DECOMPOSITION OF $K_n$ INTO DRAGONS

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ABSTRACT. It is shown that if $1 < n \equiv 0$ or $1 \pmod{2^m}$, then the edges of $K_n$ may be partitioned into isomorphic copies of a graph $D_3(m)$ and also of a graph $D_4(m)$, graphs consisting respectively of a triangle with an attached path of $m-3$ edges or a quadrilateral with an attached path of $m-4$ edges. If $m$ is a power of 2 then the above condition is shown to be necessary and sufficient for the existence of such a partition.

1. Introduction. The complete graph $K_n$ is said to have a $G$-decomposition, if it is the union of edge disjoint subgraphs each isomorphic to $G$.

An immediate and well-known necessary condition for the existence of a $G$-decomposition of $K_n$, if $G$ has $m$ edges, is

(1) $n(n-1) \equiv 0 \pmod{2^m}$.

The problem of determining the set of integers $N(G)$ for which $K_n$ has a $G$-decomposition has been solved completely or partially only for some particular graphs $G$ namely for stars, paths, circles and also for all graphs having no more than four vertices. For more detailed references see [1].

DEFINITION 1. A dragon $D_3(m)$ respectively $D_4(m)$ is a graph having $m$ edges and consisting of a triangle or a quadrilateral respectively and an attached path, called tail.

In this paper $G$ will always denote a dragon.

As a first result of this paper we will prove in Theorem 1 the sufficiency of each of the conditions

(2) $n \equiv 1 \pmod{2^m}$,
(3) $n \equiv 0 \pmod{2^m}$,

if $G$ is a dragon $D_3(m)$ or $D_4(m)$. Consequently as formulated in Theorem 2 condition (1) for dragons appears to be necessary and sufficient if $m$ is a power of 2. This establishes a complete solution of the $G$-decomposition problem for $D_3(2^a)$ and $D_4(2^a)$ namely: for

$$i = 3, 4 \quad 2^a > i$$

(4) $N(D_i(2^a)) = \{n \mid 1 < n \equiv 0 \pmod{2^{a+1}}\}$.

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This generalizes
\[ N(D_3(4)) = \{n \mid 1 < n \equiv 0 \pmod{8} \} \]
a result in [1].

2. Notation and definitions. The vertex set of \( K_n \) will be either \( Z_n \) or \( Z_{n-1} \cup \infty \) depending on whether \( n \equiv 1 \) or \( n \equiv 0 \pmod{2m} \).

A dragon \( D_3(m) \) consisting of the triangle \( \{a, b, c\} \) with tail attached to the vertex \( c \) will be denoted by \( (a, b, c; x_1, x_2, \ldots, x_{m-3}) \). Similarly \( (a, b, c, d; x_1, x_2, \ldots, x_{m-4}) \) is a dragon consisting of the quadrilateral with edges \( \{ab, bc, cd, da\} \) and with the tail attached to the vertex \( d \).

It seems to be useful to denote by \( (a, b, c; x_1, x_2, \ldots, x_{m-3}) \pmod{n} \) the set of graphs
\[
(a + j, b + j, c + j; x_1 + j, x_2 + j, \ldots, x_{m-3} + j) \quad j = 0, 1, 2, \ldots, n-1,
\]
where all the vertices are in \( Z_n \).

When the vertex set is \( Z_{n-1} \cup \infty \), denote by
\[
(a, b, c; x_1, x_2, \ldots, x_{m-4}, \infty) \pmod{n-1}
\]
the set of graphs
\[
(a + j, b + j, c + j; x_1 + j, x_2 + j, \ldots, x_{m-4} + j, \infty) \quad j = 0, 1, 2, \ldots, n-2.
\]

A similar notation will be used also for \( D_4(m) \).

Two partitions of integers, which are known [2, 3] to exist will be used in our construction and are as follows:

**DEFINITION 2**

(i) Let \( t \equiv 0, 1 \pmod{4} \). A partition of the integers \( \{1, 2, \ldots, 2t\} \) into \( t \) pairs \((p_i, q_i)\) such that \( q_i - p_i = i \) for \( i = 1, 2, \ldots, t \) will be called partition A.

(ii) Let \( t \equiv 2, 3 \pmod{4} \). A partition of the integers \( \{1, 2, \ldots, 2t - 1, 2t + 1\} \) into \( t \) pairs \((p_i, q_i)\) such that \( q_i - p_i = i \) for \( i = 1, 2, \ldots, t \) will be called partition B.

The set of pairs \( \{(p_i, q_i), i = 1, 2, \ldots, t\} \) will denote a partition A or a partition B, depending on whether \( t \equiv 0, 1 \pmod{4} \) or \( t \equiv 2, 3 \pmod{4} \).

3. Four lemmas.

**Lemma 1.** Let
\[
y_i = \begin{cases} 
\frac{m-3}{2} t & \text{if } m \text{ is odd} \\
\frac{m+2}{2} i + i & \text{if } m \text{ is even}
\end{cases} 
\]
i = 2, 3, \ldots, t
and let

\[ y_i = \begin{cases} 
\frac{m-3}{2} t, & \text{for } m \text{ odd}, \\
\frac{m+2}{2} t, & \text{for } m \text{ even}, \\
\frac{m-3}{2} t+1, & \text{for } m \text{ odd}, \\
\frac{m+2}{2} t, & \text{for } m \text{ even}.
\end{cases} \]

if \( t = 0, 1 \pmod{4} \) i.e.

\[ m-3 \quad \text{for } m \text{ odd}, \quad m+2 \quad \text{for } m \text{ even}, \]

\[ t = 2, 3 \pmod{4} \]

then the following \( t(2mt+1) \) graphs form a \( D_3(m) \)-decomposition of \( K_{2mt+1} \):

\( (v-p_i-t, v-q_i-t, 0; (m-1)t+i, t, (m-2)t+i, 2t, \ldots, y_i) \pmod{v} \)

where \( i = 1, 2, \ldots, t, \ v = 2mt+1. \)

**Lemma 2.** Let

\[ y_i = \begin{cases} 
\frac{m-3}{2} t \quad & \text{for } m \text{ odd} \\
\frac{m+2}{2} t+i-1 \quad & \text{for } m \text{ even} \\
i = 3, 4, \ldots, t
\end{cases} \]

as defined in (6), with \( i = 1, \) if

\[ y_1 = \begin{cases} 
t = 2, 3 \pmod{4} \quad & \text{if } t = 0, 1 \pmod{4} \\
\infty \quad & \text{if } t = 2, 3 \pmod{4}
\end{cases} \]

then the following \( t(2mt-1) \) graphs form a \( D_3(m) \)-decomposition of \( K_{2mt} \):

\( (v-p_i-t, v-q_i-t, 0; (m-1)t+i-1, t, (m-2)t+i-1, 2t, \ldots, y_i) \pmod{v} \)

where \( i = 1, 2, \ldots, t, \ v = 2mt-1. \)

**Lemma 3.** Let

\[ y_i = \begin{cases} 
\frac{m-4}{2} t \quad & \text{if } m \text{ is even} \\
\frac{m+3}{2} t+i \quad & \text{if } m \text{ is odd}.
\end{cases} \]

then the following \( t(2mt+1) \) graphs form a \( D_4(m) \)-decomposition of \( K_{2mt+1} \):

\( (v-2i, 2mt, v-(4t-2i+2), 0; (m-1)t+i, t, (m-2)t+i, 2t, \ldots, y_i) \pmod{v} \)

where \( i = 1, 2, \ldots, t, \ v = 2mt+1. \)
Lemma 4. Let
\[
y_i = \begin{cases} 
\frac{m-4}{2} t & \text{if } m \text{ is even} \\
\frac{m+3}{2} t + i - 1 & \text{if } m \text{ is odd}
\end{cases}
\]
for \(i = 1, 2, \ldots, t-1\), while \(y_t = \infty\), then the following \(t(2mt-1)\) graphs form a \(D_4(m)\)-decomposition of \(K_{2mt}\):
\[
(v-2i, 2mt, v-(4t-2i+2), 0; (m-1)t + i - 1, t, (m-2)t + i - 1, 2t, \ldots, y_t) \times (\text{mod } v)
\]
where \(i = 1, 2, \ldots, t, v = 2mt - 1\).

Proof of Lemmas 1–4. The direct construction exhibited in Lemmas 1–4 may be checked as follows. Every edge \((x, y)\) of \(K_n\) occurs in some graph of the claimed decomposition. Indeed, in the case of Lemma 1, \(t = 0, 1 \pmod{4}\) for instance, if \(\min\{|x-y|, n-|x-y|\} \leq 3t\) then \((x, y)\) occurs in some triangle, otherwise in the tail. For \(t = 2, 3 \pmod{4}\) the minimum \(3t\) does not occur in the triangle, but this is compensated by the change in \(y_1\). The unicity follows from the fact that the total number of edges in the decomposition is precisely the number of edges of \(K_n\).

The argument is similar in the other lemmas. Edges \((\infty, x)\) present no difficulty.

Notice that the labels used in any graph of the decomposition are different.

4. Results.

Theorem 1. Let \(i = 3 \text{ and 4 if } 1 < n = 0 \text{ or } 1 \pmod{2m}\) then \(K_n\) has a \(D_i(m)\)-decomposition.

Theorem 2. Let \(i = 3 \text{ or 4, } 2^a > i\), then \(K_n\) has a \(D_i(2^a)\) decomposition if and only if
\[
1 < n = 0 \text{ or } 1 \pmod{2^{a+1}}.
\]

Proof. Lemmas 1–4 give direct constructions for all decompositions claimed in Theorem 1.

Theorem 2 follows from condition (1) and Theorem 1 since \(m\) is a power of 2.

References

1. J. C. Bermond and J. Schonheim, \(G\)-decomposition of \(K_n\), where \(G\) has four vertices or less, Discrete Mathematics 19 (1977) 113–120.

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