ON SUBCARTESIAN SPACES
LEIBNIZ’ RULE IMPLIES THE CHAIN RULE
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Abstract. We show that derivations of the differential structure of a subcartesian space satisfy chain rule and have maximal integral curves.

1. Introduction

The structure of a smooth manifold is usually described in terms of its complete atlas. In 1967, Aronszajn [1] applied this description to Hausdorff spaces that are locally diffeomorphic to arbitrary subsets of \( \mathbb{R}^n \), which he called subcartesian spaces. In 1973, Walczak [7] showed that that subcartesian spaces of Aronszajn are special cases of differential spaces introduced by Sikorski [3]. This implied that the geometric structure of a subcartesian space \( S \) can be completely described by its ring of smooth functions \( C^\infty(S) \).

In recent years, the notion of \( C^\infty \)-rings and \( C^\infty \)-ringed spaces appeared as part of Spivak’s definition of derived manifolds [6]. Joyce [2] developed an alternative theory of derived differential geometry going beyond Spivak’s derived manifolds.

The definition of derivations of \( C^\infty \)-rings are required to satisfy chain rule, while derivations of the differential structure \( C^\infty(S) \) of a differential space \( S \) are defined algebraically in terms of Leibniz’s rule. We show that, if \( S \) is subcartesian, the derivations of \( C^\infty(S) \) also satisfy the chain rule. This ensures that subcartesian spaces do not require the additional assumption that their differential structures are \( C^\infty \)-rings. In particular, this justifies integration of derivations of differential structures of subcartesian spaces studied in [5].

2. Differential spaces

A differential structure on a topological space \( S \) is a family \( C^\infty(S) \) of real valued functions on \( S \) satisfying the following conditions.

1. The family \( \{f^{-1}(I) \mid f \in C^\infty(S), \text{ and } I \text{ is an open interval in } \mathbb{R} \} \) is a sub-basis for the topology of \( S \).
2. If \( f_1, \ldots, f_n \in C^\infty(S) \) and \( F \in C^\infty(\mathbb{R}^n) \), then \( F(f_1, \ldots, f_n) \in C^\infty(S) \).
3. If \( f : S \to \mathbb{R} \) is a function such that, for every \( x \in S \), there exists an open neighbourhood \( U \) of \( x \) and a function \( f_x \in C^\infty(S) \) satisfying \( f_x|_U = f|_U \) then \( f \in C^\infty(S) \). Here the vertical bar | denotes restriction.

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Functions $f \in C^\infty(S)$ are called smooth functions on $S$. It follows from condition 1 that smooth functions on $S$ are continuous. Condition 2 with $F(f_1, f_2) = af_1 + bf_2$, where $a, b \in \mathbb{R}$ implies that $C^\infty(S)$ is a vector space. Similarly, taking $F(f_1, f_2) = f_1f_2$, we conclude that $C^\infty(S)$ is closed under multiplication of functions. A topological space $S$ endowed with a differential structure is called a differential space.

In his original definition, Sikorski [4] defined $C^\infty(S)$ to be a family of functions satisfying condition 2. Then, he used condition 1 to define topology on $S$. Finally, he imposed condition 3 as a consistency condition.

A map $\varphi : R \to S$ is smooth if $\varphi^* f = f \circ \varphi \in C^\infty(R)$ for every $f \in C^\infty(S)$. A smooth map $\varphi$ between differential spaces is a diffeomorphism if it is invertible and its inverse is smooth.

**Proposition 2.1.** A smooth map between differential spaces is continuous.

*Proof.* See the proof of proposition 2.1.5 in [5] \qed

A differential space $S$ is subcartesian if its topology is Hausdorff and every point $x \in S$ has a neighbourhood $U$ diffeomorphic to a subset $V$ of $\mathbb{R}^n$. It should be noted that $V$ in the definition above may be an arbitrary subset of $\mathbb{R}^n$, and $n$ may depend on $x \in S$. As in the theory of manifolds, diffeomorphisms of open subsets of $S$ onto subsets of $\mathbb{R}^n$ are called charts on $S$. The family of all charts is the complete atlas on $S$. Aronszajn [1] used the notion of a complete atlas on a Hausdorff topological space in his definition of subcartesian space.

3. Derivations at a Point

Let $S$ be a subcartesian space with differential structure $C^\infty(S)$. A derivation of $C^\infty(S)$ at a point $x \in S$ is a linear map $v_x : C^\infty(S) \to \mathbb{R}$ such that

$$v_x(f_1f_2) = v_x(f_1f_2(x) + f_1(x)v_x(f_2))$$

for every $f_1, f_2 \in C^\infty(S)$.

If $M$ is a manifold, then derivations of $C^\infty(M)$ satisfy the chain rule. In other words, for every $v \in TM, f_1, ..., f_k \in C^\infty(M)$ and $F \in C^\infty(\mathbb{R}^k)$

$$vF(f_1, ..., f_k) = \partial_1F(f_1, ..., f_k)(\tau(v))v_1f_1 + ... + \partial_kF(f_1, ..., f_k)(\tau(v))v_kf_k,$$

where $\tau : TM \to M$ is the tangent bundle projection map and $\partial_1, ..., \partial_k$ are partial derivatives in $\mathbb{R}^k$. Our aim in this section is to show that the chain rule is also valid for derivations of $C^\infty(S)$, where $S$ is a subcartesian space.

**Lemma 3.1.** Let $v$ be a derivation of $C^\infty(S)$ at $x \in S$. For every open neighbourhood $U$ of $x$ and every $f \in C^\infty(S)$, $vf$ depends only on the restriction of $f$ to $U$.

*Proof.* Let $f_1, f_2 \in C^\infty(S)$ agree on a neighbourhood $U$ of $x \in S$. By lemma 2.2.1 of [5] there exists a function $h \in C^\infty(S)$ satisfying $h|_U = 1$ for some neighbourhood $V$ of $x$ contained in $U$, and $f|_W = 0$ for some open set $W$ in $S$ such that $U \cup W = S$. Then $h(f_1 - f_2) = 0$, so that $v[h(f_1 - f_2)] = 0$. Hence,

$$0 = v[h(f_1 - f_2)] = (vh)(f_1 - f_2)(x) + [v(f_1 - f_2)]h(x) = vf_1 - vf_2$$

because $f_1(x) = f_2(x)$ and $h(x) = 1$. This implies $vf_1 = vf_2$ as required. \qed
Let $\varphi : S \to R$ be a smooth map between differential spaces with differential structures $C^\infty(S)$ and $C^\infty(R)$, respectively.

**Lemma 3.2.** The map $\varphi : S \to R$ assigns to each derivation $v$ of $C^\infty(S)$ at $x \in S$ a derivation $T\varphi(v)$ of $C^\infty(R)$ at $\varphi(x) \in R$ such that, for every $f \in C^\infty(R)$,

$$T\varphi(v)f = v(\varphi^*f).$$

**Proof.** For every $f \in C^\infty(R)$, $\varphi^*f = f \circ \varphi$ is in $C^\infty(S)$, and we may evaluate the derivation $v$ on $\varphi^*f$. Note that, for $f_1, f_2 \in C^\infty(R)$ and $c_1, c_2 \in \mathbb{R}$,

$$T\varphi(v)(c_1f_1 + c_2f_2) = v(c_1\varphi^*f_1) + v(c_2\varphi^*f_2) = c_1v(\varphi^*f_1) + c_2v(\varphi^*f_2) = c_1(T\varphi(v))f_1 + c_2(T\varphi(v))f_2.$$

Hence, $f \to T\varphi(v)f$ is a linear mapping of $C^\infty(R)$ into itself. For $f_1, f_2 \in C^\infty(R)$, equation (3.3) yields

$$T\varphi(v)(f_1f_2) = v((\varphi^*f_1)(\varphi^*f_2)) = v(\varphi^*f_1)\varphi^*f_2(x) + \varphi^*f_1(x)v(\varphi^*f_2) = [T\varphi(v)f_1][f_2(\varphi(x))] + [f_1(\varphi(x))][T\varphi(v)f_2].$$

Hence $T\varphi(v)$ is a derivation of $C^\infty(R)$ at $\varphi(x)$.

**Theorem 3.3.** For each $x$ in a subcartesian space $S$, every derivation $v$ of $C^\infty(S)$ at $x$ satisfies the chain rule. In other words, for every $k \in \mathbb{N}$, $f_1, \ldots, f_k \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^k)$,

$$(3.4) \quad v[F(f_1, \ldots, f_k)] = [\partial_1F(f_1, \ldots, f_k)(x)]v_1 + \cdots + [\partial_kF(f_1, \ldots, f_k)(x)]v_k,$$

where $\partial_1, \ldots, \partial_k$ are partial derivatives in $\mathbb{R}^k$.

**Proof.** Since $S$ is subcartesian, there exists a diffeomorphism $\varphi : W \to \varphi(W) \subseteq \mathbb{R}^n$, where $W$ is an open neighbourhood of $x$ in $S$. Let $i_W : W \to S$ be the inclusion map. For every $f \in C^\infty(S)$, $i_W^*f = f|_W \in C^\infty(W)$. By lemma 3.2, $vf$ is completely determined by $f|_W$, and equation (3.3) yields

$$vF = Ti_W(v)(i_W^*f) = Ti_W(v)(f|_W).$$

Since $\varphi : W \to \varphi(W) \subseteq \mathbb{R}^n$ is a diffeomorphism, $h = (\varphi^{-1})^*f|_W \in C^\infty(\varphi(W))$ and

$$Ti_W(v)(f|_W) = [T\varphi(Ti_W(v))]h.$$

Let $i_{\varphi(W)} : \varphi(W) \to \mathbb{R}^n$ be the inclusion map. Since $T\varphi(Ti_W(v))$ is a derivation of $C^\infty(\varphi(W))$ at $\varphi(x)$, $T_{i_{\varphi(W)}}(T\varphi(Ti_W(v)))$ is a derivation of $C^\infty(\mathbb{R}^n)$ at $\varphi(x)$. Without loss of generality, we may assume that the function $h$ in equation (3.6) is the restriction to $\varphi(W)$ of a function $H \in C^\infty(\mathbb{R}^n)$. Therefore,

$$(3.7) \quad [T\varphi(Ti_W(v))]h = [T_{i_{\varphi(W)}}(T\varphi(Ti_W(v)))]H.$$ 

Derivations of $C^\infty(\mathbb{R}^n)$ satisfy the chain rule. If $H = F(H_1, \ldots, H_k)$, for some $k \in \mathbb{N}$, $H_1, \ldots, H_k \in C^\infty(\mathbb{R}^n)$ and $F \in C^\infty(\mathbb{R}^k)$, then equation (3.2) yields

$$[T_{i_{\varphi(W)}}(T\varphi(Ti_W(v)))]F(H_1, \ldots, H_k) = (3.8) \quad [\partial_1F(H_1, \ldots, H_k)(\varphi(x))]T_{i_{\varphi(W)}}(T\varphi(Ti_W(v)))[H_1] + \cdots + [\partial_kF(H_1, \ldots, H_k)(\varphi(x))]T_{i_{\varphi(W)}}(T\varphi(Ti_W(v)))[H_k].$$

Therefore,

$$vF(f_1, \ldots, f_k) = [T_{i_{\varphi(W)}}(T\varphi(Ti_W(v)))F(H_1, \ldots, H_k)]$$

$$(3.9) \quad = [\partial_1F(H_1, \ldots, H_k)(\varphi(x))]T_{i_{\varphi(W)}}(T\varphi(Ti_W(v)))[H_1] + \cdots + [\partial_kF(H_1, \ldots, H_k)(\varphi(x))]T_{i_{\varphi(W)}}(T\varphi(Ti_W(v)))[H_k].$$
\[ \cdots + \left[ \partial_k F(H_1, \ldots, H_k)(\varphi(x)) \right]|_{\varphi(W)} [T_{\varphi(W)}(T_{\varphi(T_{\varphi(W)}(v))})] H_k, \]

where \( H_i|_{\varphi(W)} = (\varphi^{-1})^* f_i|_W \) for \( i = 1, \ldots, k \).

4. THE TANGENT BUNDLE

Let \( T_x S \) be the set of all derivations of \( C^\infty(S) \) at \( x \in S \). The set \( T_x S \) is a real vector space, which is interpreted to be the tangent \textit{tangent space} to \( S \) at \( x \). Let \( TS \) be the union of tangent spaces to \( S \) at each point \( x \) of \( S \). In other words,

\[ TS = \bigcup_{x \in S} T_x S \]

The tangent bundle projection is the map \( \tau : TS \to S : v = (x, v_x) \to x \), which assigns to each derivation \( v_x \in T_x S \) at \( x \) the point \( x \in S \). The tangent bundle projection enables us to omit the subscript \( x \) in the definition of derivation at a point, and rewrite equation (3.1) in the form

\[ (4.1) \quad v(f_1 f_2) = v(f_1)f_2 + f_1 v(f_2). \]

Each function \( f \in C^\infty(S) \) gives rise to two functions on \( TS \), namely,

\[ \tau^* f : TS \to \mathbb{R} : v \mapsto f(\tau(v)) \]

and

\[ df : TS \to \mathbb{R} : v \mapsto df(v) = v(f). \]

The tangent bundle of a differential space \( S \) is \( TS \) with differential structure \( C^\infty(TS) \) generated by the family of functions \( \{ \tau^* f, df | f \in C^\infty(S) \} \). This definition of \( C^\infty(TS) \) ensures that the tangent bundle projection \( \tau : TS \to S \) is smooth. The derived map of a smooth map \( \varphi : S \to R \) is \( T\varphi : TS \to TR : v \mapsto T\varphi(v) \), where for every \( f \in C^\infty(R) \) \( [T\varphi(v)] f = v(\varphi^* f) \), see lemma 3.2. If \( \tau_S : TS \to S \) and \( \tau_R : TR \to R \) are tangent bundle projections, then

\[ (4.2) \quad \tau_R \circ T\varphi = \varphi^* \tau_R. \]

5. GLOBAL DERIVATIONS

A derivation of \( C^\infty(S) \) is a linear map \( X : C^\infty(S) \to C^\infty(S) \) : \( f \mapsto X(f) \) satisfying Leibniz’s rule

\[ (5.1) \quad X(f_1 f_2) = X(f_1)f_2 + f_1 X(f_2) \]

for every \( f_1, f_2 \in C^\infty(S) \). Let \( \operatorname{Der} C^\infty(S) \) be the space of derivations of \( C^\infty(S) \). It has the structure of a Lie algebra with the Lie bracket \([X_1, X_2]\) defined by

\[ [X_1, X_2](f) = X_1(X_2(f)) - X_2(X_1(f)) \]

for every \( X_1, X_2 \in \operatorname{Der} C^\infty(S) \) and \( f \in C^\infty(S) \). Moreover, \( \operatorname{Der} C^\infty(S) \) is a module over the ring \( C^\infty(S) \) and

\[ [f_1 X_1, f_2 X_2] = f_1 f_2[X_1, X_2] + f_1 X_1(f_2)X_2 - f_2 X_2(f_1)X_1 \]

for every \( X_1, X_2 \in \operatorname{Der} C^\infty(S) \) and \( f_1, f_2 \in C^\infty(S) \). If \( X \) is a derivation of \( C^\infty(S) \), then, for every \( x \in S \) we have a derivation \( X(x) \) of \( C^\infty(S) \) at \( x \) given by

\[ (5.2) \quad X(x) : C^\infty(S) \to \mathbb{R} : f \mapsto X(x)f = (X\varphi)(x). \]

The derivation \( X(x) \) (5.2) is called the \textit{value} of \( X \) at \( x \). Clearly, the derivation \( X \) is uniquely determined by the collection \( \{X(x) | x \in S\} \) of its values at all points of \( S \). In order to avoid confusion between a derivation of \( C^\infty(S) \) and a derivation
Theorem 5.1. Let $S$ be a differential subspace of $\mathbb{R}^n$ and $X$ a derivation of $C^\infty(S)$. For each $x \in S \subseteq \mathbb{R}^n$, there exists a neighbourhood $U$ of $x$ in $\mathbb{R}^n$ and a vector field $Y$ on $\mathbb{R}^n$ such that

$$X(F|_S)|_{U \cap S} = (Y(F))|_{U \cap S}$$

for every $F \in C^\infty(\mathbb{R}^n)$.

Proof. Let $Z$ be a derivation of $C^\infty(S)$ at $x \in S \subseteq \mathbb{R}^n$. For each $F \in C^\infty(\mathbb{R}^n)$ the restriction $F|_S$ of $F$ to $S$ is in $C^\infty(S)$. It is easy to see that the map $C^\infty(\mathbb{R}^n) \to \mathbb{R} : F \mapsto Z(F|_S)$ is a derivation at $x$ of $C^\infty(\mathbb{R}^n)$.

We denote the natural coordinate functions on $\mathbb{R}^n$ by $x^1, \ldots, x^n : \mathbb{R}^n \to \mathbb{R}$. Every derivation $Y$ of $C^\infty(\mathbb{R}^n)$ is of the form

$$Y = \sum_{i=1}^n F^i \frac{\partial}{\partial x^i},$$

where $F^i = Y(x^i)$ for $i = 1, \ldots, n$. Let $X$ be a derivation of $C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$. For each $x \in S$, the derivation $X(x)$ of $C^\infty(S)$ at $x$ gives a derivation of $C^\infty(\mathbb{R}^n)$ at $x$. Hence,

$$X(F|_S)(x) = X(x)(F|_S) = \sum_{i=1}^n \frac{\partial F}{\partial x^i}(x)(X(x)(x^i|_S))$$

$$= \sum_{i=1}^n \frac{\partial F}{\partial x^i}(x)(X(x^i|_S))(x) = \left(\sum_{i=1}^n X(x^i|_S) \frac{\partial F}{\partial x^i}\right)(x)$$

for every $x \in S$. For $i = 1, \ldots, n$, the coefficients $X(x^i|_S)$ are in $C^\infty(S)$. Since $S$ is a differential subspace of $\mathbb{R}^n$, for each $x \in S$ there exists a neighbourhood $U$ of $x$ in $\mathbb{R}^n$ and functions $F^1, \ldots, F^n \in C^\infty(\mathbb{R}^n)$ such that $X(x^i|_S)|_{U \cap S} = F^i|_{U \cap S}$ for each $i = 1, \ldots, n$. Hence,

$$X(F|_S)|_{U \cap S} = \left(\sum_{i=1}^n F^i \frac{\partial F}{\partial x^i}\right)|_{U \cap S}.$$ 

Since $F^1, \ldots, F^n$ are smooth functions on $\mathbb{R}^n$, it follows that $Y = \sum_{i=1}^n F^i \frac{\partial}{\partial x^i}$ is a vector field on $\mathbb{R}^n$. $\square$

We can rephrase theorem 5.1 by saying that every derivation on a differential subspace $S$ of $\mathbb{R}^n$ can be locally extended to a vector field on $\mathbb{R}^n$. Suppose that $S$ is closed. In this case, we can use a partition of unity on $\mathbb{R}^n$ to extend every derivation of $C^\infty(S)$ to a global vector field on $\mathbb{R}^n$.

A section of the tangent bundle projection $\tau : TS \to S$ is a smooth map $\xi : S \to TS$ such that $\tau \circ \xi = \text{id}_S$. Let $S^\infty(TS)$ be the space of sections of the tangent bundle projection $\tau : TS \to S$. Since the differential structure $C^\infty(TS)$ is generated by the collection of functions $\{\tau^*f, df | f \in C^\infty(S)\}$, it follows that a section $\xi : S \to TS$ has to satisfy the conditions that $\xi(\tau^*f)$ and $\xi^*(df)$ are in $C^\infty(S)$ for every $f \in C^\infty(S)$. The first condition holds automatically because

$$\xi(\tau^*f)(x) = (\tau^*f) \circ \xi = f \circ \tau \circ \xi = f \circ \text{id}_S = f.$$ 

On the other hand, for $x \in S$,

$$(\xi^*(df))(x) = ((df) \circ \xi)(x) = (df) | (\xi(x)) = \xi(x)f.$$
Proposition 5.2. Every global derivation \( X \) of \( C^\infty(S) \) defines a section
\[
\left(5.5\right) \quad X : S \to TS : x \mapsto X(x),
\]
where \( X(x)f = (Xf)(x) \) for every \( f \in C^\infty(S) \) and every \( x \in S \).

Proof. The section \( X : S \to TS \), defined by equation (5.5), satisfies equation (5.4) because \( X^* (df) = X(f) \in \text{Der} C^\infty(S) \) by definition of a global derivation. Conversely, if \( \xi : S \to TS \) is a section, then equation (5.4) implies that \( \xi(f) = \xi^* (df) \in \text{Der} C^\infty(S) \) for every \( f \in C^\infty(S) \). Hence, \( \xi : f \mapsto \xi(f) \) is a global derivation of \( C^\infty(S) \).

Equation (5.5) gives a bijection between the space \( S^\infty(TS) \) of sections of the tangent bundle projection and the space \( \text{Der} C^\infty(S) \). Hence proposition 5.2 leads to identification of global derivations of \( C^\infty(S) \) with the corresponding sections of the tangent bundle.

Let \( c : I \to S \) be a smooth map of an interval \( I \) in \( \mathbb{R} \) containing 0 to a differential space \( S \). We say that \( c \) is an integral curve of a derivation \( X \) of \( C^\infty(S) \) starting at \( x_0 \in S \) if \( x_0 = c(0) \) and
\[
\left(5.6\right) \quad \frac{d}{dt} f(c(t)) = X(f)(c(t))
\]
for every \( f \in C^\infty(S) \) and every \( t \in I \). In other words, \( c : I \to S \) is an integral curve of \( X \) if \( Tc(t) = X \circ c(t) \) for every \( t \in I \).

Integral curves of a given derivation \( X \) of \( C^\infty(S) \) can be ordered by inclusion of their domains. In other words, if \( c_1 : I_1 \to S \) and \( c_2 : I_2 \to S \) are two integral curves of \( X \) and \( I_1 \subseteq I_2 \), then \( c_1 \preceq c_2 \). An integral curve \( c_1 : I \to S \) of \( X \) is maximal if \( c_1 \preceq c_2 \) implies that \( c_1 = c_2 \).

Example. Let \( \mathbb{Q} \) be the set of rational numbers in \( \mathbb{R} \). Then \( C^\infty(\mathbb{Q}) \) consists of restrictions to \( \mathbb{Q} \) of smooth functions on \( \mathbb{R} \). Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), it follows that every function \( f \in C^\infty(\mathbb{Q}) \) extends to a unique smooth function on \( \mathbb{R} \) and every derivation of \( C^\infty(\mathbb{R}) \) induces a derivation of \( C^\infty(\mathbb{Q}) \). Let \( X \) be the derivation of \( C^\infty(\mathbb{Q}) \) induced by the derivative \( \frac{d}{dx} \) on \( C^\infty(\mathbb{R}) \). In other words, for every \( f \in C^\infty(\mathbb{Q}) \) and every \( x_0 \in \mathbb{Q} \),
\[
(Xf)(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},
\]
where the limit is taken over \( x \in \mathbb{Q} \). On the other hand, no two distinct points in \( \mathbb{Q} \) can be connected by a continuous curve.

Definition. Let \( \tau : TS \to S \) be the tangent bundle projection. Let \( X \) be a derivation of the differential structure \( C^\infty(S) \) of a subcartesian space \( S \). Let \( x_0 \) be a point in \( S \), and \( I \) be an interval in \( \mathbb{R} \) containing 0 in \( \mathbb{R} \) or \( I = \{0\} \). A lifted integral curve of \( X \) starting at \( x_0 \) is a map \( \gamma : I \to TS \) such that \( \gamma(0) = X(x_0) \) and
\[
\left(5.7\right) \quad \frac{d}{dt} f(\tau(\gamma(t))) = X(f)(\tau(\gamma(t)))
\]
for every \( f \in C^\infty(S) \) and \( t \in I \), if \( I \neq \{0\} \).

If \( I \neq \{0\} \), then setting \( c = \tau \circ \gamma \) we recover the definition for an integral curve of a derivation given in equation (5.6). If \( I = \{0\} \), then \( \gamma \) is a lifted integral curve of \( X \) starting at \( x_0 \), because \( \gamma(0) = X(c(0)) = X(x_0) \). Our extension of this definition
to subcartesian spaces requires lifting the curve $c : I \to S$ to the tangent bundle leading to $\gamma : I \to TS$, in order to make sense of the condition $\gamma(0) = X(c(0))$.

**Theorem 5.3.** Let $S$ be a subcartesian space and let $X$ be a derivation of $C^\infty(S)$. For every $x \in S$, there exists a unique maximal lifted integral curve $\gamma_x$ of $X$ starting at $x$.

**Proof.**

i) **Local existence.** For $x \in S$, let $\varphi$ be a diffeomorphism of a neighbourhood $V$ of $x$ in $S$ onto a differential subspace $R$ of $\mathbb{R}^n$. Let $Z = \varphi_*X|_V$ be a derivation of $C^\infty(R)$ obtained by pushing forward the restriction of $X$ to $V$ by $\varphi$. In other words,

$$Z(f) \circ \varphi = X|_V(f \circ \varphi)$$

for all $f \in C^\infty(R)$. Without loss of generality, we may assume that there is an extension of $Z$ to a vector field $Y$ on $\mathbb{R}^n$.

Let $z = \varphi(x)$ and let $c_0$ be a standard integral curve in $\mathbb{R}^n$ of the vector field $Y$ such that $c_0(0) = z$. Let $I_x$ be the connected component of $c_0^{-1}(R)$ containing 0 and let $c : I_x \to R$ the curve in $R$ obtained by the restriction of $c_0$ to $I_x$. Clearly, $c(0) = z$.

We consider first the case when $I_x = \{0\}$, which means that there exists an open neighbourhood $U_0$ of $z$ such that $c_0$ intersects $R \cap U$ only at $z$. In this case, we may consider another extension of $Z$ to a vector field on $\mathbb{R}^n$. If $I_x = \{0\}$ for every extension $Y$ of $Z$ to a vector field on $\mathbb{R}^n$, then the map $\gamma : \{0\} \to TR : 0 \to Z(z)$ is a maximal lifted integral curve of the vector field $Z$ on $R$ which starts at $z$. Since $V$ is an open neighborhood of $x$ in $S$ and $\varphi : V \to R \subseteq \mathbb{R}^n$ is a diffeomorphism, $Z = \varphi_*X$ and $z = \varphi(x)$, it follows that the such that the map $T(\varphi^{-1}) \circ \gamma : \{0\} \to TS : 0 \to TS$ is a maximal lifted integral curve of $X$ starting at $x$. Here, $\iota_V : V \to S$ is the inclusion map.

Suppose now that $I_x \neq \{0\}$. For each $t_0 \in I_x$ and each $f \in C^\infty(R)$ there exists a neighbourhood $U$ of $c(t_0)$ in $R$ and a function $F \in C^\infty(\mathbb{R}^n)$ such that $F|_U = F|_U$ and

$$\frac{d}{dt} f(c(t)) = \frac{d}{dt} F(c(t)) = Y(F)(c(t_0)) Y(F)|_U(c(t_0)) = Z(f)(c(t_0)).$$

Since $I_x \neq \{0\}$ is a connected subset of $\mathbb{R}$ containing 0, it is an interval. So $c_x = \varphi^{-1} \circ c : I_x \to V \subseteq S$ satisfies $c_x(0) = \varphi^{-1}(c(0)) = \varphi^{-1}(z) = x$. Moreover, for every $t \in I_x$ and $h \in C^\infty(S)$, we get $f = h \circ \varphi^{-1} \in C^\infty(R)$ and

$$\frac{d}{dt} h(c_x(t)) = \frac{d}{dt} h(\varphi^{-1}(c(t))) = \frac{d}{dt}(h \circ \varphi^{-1})(c(t)) = \frac{d}{dt} f(c(t))$$

$$= Z(f)(c(t)) = Z(h \circ \varphi^{-1})(\varphi \circ c_x(t)) = X(h)(c_x(t)).$$

This implies that the map $\gamma_x : I_x \to TS : t \to X(c_x(t))$ is a lifted integral curve of $X$ starting at $x$ if $I_x \neq \{0\}$. It is also an integral curve of $X$ starting at $x$ when $I_x = \{0\}$, because $\gamma_x(0) = X(x)$.

ii) **Smoothness.** From the theory of differential equations it follows that the integral curve $c_0$ in $\mathbb{R}^n$ of a smooth vector field $Y$ is smooth. Hence, $c = c_0|_{I_x}$ is smooth. Since $\varphi$ is a diffeomorphism of a neighborhood of $x$ in $S$ to $R$, its inverse $\varphi^{-1}$ is smooth and the composition $c_x = \varphi^{-1} \circ c$ is smooth. Since $X$ is a derivation,
it gives rise to a smooth section \( X : S \to TS \) of the tangent bundle projection \( \tau : TS \to S \). Moreover the composition \( \gamma_x = X \circ c_x \) is smooth.

iii) **Local uniqueness.** This follows from the local uniqueness of solutions of first order differential equations in \( \mathbb{R}^n \).

iv) **Maximality.** Suppose that \( p \leq 0 \leq q \) are the ends of the domain \( I_x \) of the integral curve \( c_x \) of \( X \) starting at \( x \) obtained in section i). If \( p = q = 0 \) and \( \gamma_x = X \circ c_x \) cannot be extended to a larger interval, then \( \gamma_x \) is maximal. If \( q > 0 \) and \( q = \infty \) or \( \lim_{t \to q} c_x(t) \) does not exist, then the curve \( c_x \) does not extend beyond \( q \). If \( x_1 = \lim_{t \to q} c_x(t) \) exists, then \( x_1 \) is unique since the topology of \( S \) is Hausdorff. We can repeat the construction of section i) by starting at the point \( x_1 \). In this way we obtain an integral curve \( c_{x_1}^1 : I_1 \to S \) of \( X \) starting at \( x_1 \).

Let \( I_1 = I \cup \{ t = q + s \in \mathbb{R} \mid s \in I_1 \cap [0, \infty) \} \) and let \( c_1 : I_1 \to S \) be given by \( c_1(t) = c_x(t) \) if \( t \in I \) and \( c_1(t) = c_{x_1}^1(t - q) \) if \( t \in \{ q + s \in \mathbb{R} \mid s \in I_1 \cap [0, \infty) \} \).

Clearly the curve \( c_1 \) is continuous. Since \( x_1 = \lim_{t \to q} c_x(t) \), it follows that the left end \( p_1 \) of \( I_1 \) is strictly less than zero. Hence, the restriction of the curve \( c_x \) to the interval \((\max(p, p_1) + q, q)\) differs from the restriction of \( c_{x_1}^1 \) to the interval \((\max(p, p_1), 0)\) by the reparametrization \( t \mapsto t - q \). Since the curves \( c_x \) and \( c_{x_1}^1 \) are smooth, it follows that the curve \( c_1 \) is smooth. Let \( q_1 \) be the right end of the interval \( I_1 \). If \( q_1 \in I \) and either \( q_1 = \infty \) or \( \lim_{t \to q_1} c_x(t) \) does not exist, then the curve \( c_1 \) does not extend beyond \( q_1 \). Otherwise, we can extend \( c_1 \) by an integral curve \( c_2 \) of \( X \) through \( x_2 = \lim_{t \to q_1} c_1(t) \). Continuing this process, we obtain a maximal extension for \( t \geq 0 \). In a similar way we can construct a maximal extension for \( t \leq 0 \).

v) **Global uniqueness.** Let \( c : I \to S \) and \( c' : I' \to S \) be maximal integral curves of \( X \) starting at \( x \). Let \( T^+ = \{ t \in I \cap I' \mid \ell > 0 \) and \( c(t) \neq c'(t) \} \). Suppose that \( T^+ \neq \emptyset \). Since \( T^+ \) is bounded from above by \( 0 \), there is a greatest lower bound \( \ell \) of \( T^+ \). This implies that \( c(t) = c'(t) \) for every \( 0 \leq t \leq \ell \) and for every \( \varepsilon > 0 \) there is a \( t_\varepsilon \) with \( \ell < t_\varepsilon < \ell + \varepsilon \) such that \( c(t_\varepsilon) \neq c'(t_\varepsilon) \). Let \( x_\varepsilon = c(\ell) = c'(\ell) \) and let \( c_\varepsilon : I_\varepsilon \to S \) be an integral curve of \( X \) starting at \( x_\varepsilon \) as constructed in i). Let \( q_\varepsilon \) be the right end of \( I_\varepsilon \). If \( q_\varepsilon > 0 \), the local uniqueness of integral curves implies that \( c(\ell) = c'(\ell) = c_\varepsilon(\ell - t) \) for all \( \ell \leq t \leq \ell + q_\varepsilon \), which contradicts the fact that \( \ell \) is the greatest lower bound of \( T^+ \). If \( q_\varepsilon = 0 \), then there is no extension of \( c_\varepsilon \). Let \( q \) and \( q' \) be the right end of \( I \) and \( I' \), respectively. Since \( c \) and \( c' \) are maximal integral curves of \( X \), it follows that \( q = q' = \ell \). Hence the set \( T^+ \) is empty, which is a contradiction.

A similar argument shows that \( T^- = \{ t \in I \cap I' \mid \ell < 0 \) and \( c(t) \neq c'(t) \} \neq \emptyset \). Therefore \( c(t) = c'(t) \) for all \( t \in I \cap I' \). If \( I \neq I' \), then this contradicts the fact that \( c \) and \( c' \) are maximal. Hence \( I = I' \) and \( c = c' \).

\[ \square \]

6. **Vector fields**

Vector fields on a manifold \( M \) are not only derivations of \( \mathcal{C}^\infty(M) \) but they also generate local one-parameter groups of local diffeomorphisms of \( M \). On a subcartesian space \( S \), not all derivations of \( \mathcal{C}^\infty(S) \) generate local one-parameter groups of local diffeomorphisms of \( S \), see example 3.2.7 in [5, p.37]. We reserve the term vector field for derivations of \( \mathcal{C}^\infty(S) \) that generate local one-parameter groups of local diffeomorphisms of \( S \). More formally, we adopt the following definition. A **vector field** on a subcartesian space \( S \) is a derivation \( X \) of \( \mathcal{C}^\infty(S) \) such that for every \( x_0 \in S \), there exists a neighbourhood \( U_{x_0} \) of \( x_0 \in S \) and \( \varepsilon_{x_0} > 0 \) such that,
for every $x \in U_{x_0}$, the interval $(-\varepsilon_{x_0}, \varepsilon_{x_0})$ is contained in the domain $I_x$ of the lifted integral curve $\gamma_x : I_x \to TS$ of $X$ and the map
\[
edot X : U_{x_0} \to S : x \mapsto \exp tX(x) = \tau \circ \gamma_x(t)
\]
is defined for every $t \in (-\varepsilon_{x_0}, \varepsilon_{x_0})$ and is a diffeomorphism of $U_{x_0}$ onto an open subset $\nedot X(U_{x_0})$ of $S$.

Note that if $X$ is a vector field on $S$, for every $x \in S$, the map $c_x : I_x \to S : t \mapsto \exp tX(x)$ is an integral curve of $S$ satisfying equation (5.6). Therefore, if we were only interested in vector fields on $S$, we could use the definition of integral curves given by equation (5.6). We have introduced the notion of lifted integral curves to obtain theorem 8, which ensures the existence and uniqueness of maximal lifted integral curves of derivations of $C^\infty(S)$. Theorem 8 is replaces theorem 3.2.1 in [5], which is incorrect. Our discussion shows that proofs of all results in [5] regarding vector fields on subcartesian spaces are not affected by errors in theorem 3.2.1. In particular, all results in section 3.4 and chapter 4 of [5] are valid.

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\[\text{et}_X(x)\text{ is a compact version of the notation } (\exp tX)(x) \text{ used in [5]}\]