WELL-POSEDNESS OF THIRD ORDER DIFFERENTIAL EQUATIONS IN HÖLDER CONTINUOUS FUNCTION SPACES

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Abstract. In this paper, by using operator-valued $\mathcal{C}^\alpha$-Fourier multiplier results on vector-valued Hölder continuous function spaces, we give a characterization of the $\mathcal{C}^\alpha$-well-posedness for the third order differential equations $au'''(t) + u''(t) = Au(t) + Bu'(t) + f(t)$, $(t \in \mathbb{R})$, where $A, B$ are closed linear operators on a Banach space $X$ such that $D(A) \subseteq D(B)$, $a \in \mathbb{C}$ and $0 < \alpha < 1$.

1. Introduction

The well-posedness of third order differential equations has been investigated by many researchers, since these differential equations describe a large number of models arising from natural phenomena, such as flexible space structures with internal damping. See [3, 4, 7, 11, 12] for more information and references therein. For example, Poblete and Pozo studied the existence and uniqueness of strong solutions for the abstract third order equation:

\begin{equation}
\begin{cases}
u'''(t) + \nu''(t) = \beta \nu(t) + \gamma \nu'(t) + f(t), & (t \in [0, 2\pi]), \\
u(0) = \nu(2\pi), \nu'(0) = \nu'(2\pi), \nu''(0) = \nu''(2\pi),
\end{cases}
\end{equation}

where $A$ and $B$ are closed linear operators defined on a Banach space $X$ with $D(A) \cap D(B) \neq \{0\}$, the constants $\alpha, \beta, \gamma \in \mathbb{R}^+$, and $f$ belongs to either Lebesgue-Bochner spaces $L^p(T; X)$, periodic Besov spaces $B^s_p(T; X)$, or periodic Triebel-Lizorkin spaces $F^s_{p,q}(T; X)$ [12]. They gave necessary and sufficient conditions for (1.1) to be $L^p$-well-posed (resp. $B^s_p$-well-posed and $F^s_{p,q}$-well-posed) by using operator-valued Fourier multipliers.

On the other hand, the well-posedness of differential equations in Hölder continuous function spaces have been extensively studied. See [5, 6, 10, 13, 14] for more information and references therein. The purpose of this paper is to study the well-posedness of the following third order differential equations on the real line:

\begin{equation}
\begin{cases}
u'''(t) + 
u''(t) = Au(t) + Bu'(t) + f(t), & (t \in \mathbb{R})
\end{cases}
\end{equation}

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on Hölder continuous function spaces $C^\alpha(\mathbb{R}; X)$, where $A$ and $B$ are closed linear operators on a complex Banach space $X$ such that $D(A) \subset D(B)$, $\alpha \in \mathbb{C}$ and $0 < \alpha < 1$ are fixed scalars.

We say that $(P)$ is $C^\alpha$-well-posed, if for every $f \in C^\alpha(\mathbb{R}; X)$, there exists a unique $u \in C^\alpha(\mathbb{R}; D(A)) \cap C^{3+\alpha}(\mathbb{R}; X)$, such that $u^\prime \in C^\alpha(\mathbb{R}; D(B))$, and $(P)$ is satisfied for all $t \in \mathbb{R}$, here we consider $D(A)$ and $D(B)$ as Banach spaces equipped with the graph norms, $C^{3+\alpha}(\mathbb{R}; X)$ is the space of all $C^3$-functions $u : \mathbb{R} \to X$ satisfying $u^\prime, u^\prime\prime, u^\prime\prime\prime \in C^\alpha(\mathbb{R}; X)$.

Using known operator-valued $C^\alpha$-Fourier multiplier results obtained by Arendt, Batty and Bu [1], we completely characterize the $C^\alpha$-well-posedness of $(P)$: when $0 < \alpha < 1$, then $(P)$ is $C^\alpha$-well-posed if and only if $i \mathbb{R} \subset \rho(P)$ and

$$
\sup_{s \in \mathbb{R}} \| s^3 [ias^3 + s^2 + A + isB]^{-1} \| < \infty,
$$

$$
\sup_{s \in \mathbb{R}} \| sB [ias^3 + s^2 + A + isB]^{-1} \| < \infty.
$$

(see Theorem 3.1 below), where $\rho(P)$ is the resolvent set defined by the problem $(P)$ (see the precise definition in the third section). Since the above estimations do not depend on the space parameter $0 < \alpha < 1$, we deduce that when $(P)$ is $C^\alpha$-well-posed for some $0 < \alpha < 1$, then it is $C^\alpha$-well-posed for all $0 < \alpha < 1$.

It is remarkable that our characterization of the $C^\alpha$-well-posedness of $(P)$ does not depend on the geometry of the underlying Banach space $X$ and the involved closed operator $A$ does not need to generate a semigroup on $X$. Our result may be regarded as generalizations of the previous known results in the simpler cases when $\alpha = 0$ and/or $B = 0$ [1].

This paper is organized as follows: in the second section, we give some preliminaries concerning $C^\alpha$-Fourier multipliers and Carleman transform for functions of subexponential growth. In section 3, we give our main result which gives a necessary and sufficient condition for the problem $(P)$ to be $C^\alpha$-well-posed. In the last section, we give a concrete example that our abstract result may be applied.

## 2. Preliminaries

Let $X$ be a complex Banach space and $0 < \alpha < 1$. We denote by $C^\alpha(\mathbb{R}; X)$ the space of all $X$-valued functions $u$ on $\mathbb{R}$ satisfying

$$
\|u\|_{\alpha} := \sup_{s \neq t} \frac{\|u(s) - u(t)\|}{|s - t|^\alpha} < \infty.
$$

Define

$$
\|u\|_{C^\alpha(\mathbb{R}; X)} := \|u(0)\| + \|u\|_{\alpha}.
$$

It is easy to see that the space $C^\alpha(\mathbb{R}; X)$ equipped with norm $\| \cdot \|_{C^\alpha(\mathbb{R}; X)}$ becomes a Banach space. The kernel of the seminorm $\| \cdot \|_{\alpha}$ on $C^\alpha(\mathbb{R}; X)$ is the space of all constant functions. The corresponding quotient space $C^\alpha(\mathbb{R}; X)$ is also a Banach space under the quotient norm. We will identify a function $u \in C^\alpha(\mathbb{R}; X)$ with its equivalent class in $C^\alpha(\mathbb{R}; X)$, that is $\dot{u} := \{v \in C^\alpha(\mathbb{R}; X) : u - v \equiv \text{constant}\}$.

Let $X$, $Y$ be Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from $X$ to $Y$. We will simply denote it by $\mathcal{L}(X)$ if $X = Y$.

We need the notion of operator-valued $C^\alpha$-multipliers which has been studied in [1].
Definition 2.1. Let \( X, Y \) be complex Banach spaces, \( m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y) \) be continuous. \( m \) is said to be a \( \dot{C}^\alpha \)-Fourier multiplier if there exists a mapping \( L : \dot{C}^\alpha(\mathbb{R}; X) \to \dot{C}^\alpha(\mathbb{R}; Y) \) such that
\[
(2.1) \quad \int_{\mathbb{R}} \mathcal{F}(\varphi)(Lf)(s) \, ds = \int_{\mathbb{R}} \mathcal{F}(\varphi m)(s) f(s) \, ds
\]
for all \( f \in C^\alpha(\mathbb{R}; X) \) and all \( \varphi \in \mathcal{D}(\mathbb{R} \setminus \{0\}) \), where \( \mathcal{D}(\mathbb{R} \setminus \{0\}) \) is the space of all \( C^\infty \)-functions on \( \mathbb{R} \setminus \{0\} \) with compact support contained in \( \mathbb{R} \setminus \{0\} \). \( \mathcal{F} \) is the Fourier transform given by
\[
(\mathcal{F}h)(s) := \hat{h}(s) := \int_{\mathbb{R}} h(t) e^{-ist} \, dt, \quad (s \in \mathbb{R})
\]
when \( h \in L^1(\mathbb{R}; X) \).

Remark 2.1. By [1, Lemma 5.1], the right-hand side of (2.1) does not depend on the representative of \( \dot{f} \) as
\[
\int_{\mathbb{R}} \mathcal{F}(\varphi m)(s) \, ds = 2\pi(\varphi m)(0) = 0.
\]
Moreover, the identity (2.1) defines \( Lf \in C^\alpha(\mathbb{R}; X) \) uniquely up to an additive constant by [1, Lemma 5.1].

The following result gives a sufficient condition for a \( C^2 \)-function to be a \( \dot{C}^\alpha \)-Fourier multiplier.

Theorem 2.1 (Arendt, Batty and Bu [1]). Let \( X, Y \) be Banach spaces and \( m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y) \) be a \( C^2 \)-function satisfying:
\[
(2.2) \quad \sup_{s \neq 0} \left( \|m(s)\| + \|sm'(s)\| + \|s^2m''(s)\| \right) < \infty.
\]
Then \( m \) is a \( \dot{C}^\alpha \)-Fourier multiplier whenever \( 0 < \alpha < 1 \).

Let \( 0 < \alpha < 1 \), we denote by \( C^{1,\alpha}(\mathbb{R}; X) \) the space of all \( X \)-valued functions \( u \) defined on \( \mathbb{R} \), such that \( u \in C^1(\mathbb{R}; X) \) and \( u' \in C^\alpha(\mathbb{R}; X) \). The space \( C^{1,\alpha}(\mathbb{R}; X) \) is equipped with the following norm
\[
\|u\|_{C^{1,\alpha}(\mathbb{R}; X)} := \|u(0)\| + \|u'\|_{C^\alpha(\mathbb{R}; X)},
\]
and it is a Banach space. It follows from [1, Lemma 6.2] that if \( u, v \in C^\alpha(\mathbb{R}; X) \), then \( u \in C^{1,\alpha}(\mathbb{R}; X) \) and \( u' = v + x \) for some \( x \in X \) if and only if
\[
(2.3) \quad \int_{\mathbb{R}} \mathcal{F}(\text{id} \cdot \varphi)(s) u(s) \, ds = \int_{\mathbb{R}} (\mathcal{F}\varphi)(s) v(s) \, ds
\]
whenever \( \varphi \in \mathcal{D}(\mathbb{R} \setminus \{0\}) \), where \( \text{id}(s) := is \) when \( s \in \mathbb{R} \).

In a similar way, \( C^{2,\alpha}(\mathbb{R}; X) \) is the space of all \( X \)-valued functions \( u \) defined on \( \mathbb{R} \), such that \( u \in C^2(\mathbb{R}; X) \) and \( u' \in C^\alpha(\mathbb{R}; X) \). \( C^{2,\alpha}(\mathbb{R}; X) \) is the space of all \( X \)-valued functions \( u \) defined on \( \mathbb{R} \), such that \( u \in C^\alpha(\mathbb{R}; X) \) and \( u' \in C^\alpha(\mathbb{R}; X) \). \( C^{3,\alpha}(\mathbb{R}; X) \) is also a Banach space equipped with the norm
\[
\|u\|_{C^{3,\alpha}(\mathbb{R}; X)} := \|u(0)\| + \|u'(0)\| + \|u''(0)\| + \|u'''(0)\|_{C^\alpha(\mathbb{R}; X)}.
\]
Let \( u \in L^1_{\text{loc}}(\mathbb{R}; X) \). We say that \( u \) is of subexponential growth, if for all \( \epsilon > 0 \)
\[
\int_{-\infty}^{\infty} e^{-\epsilon|t|} \|u(t)\| \, dt < \infty.
\]
For such function \( u \), we define its Carleman transform on \( \mathbb{C} \setminus i\mathbb{R} \) by
\[
\hat{u}(\lambda) := \left\{
\begin{array}{ll}
\int_0^\infty e^{-\lambda t}u(t)dt, & \text{Re}\lambda > 0, \\
-\int_0^\infty e^{\lambda t}u(-t)dt, & \text{Re}\lambda < 0.
\end{array}
\right.
\]

[2, (4.25), page 292]. A point \( \eta \in \mathbb{R} \) is called a regular point of \( u \) if its Carleman transform has a holomorphic extension to a neighborhood of \( i\eta \). The Carleman spectrum of \( u \) is given by \( \text{sp}_C(u) := \{ \eta \in \mathbb{R} : \eta \text{ is not regular} \} \). It is well known that \( u = 0 \) if and only if \( \text{sp}_C(u) = \emptyset \).

3. **The \( C^\alpha \)-Well-Posedness of (P)**

Let \( X \) be a complex Banach space, let \( A : D(A) \to X \) and \( B : D(B) \to X \) be closed linear operators on \( X \) satisfying \( D(A) \subset D(B) \) and let \( a \in \mathbb{C}, 0 < \alpha < 1 \). We consider the \( C^\alpha \)-well-posedness of the third order differential equations:
\[
(P) \quad au'''(t) + uu''(t) + Bu'(t) + f(t), \quad (t \in \mathbb{R}),
\]
on Hölder continuous function spaces \( C^\alpha(\mathbb{R}; X) \).

**Definition 3.1.** We say that \( (P) \) is \( C^\alpha \)-well-posed, if for all \( f \in C^\alpha(\mathbb{R}; X) \), there exists a unique \( u \in C^\alpha(\mathbb{R}; D(A)) \cap C^{3+\alpha}(\mathbb{R}; X) \), such that \( u' \in C^\alpha(\mathbb{R}; D(B)) \) and \( (P) \) is satisfied for all \( t \in \mathbb{R} \), here \( D(A) \) and \( D(B) \) are equipped with the graph norms so that they become Banach spaces.

We define the resolvent set for the problem \( (P) \) by
\[
\rho(P) := \{ z \in \mathbb{C} : az^3 + z^2 - A - zB : D(A) \to X \text{ is a bijection and} \]
\[
[a^2z^3 + z^2 - A - zB]^{-1} \in \mathcal{L}(X) \}. \]

Let \( z \in \rho(P) \). Then \( [az^3 + z^2 - A - zB]^{-1} \in \mathcal{L}(X) \) is a bijection from \( X \) onto \( D(A) \) by definition. This implies that \( A[a^2z^3 + z^2 - A - zB]^{-1} \) and \( B[a^2z^3 + z^2 - A - zB]^{-1} \) are in \( \mathcal{L}(X) \) by the closed graph theorem and the closedness of \( A \) and \( B \). In particular, \( [az^3 + z^2 - A - zB]^{-1} \in \mathcal{L}(X, D(A)) \cap \mathcal{L}(X, D(B)) \). Here again we consider \( D(A) \) and \( D(B) \) as Banach spaces equipped with the graph norms.

The following results give a necessary and sufficient condition for \( (P) \) to be \( C^\alpha \)-well-posed.

**Theorem 3.1.** Let \( X \) be a complex Banach space, \( a \in \mathbb{C}, 0 < \alpha < 1 \) and let \( A, B \) be closed linear operators on \( X \) satisfying \( D(A) \subset D(B) \). Then \( (P) \) is \( C^\alpha \)-well-posed if and only if \( i\mathbb{R} \subset \rho(P) \) and
\[
\sup_{s \in \mathbb{R}} \|s^3[ias^3 + s^2 + A + isB]^{-1}\| < \infty,
\]
\[
\sup_{s \in \mathbb{R}} \|sB[ias^3 + s^2 + A + isB]^{-1}\| < \infty.
\]

**Proof.** Assume that \( i\mathbb{R} \subset \rho(P) \) and
\[
\sup_{s \in \mathbb{R}} \|s^3[ias^3 + s^2 + A + isB]^{-1}\| < \infty,
\]
\[
\sup_{s \in \mathbb{R}} \|sB[ias^3 + s^2 + A + isB]^{-1}\| < \infty.
\]
Then \( A : D(A) \to X \) is invertible and its inverse \( A^{-1} \in \mathcal{L}(X) \) as \( 0 \in \rho(P) \) by assumption. Let
\[
m(s) := [ias^3 + s^2 + A + isB]^{-1}, \quad p(s) := s^3m(s), \quad q(s) := sBm(s)
\]
when $s \in \mathbb{R}$. It is easy to verify that $m$, $p$ and $q$ are $\mathcal{L}(X)$-valued $C^\infty$-functions on $\mathbb{R}$. We have
\begin{equation}
\sup_{s \in \mathbb{R}} \|p(s)\| < \infty, \quad \sup_{s \in \mathbb{R}} \|q(s)\| < \infty
\end{equation}
by assumption. The identity $(ias^3 + s^2)m(s) + Am(s) + isBm(s) = I_X$ and the fact that the inverse of $A : D(A) \to X$ is in $\mathcal{L}(X)$ implies that
\begin{equation}
\sup_{s \in \mathbb{R}} \|Am(s)\| < \infty, \quad \sup_{s \in \mathbb{R}} \|m(s)\| < \infty
\end{equation}
We have
\[
m'(s) = -m(s)(3ias^2 + 2s + iB)m(s),
\]
and
\[
m''(s) = 2m(s)(3ias^2 + 2s + iB)m(s) - m(s)(6ias + 2)m(s)
\]
when $s \in \mathbb{R}$. It follows that
\begin{align*}
\sup_{s \in \mathbb{R}} \|sm'(s)\| &< \infty, \quad \sup_{s \in \mathbb{R}} \|s^2m''(s)\| < \infty \\
\sup_{s \in \mathbb{R}} \|sAm'(s)\| &< \infty, \quad \sup_{s \in \mathbb{R}} \|s^2Am''(s)\| < \infty \\
\sup_{s \in \mathbb{R}} \|sBm'(s)\| &< \infty, \quad \sup_{s \in \mathbb{R}} \|s^2Bm''(s)\| < \infty
\end{align*}
by (3.1) and (3.2). Here we have used the fact that
\begin{equation}
\sup_{s \in \mathbb{R}} \|Bm(s)\| < \infty, \quad \sup_{s \in \mathbb{R}} \|s^2m(s)\| < \infty
\end{equation}
which are easy consequences of the uniform boundedness of $p, q$ and the continuity. Therefore considering $m : \mathbb{R} \to \mathcal{L}(X, D(A))$ or $m : \mathbb{R} \to \mathcal{L}(X, D(B))$, $m$ is a $C^\infty$-Fourier multiplier by Theorem 2.1. On the other hand, we have
\[
p'(s) = 3s^2m(s) - s^3m(s)(3ias^2 + 2s + iB)m(s),
\]
and
\[
p''(s) = 6sm(s) - 6s^2m(s)(3ias^2 + 2s + iB)m(s) - s^3m(s)(6ias + 2)m(s)
+ 2s^3m(s)(3ias^2 + 2s + iB)m(s)(3ias^2 + 2s + iB)m(s)
\]
by (3.1), (3.2) and (3.3). It follows that
\[
\sup_{s \in \mathbb{R}} \|sp'(s)\| < \infty, \quad \sup_{s \in \mathbb{R}} \|s^2p''(s)\| < \infty.
\]
Consequently $p$ is a $C^\infty$-Fourier multiplier by Theorem 2.1.

For $h$, we have
\[
q'(s) = Bm(s) - sBm(s)(3ias^2 + 2s + iB)m(s),
\]
and
\[
q''(s) = -2Bm(s)(3ias^2 + 2s + iB)m(s) - sBm(s)(6ias + 2)m(s)
+ 2sBm(s)(3ias^2 + 2s + iB)m(s)(3ias^2 + 2s + iB)m(s).
\]
It follows that
\[
\sup_{s \in \mathbb{R}} \|sq'(s)\| < \infty, \quad \sup_{s \in \mathbb{R}} \|s^2q''(s)\| < \infty.
\]
by (3.1), (3.2) and (3.3). Hence \( q \) is also a \( \dot{C}^\alpha \)-Fourier multiplier by Theorem 2.1.

Let \( k(s) = sm(s) \) and \( l(s) = s^2m(s) \). In a similar way, we show by using (3.1), (3.2) and (3.3) that

\[
\sup_{s \in \mathbb{R}} ||sk'(s)|| < \infty, \quad \sup_{s \in \mathbb{R}} ||s^2k''(s)|| < \infty,
\]

\[
\sup_{s \in \mathbb{R}} ||sBk'(s)|| < \infty, \quad \sup_{s \in \mathbb{R}} ||s^2Bk''(s)|| < \infty,
\]

\[
\sup_{s \in \mathbb{R}} ||sf'(s)|| < \infty, \quad \sup_{s \in \mathbb{R}} ||s^2f''(s)|| < \infty.
\]

Therefore \( l \) is a \( \dot{C}^\alpha \)-Fourier multiplier, and considering \( k : \mathbb{R} \to (X, D(B)) \), \( k \) is also a \( \dot{C}^\alpha \)-Fourier multiplier by Theorem 2.1.

Let \( f \in C^\alpha(\mathbb{R}; X) \) be fixed. Then there exist \( u_1 \in C^\alpha(\mathbb{R}; D(A)) \cap C^\alpha(\mathbb{R}; D(B)) \) and \( u_2 \in C^\alpha(\mathbb{R}; D(B)) \), \( u_3, u_4, u_5 \in C^\alpha(\mathbb{R}; X) \), such that

\[
\int_{\mathbb{R}} (\mathcal{F}\phi_1)(s)u_1(s) \, ds = -\int_{\mathbb{R}} \mathcal{F}(\phi_1m)(s)f(s) \, ds,
\]

\[
\int_{\mathbb{R}} (\mathcal{F}\phi_2)(s)u_2(s) \, ds = -i\int_{\mathbb{R}} \mathcal{F}(\phi_2k)(s)f(s) \, ds,
\]

\[
\int_{\mathbb{R}} (\mathcal{F}\phi_3)(s)u_3(s) \, ds = -i\int_{\mathbb{R}} \mathcal{F}(\phi_3q)(s)f(s) \, ds,
\]

\[
\int_{\mathbb{R}} (\mathcal{F}\phi_4)(s)u_4(s) \, ds = \int_{\mathbb{R}} \mathcal{F}(\phi_4l)(s)f(s) \, ds,
\]

\[
\int_{\mathbb{R}} (\mathcal{F}\phi_5)(s)u_5(s) \, ds = i\int_{\mathbb{R}} \mathcal{F}(\phi_5p)(s)f(s) \, ds
\]

for all \( \phi_1 \in \mathcal{D}(\mathbb{R} \setminus \{0\}) \). Since \( u_1 \in C^\alpha(\mathbb{R}; D(A)) \) and \( u_2 \in C^\alpha(\mathbb{R}; D(B)) \), we have \( Au_1, Bu_2 \in C^\alpha(\mathbb{R}; X) \). It follows from (3.4) and (3.5) that

\[
\int_{\mathbb{R}} (\mathcal{F}\phi_1)(s)Au_1(s) \, ds = -\int_{\mathbb{R}} \mathcal{F}(\phi_1Am)(s)f(s) \, ds,
\]

\[
\int_{\mathbb{R}} (\mathcal{F}\phi_2)(s)Bu_2(s) \, ds = -i\int_{\mathbb{R}} \mathcal{F}(\phi_2q)(s)f(s) \, ds
\]

when \( \phi_1, \phi_2 \in \mathcal{D}(\mathbb{R} \setminus \{0\}) \) by the closedness of \( A \) and \( B \). Choosing \( \phi_1 = \text{id} \cdot \phi_2 \) in (3.4), where \( \text{id}(s) := is \) when \( s \in \mathbb{R} \), we obtain from (3.5) that

\[
\int_{\mathbb{R}} \mathcal{F}(\text{id} \cdot \phi_2)(s)u_1(s) \, ds = \int_{\mathbb{R}} (\mathcal{F}\phi_2)(s)u_2(s) \, ds
\]

whenever \( \phi_2 \in \mathcal{D}(\mathbb{R} \setminus \{0\}) \). Thus \( u_1 \in C^{1+\alpha}(\mathbb{R}; D(B)) \) and \( u'_1 = u_2 + y_1 \) for some \( y_1 \in D(B) \) by [1, Lemma 6.2]. Here we have used the facts that \( u_1, u_2 \in C^\alpha(\mathbb{R}; D(B)) \).

Similarly choosing \( \phi_2 = \text{id} \cdot \phi_4 \) in (3.5), we deduce that \( u_2 \in C^{1+\alpha}(\mathbb{R}; X) \) and \( u'_2 = u_4 + y_2 \) for some \( y_2 \in X \) by [1, Lemma 6.2] and (3.7). Taking \( \phi_4 = \text{id} \cdot \phi_5 \) in (3.7), we deduce that \( u_4 \in C^{1+\alpha}(\mathbb{R}; X) \) and \( u'_4 = u_5 + y_3 \) for some \( y_3 \in X \) by [1, Lemma 6.2] and (3.8). Thus \( u_1 \in C^{1+\alpha}(\mathbb{R}; X) \) and \( u''_1 = u_5 + y_3 \).

Now the identity

\[
ias^3m(s) + s^2m(s) = -Am(s) - isBm(s) + I_X, \quad (s \in \mathbb{R})
\]
implies that
\[ \int_{\mathbb{R}} \mathcal{F}(iap)(s)f(s)\,ds + \int_{\mathbb{R}} (\mathcal{F}l)(s)f(s)\,ds = \cdots \]
for all \( \phi \in \mathcal{D}(\mathbb{R} \setminus \{0\}) \) by (3.9) and (3.10), or equivalently,
\[ au'' + u'' = Au_1 + Bu_2 + f + y \]
for some \( y \in X \) by [1, Lemma 5.1].

Let \( A^{-1} \) be the inverse of \( A : D(A) \to X \) and let \( x = A^{-1}y \). Then \( x \in D(A) \) and \( u = u_1 + x \) solves \( (P) \). This shows the existence.

To show the uniqueness, we let \( u \in C^\alpha(\mathbb{R}; D(A)) \cap C^{3+\alpha}(\mathbb{R}; X) \) be such that \( u' \in C^\alpha(\mathbb{R}; D(B)) \) and
\[ au''(t) + u''(t) = Au(t) + Bu'(t) \]
for all \( t \in \mathbb{R} \). Taking the Carleman transform \( \hat{u} \) of \( u \), we have \( \hat{u}(\lambda) \in D(A) \cap D(B) \) and
\[ \hat{Au}(\lambda) = A\hat{u}(\lambda), \quad \hat{Bu}'(\lambda) = \lambda B\hat{u}(\lambda) - Bu(0), \]
\[ \hat{u''}(\lambda) = \lambda^3 \hat{u}(\lambda) - \lambda^2 u(0) - \lambda u'(0) - u''(0), \]
for all \( \lambda \in \mathbb{C} \setminus i\mathbb{R} \). This implies that the Carleman spectrum \( sp_C(u) \) of \( u \) is empty as \( i\mathbb{R} \subset \rho(P) \) by assumption. Therefore \( u = 0 \) by [2, Theorem 4.8.2]. Hence \( (P) \) is \( C^\alpha \)-well-posed.

Conversely, assume that \( (P) \) is \( C^\alpha \)-well-posed. Let \( L : C^\alpha(\mathbb{R}; X) \to S(\mathbb{R}; X) \) be the solution operator which associates for each \( f \in C^\alpha(\mathbb{R}; X) \), the unique solution \( L(f) \) of \( (P) \), where \( S(\mathbb{R}; X) \) is the solution space of \( (P) \) consisting of all \( u \in C^\alpha(\mathbb{R}; D(A)) \cap C^{3+\alpha}(\mathbb{R}; X) \), such that \( u' \in C^\alpha(\mathbb{R}; D(B)) \) and \( S(\mathbb{R}; X) \) equipped with the norm
\[ (3.12) \|u\|_{S(\mathbb{R}; X)} := \|u\|_{C^\alpha(\mathbb{R}; D(A))} + \|u\|_{C^{3+\alpha}(\mathbb{R}; X)} + \|Bu\|_{C^\alpha(\mathbb{R}; X)} \]
is a Banach space. It is easy to show that \( L \) is linear and bounded by the closed graph theorem.

Let \( s \in \mathbb{R} \) be fixed, we are going to show that \( is \in \rho(P) \). Let \( x \in D(A) \) be such that \( (-ias^3 - s^2)x = Ax + isBx \) and let \( u = e_t \otimes x \), where \( e_t \otimes x \) is the indicator function of \( e^{it}x \) when \( t \in \mathbb{R} \). Then \( u \in C^\alpha(\mathbb{R}; D(A)) \cap C^{3+\alpha}(\mathbb{R}; X) \), \( u' \in C^\alpha(\mathbb{R}; D(B)) \) and
\[ au''(t) + u''(t) = Au(t) + Bu'(t) \]
for all \( t \in \mathbb{R} \). This means that \( u \in S(\mathbb{R}; X) \) and \( u \) solves \( (P) \) when taking \( f = 0 \). Hence \( u = 0 \) by the uniqueness of the solution of \( (P) \). Consequently \( x = 0 \). We have shown that \( ias^3 + s^2 + A + isB \) is injective.
To show that $\text{ias}^3 + s^2 + A + isB$ is also surjective, we let $y \in X$ and consider $f = e_s \otimes y$. Then $f \in \mathcal{C}^\infty(\mathbb{R}; X)$. Let $u \in S(\mathbb{R}; X)$ be the unique solution of $(P)$, i.e.,

$$au''(t) + u''(t) = Au(t) + Bu'(t) + f(t)$$

for all $t \in \mathbb{R}$. For fixed $\xi \in \mathbb{R}$, we consider the function $u_\xi$ given by $u_\xi(t) = u(t + \xi)$ when $t \in \mathbb{R}$. Then both functions $u_\xi$ and $e^{i\xi}u$ are in $S(\mathbb{R}; X)$ and solve the problem

$$au''(t) + u''(t) = Av(t) + Bs(t) + e^{i\xi}f(t).$$

We deduce from the uniqueness that $u_\xi = e^{i\xi}u$, that is $u(t + \xi) = e^{i\xi}u(t)$ for $t, \xi \in \mathbb{R}$. Let $x = u(0)$. Then $x \in D(A)$ and $u = e_s \otimes x$. Since $u$ solve

$$au''(t) + u''(t) = Au(t) + Bu'(t) + f(t),$$

we have $(-\text{ias}^3 - s^2)e_s \otimes x = e_s \otimes Ax + ise_s \otimes Bx + e_s \otimes y$. Letting $t = 0$, we obtain $(-\text{ias}^3 - s^2 - A - isB)x = y$. This shows that $\text{ias}^3 + s^2 + A + isB$ is surjective. Thus $\text{ias}^3 + s^2 + A + isB$ is a bijection from $D(A)$ onto $X$ and $x = -(\text{ias}^3 + s^2 + A + isB)^{-1}y$. We have shown that

$$u = -e_s \otimes (\text{ias}^3 + s^2 + A + isB)^{-1}y.$$ 

Consequently

$$\|\text{ias}^3 + s^2 + A + isB)^{-1}y\| = \|u(0)\| \leq \|L\| \|f\|_{C^\infty(\mathbb{R}, X)} = \|L\| \|1 + \gamma_0 \|s\|^{\alpha}\|y\|$$

for some constant $\gamma_0 > 0$ depending only on $\alpha$ [1, (3.1)]. Thus $(\text{ias}^3 + s^2 + A + isB)^{-1}$ is a bounded linear operator for every $s \neq 0$. That is $is \in \rho(P)$ for all $s \in \mathbb{R} \setminus \{0\}$.

On the other hand, we note that

$$\gamma_0 \|s\|^{\alpha} \|\text{ias}^3 + s^2 + A + isB)^{-1}y\|
= \|s^3 e_s \otimes (\text{ias}^3 + s^2 + A + isB)^{-1}y\|_a
\leq \|L\| \|f\|_{C^\infty} = \|L\| (1 + \gamma_0) \|s\|^{\alpha}\|y\|.$$ 

by [1, (3.1)]. It follows that when $s \neq 0$

$$\|\text{ias}^3 + s^2 + A + isB)^{-1}y\| \leq \|L\| (1 + \gamma_0) \|s\|^{-\alpha}.$$ 

Similarly using the inequality $\|Bu'\|_{C^\infty} \leq \|L\| \|f\|_{C^\infty}$, one obtains

$$\|B(is\text{ias}^3 + s^2 + A + isB)^{-1}y\| \leq \|L\| (1 + \gamma_0) \|s\|^{-\alpha}$$

when $s \neq 0$.

When $s = 0$, $f$ is the constant function $y$ and the corresponding solution $u$ is the constant function $-A^{-1}y$. Then the inequality $\|u\|_{C^\infty} \leq \|L\| \|f\|_{C^\infty}$ implies that

$$\|A^{-1}y\| \leq \|L\| \|y\|.$$ 

that is $0 \in \rho(P)$. Hence we have $i\mathbb{R} \subset \rho(P)$. It is not hard to verify that $\rho(P)$ is an open subset of $\mathbb{C}$ and the functions defined on $\mathbb{R}$ by

$$s \to \|i\text{ias}^3 + s^2 + A + isB)^{-1}y\|, \quad s \to \|B(is\text{ias}^3 + s^2 + A + isB)^{-1}y\|$$

are continuous. It follows that

$$\|i\text{ias}^3 + s^2 + A + isB)^{-1}y\| < \infty, \quad \|B(is\text{ias}^3 + s^2 + A + isB)^{-1}y\| < \infty$$

by continuity, (3.14) and (3.15). This completes the proof. \qed
Since the necessary and sufficient condition for the problem \((P)\) to be \(C^\alpha\)-well-posed obtained in Theorem 3.1 does not depend on the space parameter \(0 < \alpha < 1\). We have the following immediate corollary.

**Corollary 3.1.** If the problem \((P)\) is \(C^\alpha\)-well-posed for some \(0 < \alpha < 1\), then it is \(C^\alpha\)-well-posed for all \(0 < \alpha < 1\).

### 4. Applications

In the last section, we give an example that our abstract results may be applied. We recall that a closed densely defined operator \(A\) on a Banach space \(X\) is sectorial of angle \(\beta \in (0, \pi)\), if \(\sigma(A) \subset \Sigma_\beta\), and if for every \(\beta' \in (\beta, \pi)\)
\[
\sup_{z \in \mathbb{C} \setminus \Sigma_{\beta'}} \|z(z - A)^{-1}\| < \infty,
\]
where \(\Sigma_\beta := \{z \in \mathbb{C} : |\arg(z)| < \beta\}\). For a sectorial operator \(A\) we define the sectorial angle \(\omega(A)\) by
\[
\omega(A) := \inf \{\beta \in (0, \pi) : A\text{ is sectorial of angle } \beta\}.
\]
For every \(\beta \in (0, \pi)\), we put
\[
H^\infty(\Sigma_\beta) := \{f : \Sigma_\beta \to \mathbb{C} \text{ holomorphic} : \|f\|_\infty := \sup_{z \in \Sigma_\beta} \|f(z)\| < \infty\},
\]
\[
H_0^\infty(\Sigma_\beta) := \{f \in H^\infty(\Sigma_\beta) : \text{there exists } \epsilon > 0, \text{ such that } \sup_{z \in \Sigma_\beta} |f(z)||\frac{1 + z^2}{z}| < \infty\}.
\]
If \(A\) is sectorial operator of angle \(\beta \in (0, \pi)\), then
\[
f(A) := \frac{1}{2\pi i} \int_{\partial \Sigma_{\beta'}} f(z)(z - A)^{-1}dz
\]
defines a functional calculus from \(H_0^\infty(\Sigma_\beta')\) into \(\mathcal{L}(X)\) for all \(\beta' > \beta\). This functional calculus may be extended in a natural way in order to define the fractional powers \(A^\epsilon\) for all \(\epsilon > 0\). It is known that \(A^\epsilon\) is still a sectorial operator and \(D(A) \subset D(A^\epsilon)\) when \(0 < \epsilon < 1\). [8]

We say that a sectorial operator \(A\) admits a bounded \(H^\infty\)-functional calculus of angle \(\beta \in \omega(A), \pi)\), if the functional calculus on \(H_0^\infty(\Sigma_{\beta'})\) defined above extends to a bounded linear operator on \(H^\infty(\Sigma_{\beta'})\) for all \(\beta' \in (\beta, \pi)\). The infimum of all such \(\beta\) is denoted by \(\omega_f(A)\). When \(A\) admits a bounded \(H^\infty\)-functional calculus of angle \(\beta\), then there exists a constant \(C_\beta \geq 0\), such that for all \(f \in H^\infty(\Sigma_\beta)\), one has
\[
(4.1) \quad \|f(A)\| \leq C_\beta \|f\|_\infty.
\]
We refer to [8, 9] for the concepts of \(H^\infty\)-functional calculus for sectorial operators.

**Example 4.1.** Let \(a > 0\), \(0 < \alpha < 1\) be fixed and \(X\) be a Banach space. We consider the following second order differential equation:
\[
(P') \quad au''(t) + u''(t) = Au(t) + \gamma A^{m/n}u'(t) + f(t), \quad (t \in \mathbb{R})
\]
where \(A\) is a sectorial operator on \(X\) admitting a bounded \(H^\infty\)-functional calculus of angle \(\beta\) for some \(\beta \in (0, \pi)\), \(m, n\) are given positive integers satisfying \(0 < m/n < 2/3\), \(i\mathbb{R} \subset \rho(P')\) and \(\gamma\) is a fixed scalar number. Then \((P')\) is \(C^\alpha\)-well-posed.
Proof. It is clear that (P′) is a special case of (P) when $B = \gamma A^m/n$. Let $s \in \mathbb{R}$, we consider the polynomial $g_k(z) := z^n + is\gamma z^m + s^2 + as^3$. Let $z_k$ be one of the roots of $g_k(z) = 0$, that is

\begin{equation}
  z_k^n + is\gamma z_k^m + s^2 + as^3 = 0.
\end{equation}

It follows from (4.2) that there exists no real sequence $(s_k)k\in\mathbb{Z}$ converging to $\infty$, such that $(z_k)k\in\mathbb{Z}$ is bounded. This implies that $\lim_{s\to\infty} |z_k| = +\infty$. Let $y_k := z_k^n$. Then by (4.2) there exists no real sequence $(s_k)k\in\mathbb{Z}$ converging to $\infty$, such that $(|y_k|)k\in\mathbb{Z}$ converges to $+\infty$ as $0 < m/n < 2/3$ by assumption. Therefore there exists a constant $C \geq 0$, such that $|y_k| \leq C$ for all $s \in \mathbb{R}$. We deduce from (4.2) and the assumption $0 < m/n < 2/3$ that

\begin{equation}
  \lim_{s\to\infty} y_k = -a. \quad (4.3)
\end{equation}

Let $z_{1,s}, z_{2,s}, \cdots, z_{n,s}$ be the roots of $g_k(z) = 0$. Then $g_k(z) = \prod_{j=1}^n(z - z_{j,s})$. Consider the function $h_k(z) = z + is\gamma z^m/n + s^2 + as^3$, where $z^m/n := e^{i\pi \log z}$ by using the main branch of the logarithm. We have $h_k(z) = \prod_{j=1}^n(z - z_{j,s} - 1/n)$. Then

\begin{equation}
  \sup_{z \in \Sigma_\beta} \left| \frac{z}{h_k(z)} \right| = \sup_{z \in \Sigma_\beta} \left| \frac{1}{\prod_{j=1}^n(z_{j,k} - 1/n)} \right| < \infty, \quad (4.4)
\end{equation}

\begin{equation}
  \sup_{z \in \Sigma_\beta} \left| \frac{z^2}{h_k(z)} \right| = \sup_{z \in \Sigma_\beta} \left| \frac{1}{\prod_{j=1}^n(z_{j,k} - 1)} \right| < \infty, \quad (4.5)
\end{equation}

when $|s|$ is big enough by (4.3). An immediate consequence of (4.4) is

\[
  \sup_{z \in \Sigma_\beta} \left| \frac{s^2}{h_k(z)} \right| < \infty
\]

when $|s|$ is big enough. This implies that

\[
  \sup_{z \in \Sigma_\beta} \left| \frac{sz^{m/n}}{h_k(z)} \right| < \infty
\]

when $|s|$ is big enough by the definition of $h_k$. Consequently the functions $f_{1,s}$ and $f_{2,s}$ defined by

\begin{equation}
  f_{1,s}(z) = \frac{s^2}{z + is\gamma z^m/n + s^2 + as^3}, \quad f_{2,s}(z) = \frac{sz^{m/n}}{z + is\gamma z^m/n + s^2 + as^3}
\end{equation}

belong to $H^{\infty}(\Sigma_\beta)$ when $|s|$ is big enough.

For all $s \in \mathbb{R}$, the operator $as^3 + s^2 + A + i\gamma A^m/n$ is a bijection from $D(A)$ onto $X$ and $(as^3 + s^2 + A + i\gamma A^m/n)^{-1} \in L(X)$ as $i\mathbb{R} \subset \rho(P)$ by assumption. One can verify, using for example [8, Chapter 1], that

\[
  f_{1,s}(A) = s^2(as^3 + s^2 + A + i\gamma A^m/n)^{-1},
\]

\[
  f_{2,s}(A) = sA^{m/n}(as^3 + s^2 + A + i\gamma A^m/n)^{-1}
\]

when $|s|$ is big enough. The assumption that $A$ admits a bounded $H^{\infty}$-functional calculus of angle $\beta$ implies that the sets

\[
  \{s^2(as^3 + s^2 + A + i\gamma A^m/n)^{-1} : s \in \mathbb{R}\}
\]

\[
  \{sA^{m/n}(as^3 + s^2 + A + i\gamma A^m/n)^{-1} : s \in \mathbb{R}\}
\]

are relatively compact.
are bounded by (4.6) and the assumption that $i\mathbb{R} \subset \rho(P')$. Consequently, $(P')$ is $C^\alpha$-well-posed by Theorem 3.1.

We do not know whether the result remains true if we replace the component $m/n$ in $(P')$ by arbitrary $0 < \epsilon < 2/3$.

Similar argument shows that when $\alpha = 0$, then the result remains true when the assumption $0 < m/n < 2/3$ is replaced by $0 < m/n < 1/2$.

References


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