{1/r}x) \to 0 \quad \text{as } y \to \infty.$$ But, from (1) and (4),

$$\lim_{n \to \infty} k_n^{(r-2)/r} \int_{|y| < k_n^{1/r}} y^2 \, dF(y) = 2 \lim_{n \to \infty} k_n^{(r-2)/r} \int_{0}^{k_n^{1/r}} yG(y) \, dy.$$ Making use of (7), one obtains easily that the last limit equals zero unless $r = 2$. But the limit cannot be zero because of (5). Hence $r = 2$.

Therefore, again making use of (5), $E(X^2) < \infty$, which is the necessary and sufficient condition that $\{X_i\}$ be normally attracted to the normal law (cf. [1, p. 181]). This conclusion is also a direct consequence of (7), and the fact that (1), (4), and (5) with $k_n$ replaced by $n$, and taking $\alpha = 2$, provide the necessary and sufficient conditions for the convergence in distribution of $S_n/n^{1/r}$ to the normal law.

Finally, suppose that $k_n/k_{n+1} \to 1$. Then, by (6) and (1), $yx^r G(y^{1/r}x) \to L(x)x^r$. Therefore, in particular, $yG(y^{1/r}) \to L(1)$.

Thus, $L(x) = L(1)/x^r$. This, together with (1), (2), (3), (4), and (5), completes the proof of the theorem.

**Reference**