GAUSSIAN PROCESSES WITH MARKOVIAN COVARIANCES

BY

DUDLEY PAUL JOHNSON

ABSTRACT. We show that any Gaussian process can be derived in a simple manner from a Markov process if it has zero mean and covariance identical to the covariance of a real valued function of a temporally homogeneous Markov process.

Suppose that \( M_T \), \( T \) being either the nonnegative integers or the nonnegative real numbers, is a temporally homogeneous Markov process on a measurable space \( (S, \Sigma) \) with initial distribution \( P(\cdot) \) and transition probability function \( P_t(\cdot, \cdot) \). Let \( f \) be a mapping of \( T \times S \) into the real numbers which is square integrable with respect to the measure \( P_t(a, \cdot) \) for each \( t \in T \) and \( a \in S \).

Suppose now that \( X_T \) is a real Gaussian process with zero expectations and covariance

\[
\Gamma_{st} = \int_S P(du) \int_S f(s, v)P_s(u, dv) \int_S f(t - s, w)P_{t-s}(v, dw)
\]

identical to the covariance of \( f(t, M_t) \), \( t \in T \). Let \( X_T^\ast(\Sigma) \) be the generalized Gaussian random field (see [2]) on \( T \times \Sigma \) with zero expectations and covariance function

\[
\Gamma^\ast_{st}(U, V) = \int_S P(du) \int_U P_s(u, dv) \int_U P_{t-s}(v, dw)
\]

identical to the covariance of the random field

\[
I_U(M_t), \quad t \in T, \quad U \in \Sigma
\]

where \( I_U \) is the indicator function of \( U \). Then if \( F(\Sigma) \) is the set of all functions mapping \( \Sigma \) into the real numbers we have the following

**Theorem.** Under the above conditions \( X_T^\ast \) is a Markov process on \( F(\Sigma) \) and

\[
X_T = \int_S f(t, u)X_t^\ast(du)
\]

where the integral is the Wiener-Ito stochastic integral.

**Proof.** Since conditioning and projections are the same for a Gaussian field (see [1] or [3]), it follows that \( X_T \) is a Markov process on \( F(\Sigma) \) if for each \( s \) and
In $T$ and $V \in \Sigma$, the projection $E_u(X^*_{s+t}(V) \mid X^*_s(U), U \in \Sigma)$ of $X^*_{s+t}$ on the closed linear span of the functions $X^*_s(U)$, $U \in \Sigma$ is equal to the projection $E_u(X^*_{s+t}(V) \mid X^*_s(U), r \leq s, U \in \Sigma)$ of $X^*_{s+t}$ on the closed linear span of the functions $X^*_r(U)$, $r \leq s$ and $U \in \Sigma$. To do this we show that for each $V \in \Sigma$,

$$E_u(X^*_{s+t}(V) \mid X^*_s(U), U \in \Sigma) = \int_S P_t(u, V)X^*_s(du)$$

That is, we show that for each $r \leq s$, $U \in \Sigma$ and $V \in \Sigma$,

$$X^*_r(U) \perp X^*_{s+t}(V) - \int_S P_t(w, V)X^*_r(dw).$$

But

$$EX^*_r(U)X^*_{s+t}(V) - \int_S P_t(w, V)X^*_r(dw)$$

$$= EX^*_r(U)X^*_{s+t}(V) - \int_S P_t(w, V)EX^*_r(U)X^*_r(dw)$$

$$= \int_S P(du)\int_U P_t(u, dv)P_{s+t-r}(v, dw) - \int_S P_t(w, V)$$

$$\times \int_U P(du)\int_U P_t(u, dv)P_{s+t-r}(v, dw)$$

$$= \int_S P(du)\int_U P_t(u, dv)P_{s+t-r}(v, V) - \int_S P(du)\int_U P_t(u, dv)$$

$$\times \int_S P_{s+t-r}(v, dw)P_t(w, V) = \int_S P(du)\int_U P_t(u, dv)P_{s+t-r}(v, V)$$

$$- \int_S P(du)\int_U P_t(u, dv)P_{s+t-r}(v, V) = 0$$

and so $X^*_s(U)$ is a Markov process on $F(\Sigma)$.

To complete the proof, we need only show that

$$X_t = \int_S f(t, u)X^*_t(du), \quad t \in T.$$  

Clearly both $X_t$ and $\int_S f(t, u)X^*_t(du)$ are Gaussian with zero expectation. To show that they are equal in distribution we need only show that their covariances are
equal. Since the covariance of $X_t$ is $\Gamma'_t$ and since

$$E \int_S f(s, v)X_t^*(dv) \int_S f(t, w)X_t^*(dw)$$

$$= \int_S \int_S f(s, v)f(t, w)EX_t^*(dv)X_t^*(dw)$$

$$= \int_S \int_S f(s, v)f(t, w)\int_S P_4(du)P(u, dv)P_{t-u}(v, dw)$$

$$= \int_S P(du)\int_S f(s, v)P_4(u, dv)\int_S f(t, w)P_{t-u}(v, dw) = \Gamma_{tt}$$

the theorem is proved.

REFERENCES

