DERIVATIVES OF BLASCHKE PRODUCTS AND MODEL SPACE FUNCTIONS

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Abstract. The relationship between the distribution of zeros of an infinite Blaschke product $B$ and the inclusion in weighted Bergman spaces $A^p_\alpha$ of the derivative of $B$ or the derivative of functions in its model space $H^2 \ominus BH^2$ is investigated.

1. Preliminaries

If $f$ is analytic in the open unit disc $U$ and $0 < p < \infty$, then

$$M_p(r; f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right\}^{1/p}$$

is defined for each positive $r < 1$. The Hardy space $H^p$ is the set of all functions $f$, analytic in $U$, for which $\|f\|_{H^p} = \sup_{0<r<1} M_p(r; f)$ is finite. Let $dA(z)$ denote Lebesgue area measure. If $f$ is analytic in $U$, $0 < p < \infty$ and $\alpha > -1$, then $f$ is said to be in the space $A^p_\alpha$ if

$$\|f\|_{A^p_\alpha} = \left\{ \frac{1}{\pi} \iint_U |f(re^{it})|^p (1-r)^\alpha \, dA(z) \right\}^{1/p}$$

is finite. Put $A^p = A^p_0$.

If $\{a_n\}$ is a sequence of complex numbers such that $0 < |a_n| < 1$ for all $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$, the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{a_n z}{a_n - z}$$

is an analytic function in $U$ with zeros $\{a_n\}$. A sequence $\{a_n\}$ of points in $U$ is said to be separated or uniformly discrete if there is a constant $\delta > 0$ such that $\rho(a_m, a_n) \geq \delta$ for all $m \neq n$, where $\rho$ is the pseudohyperbolic metric in $U$ and is given by

$$\rho(z, w) = \frac{|z - w|}{|1 - wz|}, \quad z, w \in U.$$ 

The sequence $\{a_n\}$ is said to be uniformly separated if there is a constant $\delta > 0$ such that

$$\inf_n \prod_{m \neq n} \rho(a_m, a_n) \geq \delta.$$
A Blaschke product whose zeros are uniformly separated is called an interpolating Blaschke product. It is clear that uniformly separated sequences form a proper subset of the set of all uniformly discrete sequences.

For any Blaschke product $B$, let $(BH^2)^\perp = H^2 \ominus BH^2$ be the orthogonal complement of the invariant subspace $BH^2$ in $H^2$. $(BH^2)^\perp$ is called the model space or star-invariant subspace for $B$ in $H^2$. Here are a few standard results about $(BH^2)^\perp$ (see, for example, [9] and [11]) that we will be using. For $z \in U$, the reproducing kernel for $(BH^2)^\perp$ is

$$K_\lambda(u) = \frac{1 - \overline{B(u)}B(u)}{1 - \overline{u}u}.$$ 

$u \in U$. That is, $\langle f, K_\lambda \rangle = f(z)$ for all $f \in (BH^2)^\perp$. Next, for any Blaschke product $B$ with zeros $\{a_n\}$, let $B_1 = 1$ and $B_n$ be the subproduct of $B$ with zeros $a_1, \ldots, a_{n-1}$, $n = 2, 3, \ldots$. If

$$g_n(z) = B_n(z)(1 - |a_n|^2)^{1/2}/(1 - a_nz),$$

then $\{g_n\}$ is an orthonormal basis for $(BH^2)^\perp$. If the sequence $\{a_n\}$ is uniformly separated, then $\{h_n\}$ is a Riesz basis for $(BH^2)^\perp$, where

$$h_n(z) = (1 - |a_n|)^{1/2}/(1 - a_nz).$$

We will be considering the relationship between the condition $\sum_n (1 - |a_n|)^\beta < \infty$ and the inclusion of $B'$ and the derivative of $(BH^2)^\perp$ functions in various spaces $A^\alpha$. In [10], A. Gluchoff proved that an inner function has its first derivative in $A^\alpha$ if and only if it is a finite Blaschke product. So, we will restrict our attention to $\alpha > -1, p < \alpha + 2$ since we are interested only in infinite Blaschke products.

In any theorem concerned with a function being in $A^\alpha$ for certain points $(\alpha, p)$, we will say that the scope of the theorem is the set of all such points. Also, we write $K_1 \lesssim K_2$ or $K_2 \gtrsim K_1$ if there exists a constant $C > 0$ such that $K_1 \leq CK_2$ for all values of $K_1$ and $K_2$ under consideration, and we write $K_1 \asymp K_2$ if $K_1 \lesssim K_2$ and $K_1 \gtrsim K_2$.

2. The Derivative of a Blaschke Product

In [12], H. O. Kim proved that if $\alpha > -1, \max((\alpha + 2)/2, \alpha + 1) < p < \alpha + 2$, and $B$ is a Blaschke product with zeros $\{a_n\}$ such that

$$(1) \quad \sum_n (1 - |a_n|)^{2 - p + \alpha} < \infty,$$

then $B' \in A^\alpha$. In the other direction, Gluchoff proved in [10, Theorem 6] that if $B$ is a Blaschke product with zeros $\{a_n\}$ that are uniformly separated and if $B' \in A^\alpha$ where $\alpha > -1$, $p \geq 1$, and $\alpha + 1 < p < \alpha + 2$, then condition (1) holds. In [16, Theorem 3], the scope of Gluchoff's result was extended to $p > 0$. On the other hand, in [5, Theorem 2(ii)], A. Aleman and D. Vukotić generalized Gluchoff’s result to uniformly discrete zeros and certain normal weights, and in [15, Theorem 1], F. Pérez-González, A. Reijonen, and J. Rättyä further generalized this to doubling weights. Both of these last two generalizations are stated for $p > 1/2$.

However, the proofs of both can be seen to show that condition (1) holds even with the hypothesis $p > 1/2$ dropped. In the following theorem, we state the result for the classical weights $(1 - r)^\alpha$ that holds for uniformly discrete zeros and scope extended to $p > 0$, and we supply a straightforward proof.
Theorem 1. Let $B$ be an infinite Blaschke product with zeros $\{a_n\}$ that are uniformly discrete. If $B' \in A_p^\alpha$ where $\alpha > -1$ and $\alpha + 1 < p < \alpha + 2$, then condition (1) holds.

Proof. We have $\rho(a_i, a_j) \geq \delta > 0$ for all $j \neq i$. For each $n = 1, 2, \ldots$, put

$$\Delta_n = \{z : \rho(z, a_n) < R\},$$

where $R = \frac{4}{\delta}$. It is known (see [8]) that $\Delta_n$ is a Euclidean disk with radius equal to $R(1 - |a_n|^2)/(1 - R^2|a_n|^2)$, and that there exists a constant $C$ (independent of $n$) such that $1 - |z| \leq C(1 - |a_n|)$ for all $z \in \Delta_n$. Also note that for each $n$, $z \in \Delta_n \Rightarrow |B(z)| \leq \frac{|z - a_n|}{1 - \bar{a}_n z} < R$. Thus,

$$\begin{align*}
\int\int_{\Delta_n} \left(\frac{1 - |B(z)|}{1 - |z|}\right)^p (1 - |z|)^\alpha dA(z) & \geq (1 - R)^p \int\int_{\Delta_n} (1 - |z|)^{\alpha - p} dA(z) \\
& \geq (1 - R)^p C^{\alpha - p}(1 - |a_n|)^{\alpha - p} \left(\frac{R(1 - |a_n|^2)}{1 - R^2|a_n|^2}\right)^2 \\
& \times (1 - |a_n|)^{2 - p + \alpha}.
\end{align*}$$

Then since the disks $\Delta_n$ are pairwise disjoint,

$$\sum_n (1 - |a_n|)^{2 - p + \alpha} \lesssim \int\int_U \left(\frac{1 - |B(z)|}{1 - |z|}\right)^p (1 - |z|)^\alpha dA(z).$$

The result follows since, as proved by P. Ahern in [2], $B' \in A_p^\alpha$ if and only if

$$\int\int_U \left(\frac{1 - |B(z)|}{1 - |z|}\right)^p (1 - |z|)^\alpha dA(z) < \infty$$

when $\alpha > -1$, $p > \alpha + 1$. □

We now investigate what can be deduced when the hypothesis that the zeros of $B$ be uniformly discrete is dropped. In [4, Theorem 6], P. Ahern and D. Clark proved that if $B$ is any Blaschke product with zeros $\{a_n\}$ such that $B' \in A_p^\alpha$, for $-1 < \alpha < -\frac{1}{2}$, then $\sum_n (1 - |a_n|)^{\beta} < \infty$ for all $\beta > \frac{2 + \alpha}{p + \alpha}$. This was generalized in [17] to $\sum_n (1 - |a_n|)^{\beta} < \infty$ for all $\beta > \frac{2 + \alpha}{p - \alpha}$ if $-1 < \alpha < -\frac{1}{2}$, $\frac{1}{2} < \alpha < p \\leq 1$, and $B' \in A_p^\alpha$. We now prove a further generalization that increases the scope and is valid for $\beta = \frac{2 + p + \alpha}{p + \alpha - 1}$.

Theorem 2. Let $B$ be an infinite Blaschke product with zeros $\{a_n\}$. If $B' \in A_p^\alpha$ where $\alpha > -1$ and $\frac{1}{2} < \alpha < p < 2 + \alpha$, then

$$\sum_n (1 - |a_n|)^{\frac{2 + p + \alpha}{p + \alpha - 1}} < \infty.$$

Proof. First note that $0 < \frac{2 + p + \alpha}{p + \alpha - 1} < 1$ since $\alpha > -1$ and $\frac{1}{2} + \alpha < p < 2 + \alpha$. Suppose $B' \in A_p^\alpha$. By Theorem 5.1 of [1], $B' \in A_{2p + 1}^\alpha$ since $1 + \alpha < p < 2 + \alpha$, and then by Theorem 6.2 of [1], $B' \in H^{p - 1 - \alpha}$ since $\frac{1}{2} < p - 1 - \alpha < 1$. Theorem 8 of [3] then says that $\sum_n (1 - |a_n|)^{\frac{2 + p + \alpha}{p + \alpha - 1}} < \infty$, again since $\frac{1}{2} < p - 1 - \alpha < 1$. □
3. The Derivative of Model Space Functions

For a given infinite Blaschke product $B$, we will be investigating conditions that imply that $f' \in A_\alpha^p$ for all $f \in (BH^2)^\perp$. (In Theorem 4 of [6], W. Cohn proved a result of this sort, but for $f'$ being in a Hardy space.) We will start, however, with a condition on $(\alpha, p)$ that ensures that $f' \in A_\alpha^p$ for all $f \in H^2$.

**Theorem 3.** Let $f$ be any function in $H^2$. If $\alpha > 1$ and $0 < p < \frac{4}{\alpha} + \frac{2}{\alpha}^2$, then $f' \in A_\alpha^p$.

**Proof.** Let $f \in H^2$. Assume for now that $p > 2$. By two theorems of Hardy and Littlewood, $M_p(r; f) \lesssim 1/(1-r)^{\frac{3}{2} + \frac{1}{3}}$ (see [7, Theorem 5.9]) and then $M_p(r; f') \lesssim 1/(1-r)^{\frac{3}{2} + \frac{1}{3}}$ (see [7, Theorem 5.5]). So,

$$\|f'\|^p_{A_\alpha^p} \lesssim \int_0^1 (1-r)^{-\frac{3}{2} + 1 + \alpha} dr < \infty$$

for all $(\alpha, p)$ with $\alpha > 1$ and $2 < p < \frac{4}{\alpha} + \frac{2}{\alpha}^2$. Then, $\|f'\|^p_{A_\alpha^p} < \infty$ for all $(\alpha, p)$ with $\alpha > 1$ and $0 < p < \frac{4}{\alpha} + \frac{2}{\alpha}^2$ since the $A_\alpha^p$ spaces expand as $p$ decreases. \tag{\textsuperscript{□}}

We note that J. Littlewood and R. Paley proved the last result and its converse for $(\alpha, p) = (1, 2)$ in [13]. The next two theorems enable us to extend the region of points $(\alpha, p)$ where $f \in (BH^2)^\perp \Rightarrow f' \in A_\alpha^p$ for every Blaschke product $B$, beyond the scope of Theorem 3.

**Theorem 4.** Let $B$ be any Blaschke product. If $\alpha > -1$ and $0 < p < \frac{4}{\alpha} + \frac{2}{\alpha}^2$, then $f' \in A_\alpha^p$ for all $f \in (BH^2)^\perp$.

**Proof.** Let $f \in (BH^2)^\perp$. Then,

$$|f(z)| = |(f, K_z)| \leq \|f\|_{H^2} \|K_z\|_{H^2} = \|f\|_{H^2} \left(\frac{1 - |B(z)|^2}{1 - |z|^2}\right)^{\frac{1}{2}},$$

and so $|f(z)| \lesssim 1/(1 - |z|)^{\frac{4}{3}}$. Then $M_p(r; f) \lesssim 1/(1-r)^{\frac{4}{3}}$, which implies by [7, Theorem 5.9], that $M_p(r; f') \lesssim 1/(1-r)^{\frac{4}{3}}$. Therefore,

$$\|f'\|^p_{A_\alpha^p} \lesssim \int_0^1 (1-r)^{-\frac{4}{3} + \alpha} dr < \infty,$$

since $0 < p < \frac{4}{\alpha} + \frac{2}{\alpha}^2$. \tag{\textsuperscript{□}}

We now present a well known proposition (see [14, Lemma 4.3], for example), which will be used a number of times in what follows.

**Lemma.** Let $a \in U$, $\alpha > -1$, and $p > 0$. Then

$$\int_U \frac{(1 - |z|)^\alpha}{|1 - \bar{a}z|^{2p}} dA(z) \asymp \begin{cases} 
1 & \text{if } 0 < 2p < \alpha + 2
\log \frac{1}{1 - |a|} & \text{if } 2p = \alpha + 2
\frac{1}{|1 - |a||^{2p - \alpha + 2}} & \text{if } 2p > \alpha + 2
\end{cases}$$

**Theorem 5.** Let $B$ be any Blaschke product. If $\alpha > 0$ and $0 < p < 1 + \frac{1}{2}\alpha$, then $f' \in A_\alpha^p$ for all $f \in (BH^2)^\perp$. 
Proof. Let $f \in (BH^2)^\perp$. Assume for now that $p \geq 1$. \{gn\} is an orthonormal basis for $(BH^2)^\perp$, where $g_n(z) = B_n(z)(1 - |a_n|^2)/(1 - \overline{a}_n z)$ and \{an\} is the sequence of zeros of $B$. So,

$$f(z) = \sum_n c_n B_n(z) \frac{(1 - |a_n|^2)^{1/2}}{1 - \overline{a}_n z},$$

where $\sum_n |c_n|^2 = \|f\|_{BH^2}^2 < \infty$. Then,

$$f'(z) = \sum_n c_n B_n(z) \overline{a}_n \frac{(1 - |a_n|^2)^{1/2}}{(1 - \overline{a}_n z)^2} + \sum_n c_n B'_n(z) \frac{(1 - |a_n|^2)^{1/2}}{1 - \overline{a}_n z}.$$  \hfill (2)

Because we are assuming that $p \geq 1$, we can apply Minkowski’s inequality to (2). We get

$$\|f'|\|_{A^p_B} \lesssim \sum_n |c_n|(1 - |a_n|^2)^{1/2} \left\{ \int_I \frac{(1 - |z|)^\alpha}{(1 - |\overline{a}_n z|)^{2p}} dA(z) \right\}^{1/2} + \sum_n |c_n|(1 - |a_n|^2)^{1/2} \left\{ \int_I \frac{(1 - |z|)^{\alpha - p}}{|1 - \overline{a}_n z|^{p}} dA(z) \right\}^{1/2}$$  \hfill (3)

since $|B_n(z)| \leq 1$ and $|B'_n(z)| \leq 1/(1 - |z|)$ for all $n$. Let $I_1(a_n)$ be the first integral and $I_2(a_n)$ be the second integral in formula (3). By the Lemma, $I_1(a_n) \approx 1$ since $\alpha > 0 > -1$ and $2p < \alpha + 2$, while $I_2(a_n) \approx 1$ since $\alpha - p > \frac{\alpha}{2} - p > -1$ and $p < \alpha - p + 2$. Thus,

$$\|f\|_{A^p_B} \lesssim \sum_n |c_n|(1 - |a_n|^2)^{1/2} \left( \sum_n |c_n|^2 \right)^{1/2} \left( \sum_n (1 - |a_n|^2) \right)^{1/2} < \infty$$

by Hölder’s inequality. This gives the result for $p \geq 1$. The result for $p > 0$ then follows immediately. \hfill \Box

Theorems 3, 4 and 5 give us values of $(\alpha, p)$ at which the derivative of every model space function for every infinite Blaschke product is in $A^p_B$. Notice that the scopes of these three theorems overlap. When $-1 < \alpha \leq 0$, only Theorem 4 applies. When $0 < \alpha \leq 1$, Theorem 5 and Theorem 4 both apply with Theorem 5 being the stronger of the two. When $\alpha > 1$ all three apply, with Theorem 3 being the strongest.

We now look for conditions on zero sequences \{an\} that will imply that the derivative of every model space function corresponding to certain Blaschke products, but not necessarily all Blaschke products, is in $A^p_B$ for additional values of $(\alpha, p)$.

**Theorem 6.** Let $B$ be an infinite Blaschke product with zeros \{an\}. If $-1 < \alpha \leq 0$ and $\frac{\alpha}{2} + \frac{2}{1 + \alpha} \leq p < 1 + \alpha$, and if

$$\sum_n (1 - |a_n|)^{\frac{1}{1 + \alpha}} < \infty,$$

then $f' \in A^p_B$ for all $f \in (BH^2)^\perp$.

**Proof.** By Theorem 4 the conclusion holds for all Blaschke products $B$ when $0 < p < \frac{2}{3} + \frac{2}{3} \alpha$, which is why we restrict our attention to $p \geq \frac{2}{3} + \frac{2}{3} \alpha$. Let $f \in (BH^2)^\perp$. 

Proof.
As in the proof of Theorem 5, equation (2) holds. Note that now $0 < p < 1$. Thus, 
\[ |x + y|^p \leq (|x| + |y|)^p \leq |x|^p + |y|^p \] for any $x$ and $y$, and so,
\[ \|f''\|_{A^p}^p \leq \sum_n |c_n|^p(1 - |a_n|^2)^{p/2} I_1(a_n) + \sum_n |c_n|^p(1 - |a_n|^2)^{p/2} I_2(a_n), \]
where $I_1(a_n)$ and $I_2(a_n)$ are as defined in the proof of Theorem 5. By the Lemma, $I_1(a_n) \approx 1$ since $\alpha > -1$ and $2p < \alpha + 2$, while $I_2(a_n) \approx 1$ since $\alpha - p > -1$ and $p < \alpha - p + 2$. Thus,
\[ \|f''\|_{A^p}^p \leq \sum_n |c_n|^p(1 - |a_n|^2)^{p/2} \leq \left( \sum_n |c_n|^2 \right)^{\frac{p}{2}} \left( \sum_n (1 - |a_n|^2)^{-\frac{p}{2}} \right) \frac{2}{2 - \alpha} < \infty \]
by Hölder’s inequality since \( \left( \frac{2}{p} \right) - 1 + \left( \frac{2}{2 - \alpha} \right) - 1 = 1 \).

**Theorem 7.** Let $B$ be an infinite Blaschke product with zeros \( \{a_n\} \). If $0 < \alpha \leq 1$ and $1 + \frac{1}{2} \alpha < p < 1 + \alpha$ and if
\[ \sum_n (1 - |a_n|) \frac{4 - 3p + 2\alpha}{p} < \infty, \]
then $f' \in A^p_f$ for all $f \in (BH^2)^\perp$.

**Proof.** By Theorem 5 the conclusion holds for all Blaschke products $B$ when $\alpha > 0$ and $0 < p < 1 + \frac{1}{2} \alpha$, which is why we restrict our attention to $p \geq 1 + \frac{1}{2} \alpha$. Let $f \in (BH^2)^\perp$. Since $p > 1$, we can proceed as we did in the proof of Theorem 5. We get
\[ \|f''\|_{A^p}^p \leq \sum_n |c_n|(1 - |a_n|^2)^{1/2} I_1(a_n)^{1/p} + \sum_n |c_n|(1 - |a_n|^2)^{1/2} I_2(a_n)^{1/p}. \]
But $2p > \alpha + 2$, $\alpha > -1$, and $\alpha > -1+p$. So the Lemma says $I_1(a_n) \propto \frac{1}{1 - |a_n|^{2p - \alpha}}$ and $I_2(a_n) \propto \frac{1}{(1 - |a_n|^{2p - \alpha})^2}$. Thus,
\[ \|f''\|_{A^p}^p \leq \sum_n |c_n|(1 - |a_n|^2)^{\frac{4 - 3p + 2\alpha}{2}} \leq \left( \sum_n |c_n|^2 \right)^{\frac{1}{2}} \left( \sum_n (1 - |a_n|^2)^{\frac{4 - 3p + 2\alpha}{2}} \right) \frac{2}{2 - \alpha} < \infty \]
by Hölder’s inequality.

Our next step is to see what additional information can be deduced when the sequence of zeros \( \{a_n\} \) is assumed to be uniformly separated (or the union of finitely many uniformly separated sequences). In Theorem 4 of [5], A. Aleman and D. Vukotić proved a result for normal weights which in the case of standard weights says in part that if $\alpha > -1$, $p > 1$, and $1 + \alpha < p < \frac{4}{3} + \frac{\alpha}{2}$, then
\[ \sum_n (1 - |a_n|) \frac{4 - 3p + 2\alpha}{p} < \infty \]
if and only if $f' \in A^p_f$ for all $f \in (BH^2)^\perp$. The next theorem shows that the scope of Theorem 6 (with Theorem 4) can be increased to $-1 < \alpha \leq 0$, $0 < p < 1 + \frac{1}{2} \alpha$ when the zeros of $B$ are assumed to be uniformly separated.
Theorem 8. Let $B$ be an infinite Blaschke product with uniformly separated zeros \( \{a_n\} \). If \(-1 < \alpha \leq 0 \) and \( 1 + \alpha \leq p < 1 + \frac{1}{2} \alpha \) and if
\[
\sum_n (1 - |a_n|)^{\frac{1}{2p}} < \infty,
\]
then \( f' \in A^p_\alpha \) for all \( f \in (BH^2)^\perp \).

Proof. Let \( f \in (BH^2)^\perp \). Since \( \{a_n\} \) is uniformly separated, \( f \) can be expressed as
\[
f(z) = \sum_n c_n (1 - |a_n|)^{\frac{1}{2}}/(1 - \overline{a_n} z) \quad \text{where} \quad \|f\|_{2p}^p \approx \sum_n |c_n|^2.
\]
So, \( |f'(z)| = \left| \sum_n \overline{c_n} (1 - |a_n|)^{\frac{1}{2}}/(1 - \overline{a_n} z)^2 \right| \leq \sum_n |c_n| (1 - |a_n|)^{\frac{1}{2}}/(1 - \overline{a_n} z)^2 \).

Now as in the proof of Theorem 6, we use \( p \) being less than 1, the Lemma with \( 2p < \alpha + 2 \), and Hölder’s inequality with \( (\frac{2}{p})^{-1} + (\frac{2}{2p})^{-1} = 1 \). We get
\[
\|f''\|_{A^p_\alpha}^p \leq \iint_U \left( \sum_n |c_n| (1 - |a_n|)^{\frac{1}{2}}/(1 - \overline{a_n} z)^2 \right)^p (1 - |z|)^\alpha dA(z)
\leq \sum_n |c_n|^p (1 - |a_n|)^{\frac{1}{2}} \iint_U (1 - |z|)^\alpha/(1 - \overline{a_n} z)^{2p} dA(z)
\leq \left( \sum_n |c_n|^2 \right)^{\frac{p}{2}} \left( \sum_n (1 - |a_n|)^{\frac{1}{2}} \right)^{\frac{2}{2p}} < \infty.
\]
\( \square \)

Theorem 9. Let \( B \) be an infinite Blaschke product with uniformly separated zeros \( \{a_n\} \). If \(-1 < \alpha \leq 0 \) and \( 1 + \frac{3}{4} \alpha < p < \min(1, \frac{3}{2} + \frac{1}{2} \alpha) \), and if
\[
\sum_n (1 - |a_n|)^{\frac{4 - 2\alpha + 2\alpha}{2p}} < \infty,
\]
then \( f' \in A^p_\alpha \) for all \( f \in (BH^2)^\perp \).

Proof. Let \( f \in (BH^2)^\perp \). Since \( \{a_n\} \) is uniformly separated and \( p < 1 \) as in Theorem 8,
\[
|f''(z)| \leq \sum_n |c_n| (1 - |a_n|)^{\frac{1}{2}}/(1 - \overline{a_n} z)^2
\]
and then
\[
\|f''\|_{A^p_\alpha}^p \leq \iint_U \left( \sum_n |c_n| (1 - |a_n|)^{\frac{1}{2}}/(1 - \overline{a_n} z)^2 \right)^p (1 - |z|)^\alpha dA(z)
\leq \sum_n |c_n|^p (1 - |a_n|)^{\frac{1}{2}} \iint_U (1 - |z|)^\alpha/(1 - \overline{a_n} z)^{2p} dA(z).
\]
But now since \( 2p > \alpha + 2 \),
\[
\iint_U (1 - |z|)^\alpha/(1 - \overline{a_n} z)^{2p} dA(z) \approx \frac{1}{(1 - |a_n|)^{2p - \alpha - 2}}.
\]
Thus,
\[ \|f'\|_{A^p_\alpha}^p \leq \sum_n |c_n|^p (1 - |a_n|)^{1 - \frac{p+2\alpha}{2}} \]
\[ < \left( \sum_n |c_n|^2 \right)^{\frac{p}{2}} \left( \sum_n (1 - |a_n|)^{1 - \frac{p+2\alpha}{2}} \right)^{\frac{2-p}{2}} < \infty. \]

by Hölder’s inequality since \( \left( \frac{2}{p} \right)^{-1} + \left( \frac{2}{2-p} \right)^{-1} = 1. \)

\[ \square \]

Figure 1 summarizes what the results in this section say about at which points \((\alpha, p), f' \in A^p_\alpha\) for all \(f \in (BH^2)^{+}\). Region A, being the union of the scopes of Theorems 3, 4, and 5, gives us the points \((\alpha, p)\) at which we have shown that the desired conclusion holds for every infinite Blaschke product \(B\). Regions B and C, corresponding to the scopes of Theorems 6 and 7, involve points at which the result holds for certain Blaschke products. Regions D, E and F, corresponding to the scopes of the theorem of Aleman and Vukotić, Theorem 8 and Theorem 9, are restricted to interpolating Blaschke products. What happens in the region between \(p = \frac{4}{3} + \frac{2}{3} \alpha\) and \(p = 2 + \alpha\) is an open question.

Figure 1. Where \(f' \in A^p_\alpha\) for all \(f \in (BH^2)^{+}\)

We finish with one more way of restricting the zeros \(\{a_n\}\) of a Blaschke product. For any \(\xi \in \partial U\) and any \(\eta > 1\), \(\Omega_\eta(\xi) = \{z \in U : |1 - \bar{\xi}z| \leq \eta (1 - |z|)\}\) is a
Stolz domain or Stolz angle. In Theorem 1(c) of [18] (when applied to the weights 
(1 − r)^α), A. Reijonen, proved that if B is a Blaschke product with zeros \( \{a_n\} \) in a 
Stolz domain, and if \( \alpha > -1 \) and \( \frac{1}{2} < p < \frac{3}{2} + \alpha \), then \( B' \in A_p^\alpha \). We prove a similar 
result for the derivative of model space functions in \( A_p^\alpha \).

**Theorem 10.** Let \( B \) be a Blaschke product with zeros \( \{a_n\} \) in a Stolz domain. If 
\( \alpha > -1 \) and \( 0 < p < 1 + \frac{2}{3} \alpha \), then \( f' \in A_p^\alpha \) for all \( f \in (BH^2)^\perp \).

**Proof.** Let \( f \in (BH^2)^\perp \). As in the proof of Theorem 4, \( |f(z)| \leq \left( \frac{1-|B(z)|^2}{1-|z|} \right)^{\frac{3}{2}} \) for 
all \( |z| < 1 \), and so

\[
M_p(r; f) \lesssim \left\{ \int_0^{2\pi} \left( \frac{1-|B(re^{i\theta})|^2}{1-r^2} \right)^{\frac{3}{2}} \, d\theta \right\}^{\frac{1}{3}}.
\]

But, Reijonen proved in Proposition 3.2(ii) of [19] that \( f_{2\pi} (\frac{1-|B(re^{i\theta})|^2}{1-r^2})^{\frac{3}{2}} \, d\theta \lesssim 
(1-r)^{\frac{1}{3}-\frac{3}{2}p} \) for \( \{a_n\} \) being contained in a Stolz domain and \( p > 1 \). Then \( M_p(r; f) \lesssim (1-r)^{\frac{1}{3}-\frac{3}{2}p} \), and so \( M_p(r; f') \lesssim (1-r)^{\frac{1}{3}-\frac{3}{2}p} \) by [7, Theorem 5.5]. Therefore, 
\( |f'|_{A_p^\alpha}^p \lesssim \int_0^1 (1-r)^{\frac{1}{3}-\frac{3}{2}p+\alpha} \, dr < \infty \) since \( p < 1 + \frac{2}{3} \alpha \). \( \square \)

**References**


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