Let \( X \) be a random variable having the extreme value density of the form

\[
f(x; \theta) = \begin{cases} q(\theta) r(x), & 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}
\]

where \( r \) is assumed to be a positive Lebesgue measurable function of \( x \) and the function \( q \) is defined by

\[
\frac{1}{q(\theta)} = \int_0^\theta r(x) \, dx < \infty
\]

for all \( \theta \) in \( \Omega = (0, \infty) \). It is further assumed that \( q(\theta) \) approaches zero as \( \theta \to \infty \).

In this note we are concerned with estimating parametric functions \( g(\theta) = [1/q(\theta)]^a \), \( a \) any real number. The loss function is assumed to be squared error and the estimators are assumed to be functions of a single observation \( X \). The case of estimators based on a sample of size \( n \geq 1 \) is discussed in Remark 1.

In our search for a 'good' estimator for \( g(\theta) = [1/q(\theta)]^a \) we calculate \( E[1/q(X)]^a = \int_0^\theta [1/q(x)]^a q(\theta) r(x) \, dx \). Since \( r(x) = -q'(x)/q^2(x) \) almost everywhere we find that for every \( \alpha > -1 \), \( E[1/q(X)]^a \) exists and is given by \( E[1/q(X)]^a = (1/\alpha + 1)[1/q(\theta)]^a \).

This leads us to consider the class \( \Lambda_a = \{ \delta_K(X) = K[1/q(X)]^a : K \text{ real} \} \) of estimators, which are constant multiples of \( [1/q(X)]^a \), for estimating the given parametric function \( [1/q(\theta)]^a \). Which of these estimators in \( \Lambda_a \) has the smallest risk uniformly for all \( \theta \) in \( \Omega \)? Since \( E[1/q(X)]^a = (1/(l+1))^a[1/q(\theta)]^a \) if \( l > -1 \) and \( = \infty \) if \( l \leq -1 \), it follows easily that for any \( \delta_K \) in \( \Lambda_a \),

\[
R(\delta_K, \theta) = E[K(1/q(X))^a - (1/q(\theta))^a]^2
\]

(2)

\[
= \begin{cases} [1/q(\theta)]^{2a}, & K = 0, \quad \text{all } \alpha \\
\frac{K^2}{2\alpha + 1} - \frac{2K}{\alpha + 1} + 1 \cdot [1/q(\theta)]^{2a}, & \alpha > -\frac{1}{2}, \quad \text{all } K \\
\infty, & \alpha \leq -\frac{1}{2}, \quad K \neq 0
\end{cases}
\]

where throughout this paper \( \infty \) stands for \( +\infty \). If \( \alpha > -\frac{1}{2} \), the quadratic expression \( [K^2/(2\alpha + 1)] - [2K/(\alpha + 1)] + 1 \) in \( K \) achieves its minimum at \( K = (2\alpha + 1)/(\alpha + 1) \). It follows from this that for estimating \( [1/q(\theta)]^a \), \( \alpha > -\frac{1}{2} \), the minimum risk estimator in \( \Lambda_a \) is \( T_a(X) = [(2\alpha + 1)/(\alpha + 1)][1/q(X)]^a \) corresponding to \( K = (2\alpha + 1)/
(α+1) with risk

\[ R(\delta, \theta) = \frac{\alpha}{\alpha+1} \frac{1}{[q(\theta)]^2}. \]

Is \( T_* \) an admissible estimator of \([1/q(\theta)]^2\) for all \( \alpha > \)? We have the following

**Theorem 1.** Let the random variable \( X \) have density (1) and let the loss be quadratic. Then the estimator

\[ T_*(X) = \frac{2\alpha+1}{\alpha+1} [1/q(X)]^2 \]

is admissible for estimating \([1/q(\theta)]^2\) for every \( \alpha > -\frac{1}{2} \) and is inadmissible for all \( \alpha \leq -\frac{1}{2} \).

**Proof.** Assume \( \alpha > -\frac{1}{2} \). Let \( T \) be any estimator satisfying the inadmissibility inequality for \( T_* \):

\[ E[T-(1/q(\theta))^2] \leq E[T_*-(1/q(\theta))^2] \]

Writing \( m(\theta) \) for \( E(T) \) and \( m^*(\theta) \) for \( E(T_*) \) we have the following equivalent inequalities:

\[ E[T-m(\theta)]^2 + [m(\theta)-(1/q(\theta))^2] \leq [\alpha/(\alpha+1)]^2 [q(\theta)]^{-2x}, \]

and

\[ E(T-T_*)^2 + 2E[[T-T_*][T_*-(1/q(\theta))^2]] \leq 0. \]

Inequality (5) implies that

\[ [m(\theta)-(1/q(\theta))^2] \leq [\alpha/(\alpha+1)]^2 [1/q(\theta)]^{2x} \]

from which we get the bounds for the function \( m \) as

\[ \frac{2\alpha+1}{\alpha+1} \frac{1}{[q(\theta)]^2} \leq m(\theta) \leq \frac{1}{\alpha+1} \frac{1}{[q(\theta)]^2} \quad \text{if} \quad -\frac{1}{2} < \alpha < 0 \]

\[ \frac{1}{\alpha+1} \frac{1}{[q(\theta)]^2} \leq m(\theta) \leq \frac{2\alpha+1}{\alpha+1} \frac{1}{[q(\theta)]^2} \quad \text{if} \quad \alpha \geq 0. \]

Since \( 1/q(\theta) \) tends to zero as \( \theta \) tends to zero, it is clear from (7) that \( m(\theta)/[q(\theta)]^{x-2} \to 0 \) for every \( \delta > 0 \). Now the hypothesis \( \alpha > -\frac{1}{2} \) guarantees some \( \delta > 0 \) such that \( \alpha = (\delta/2)-(\frac{1}{3}) \) i.e., \( \alpha+1 = \delta \), i.e., \( \alpha+1 = \delta - \alpha \). Thus it follows that

\[ m(\theta)/[q(\theta)]^{x+1} \to 0 \quad \text{as} \quad \theta \to 0 \]

The rest of the proof consists in showing that the only solution of the inadmissibility inequality (6) is \( m = m^* \). For this it is enough to show that \( m = m^* \) is the only solution to the inequality

\[ [m(\theta)-m^*(\theta)]^2 + 2E[[T-T_*][T_*-(1/q(\theta))^2]] \leq 0 \]
which is relaxation of (6) obtained after replacing its LHS by something smaller. But (9) still has $T$ in it. To express it in terms of $m$ we use the identity $m(\theta) = q(\theta) \int_1^x T(x) r(x) \, dx$ to provide us the relation

$$T(x) = \frac{m'(x)}{q(x)r(x)} + m(x)$$

Substituting this value of $T$ in (9) and performing the expectation of the expression therein, we obtain the inequality

$$[m(\theta) - m^*(\theta)]^2 + \frac{2\alpha}{\alpha + 1} \frac{m(\theta)}{[q(\theta)]^x} \frac{2\alpha(2\alpha + 1)}{\alpha + 1} E\left\{ \frac{m(X)}{[q(X)]^x} \right\} \leq 0$$

where in this derivation integration by parts and result (8) is used. This inequality still contains the integral $E\{m(X)/[q(X)]^x\}$. If we write

$$u(\theta) = E\left\{ \frac{m(X)}{[q(X)]^x} \right\} = q(\theta) \int_0^1 \frac{m(x)}{[q(x)]^x} r(x) \, dx$$

we have

$$m(\theta) = u(\theta) q^x(\theta) - \left[ \frac{q^{x+1}(\theta)}{q'(\theta)} \right] u'(\theta).$$

Introducing $u(\theta)$ in (11) we have the inequality

$$[m(\theta) - m^*(\theta)]^2 - \left[ \frac{4\alpha^x}{\alpha + 1} \right] u(\theta) - \frac{2\alpha}{\alpha + 1} \left[ \frac{q(\theta)}{q'(\theta)} \right] u'(\theta) \leq 0$$

wherein $m(\theta)$ is to be replaced by its value in terms of $u(\theta)$ from (12). It is now shown that $u^*(\theta) = [1/(1 + \alpha)^x][1/q(\theta)]^{2x}$, corresponding to $m = m^*$, is the unique solution of (13). For convenience we write

$$[1/q(\theta)]^{2x} v(\theta) = u(\theta) - \frac{1}{(1 + \alpha)^x} [1/q(\theta)]^{2x}$$

in (13) which becomes

$$\left[ (1 + 2\alpha) v(\theta) - \frac{q(\theta)}{q'(\theta)} v'(\theta) \right]^2 - \frac{2\alpha}{\alpha + 1} \frac{q(\theta)}{q'(\theta)} v'(\theta) \leq 0.$$
which after multiplying through by \([q(\theta)r(x)]/[q^*(x)]\) and integrating from 0 to \(\theta\) becomes

\[
\frac{2\alpha+1}{\alpha+1} q(\theta) \int_0^\theta \frac{-q(x)}{[q(x)]^{2\alpha+2}} \, dx \leq u(\theta) \leq \frac{q(\theta)}{\alpha+1} \int_0^\theta \frac{-q(x)}{[q(x)]^{2\alpha+2}} \, dx
\]

i.e.

\[
\frac{1}{\alpha+1} \left[1/q(\theta)\right]^{2\alpha} \leq u(\theta) \leq \frac{1}{(\alpha+1)(2\alpha+1)} \left[1/q(\theta)\right]^{2\alpha}
\]

Expressed in terms of \(v(\theta)\), it becomes

\[
\alpha[1+\alpha]^{-2} \leq v(\theta) \leq -\alpha[(1+2\alpha)(1+\alpha)]^{-2}
\]

showing that \(v(\theta)\) is bounded. The boundedness of \(v(\theta)\) for \(\alpha \geq 0\) follows likewise.

(c) \([q(\theta)/q'(\theta)]v'(\theta)\) is not bounded away from zero as \(\theta \to 0\). For suppose there exists \(\epsilon > 0\) and \(\theta_0 > 0\) such that \([q(\theta)/q'(\theta)]v'(\theta) < -\epsilon\) for \(\theta < \theta_0\). That is, 

\[-v'(x) < -\epsilon[q'(x)/q(x)]\]

for all \(x < \theta_0\). Integrating this from \(\theta\) to \(\theta_0\) we get \(v(\theta) - v(\theta_0) < -\epsilon \ln[q(\theta_0)/q(\theta)]\) which shows that \(v(\theta) \to -\infty\) as \(\theta \to 0\). This violates (b). Thus there exists a sequence \(\theta_i \to 0\) along which

\([q(\theta_i)/q'(\theta_i)]v'(\theta_i) \to 0\).

Similarly we can show

(d) \([q(\theta)/q'(\theta)]v'(\theta)\) is not bounded away from zero as \(\theta \to \infty\).

Now from (c) and (d) there are sequences \(\theta_i \to 0\) and \(\theta_i \to \infty\) along which \([q(\theta)/q'(\theta)]v'(\theta) \to 0\). From (14) it follows that \(v(\theta) \to 0\) along these sequences. Hence from (a) it follows that \(v(\theta) \equiv 0\). This completes the proof of admissibility of \(T_\alpha\) for \(\alpha > -\frac{1}{2}\). That \(T_\alpha\) is inadmissible for \(\alpha \leq -\frac{1}{2}\) follows from the fact that its risk (as shown in (2)) is finite for each such \(\alpha\).

REMARKS 1. If \(X_1, \ldots, X_n\) are independent random variables each having density (1) then the sufficient statistic \(T = \max X_i\) has density given by

\[
[q(\theta)]^n \left[\int_0^t r(x) \, dx \right]^{n-1} r(t) \text{ for } 0 \leq t \leq \theta
\]

which is a density of the form (1) with \(q(\theta)\) replaced by \([q(\theta)]^n\) and \(r(x)\) replaced by \(n[\int_0^x f(v) \, dv]^{n-1} r(x)\). So from Theorem 1 we have the conclusion that

\[
\frac{2\beta+1}{\beta+1} \left[n \left[\int_0^\infty r(v) \, dv \right]^{n-1} r(X)\right]^{-\beta}
\]

is an admissible estimator of \([q(\theta)]^{-n\beta}\) if and only if \(\beta > -\frac{1}{2}\). That is, writing \(\alpha\) for \(n\beta\), we conclude that

\[
\frac{2\alpha+n}{\alpha+n} \left[n \left[\int_0^\infty r(v) \, dv \right]^{n-1} r(x)\right]^{-\alpha/n}
\]
is an admissible estimator of \([q(\theta)]^{-a}\) if and only if \(a > -n/2\). So for a given \(\alpha\) we have admissibility for all sufficiently large sample sizes \(n\).

2. Proof of Theorem 1 parallels the Blyth-Roberts [2] proof of the special case of the density (1) as

\[
\begin{align*}
    f(x; \theta) &= \begin{cases} 
        n\theta^{-n}x^{n-1}, & 0 \leq x \leq \theta \\
        0, & \text{otherwise}
    \end{cases}
\end{align*}
\]

In [2], the parametric function of interest is \(g(\theta) = \theta\). If \(g(\theta) = \theta^a\) then according to Theorem 1 the estimator \((n+2s)/(n+s)X^s\) is admissible (with respect to quadratic loss) for estimating \(\theta^s\) for every \(s > -n/2\) and is inadmissible for \(s \leq -n/2\).

3. In [5] Karlin proved Theorem 1 (of this paper) for all \(\alpha > 0\) (see his Theorem 2, p. 418). His proof makes use of the fact that \(\alpha > 0\). Theorem 1 of the present paper settles the question of the admissibility of \(T_\alpha\) for all values of \(\alpha\).

4. An attempt was made in [6] to extend Karlin’s Theorem 2 to all values of \(\alpha\) but this was successful only for some special extreme value densities such as (15). The approach there is the limiting Bayes method, used by Blyth [1] and Karlin [5].

5. The following theorem extends Theorem 3 of Karlin [5] to all other values of \(\alpha\).

**Theorem 2.** Let \(X\) have density

\[
f(x; \theta) = \begin{cases} 
    q(\theta)r(x), & x \geq \theta \\
    0, & 0 < x < \theta,
\end{cases}
\]

where \(q^{-1}(\theta) = \int_0^\theta r(x) \, dx\) and \(q(\theta_0) = 0\). Then (with quadratic loss) the estimator \(T_\alpha = [(2\alpha+1)/(\alpha+1)][1/q(\theta)]^\alpha\) is admissible for estimating \([1/q(\theta)]^\alpha\) for all \(\alpha > -\frac{1}{2}\) and inadmissible for all \(\alpha \leq -\frac{1}{2}\).

6. If the loss function is given by \(L_\alpha(\delta, g) = |(\delta - g)/g|^\alpha\), the estimator \(T_\alpha\) is minimax and admissible for estimating \([1/q(\theta)]^\alpha\) for all \(\alpha > -\frac{1}{2}\).

7. The estimator \((\alpha + 1)[1/q(X)]^\alpha\) is the uniformly minimum variance unbiased estimator of \([1/q(\theta)]^\alpha\) for all \(\alpha > -\frac{1}{2}\). This estimator, however, is inadmissible for it is uniformly improved upon by the estimator \(T_\alpha\).

8. In addition to the example of the density (15), Theorems 1 and 2 have the following applications:

(i) **Pareto distribution.** Let \(X\) have density of the form

\[
f(x; \theta) = \begin{cases}
    c\theta^x \frac{1}{x^{\theta+1}}, & x \geq \theta \\
    0, & \text{otherwise},
\end{cases}
\]

where \(c \geq 0\) is known and \(g(\theta) = \theta^x\). If we take \(r(x) = c/x^{\theta+1}\) then (17) is a special case of (16).
(ii) Let $X$ have density

$$f(x; \theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta \in (-\infty, \infty)$ and $g(\theta) = \theta^*$. If we set $r(x) = e^{-x}$ the (18) is a special case of (16).

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