



Almost Gorenstein rings arising from fiber products

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Abstract. The purpose of this paper is, as part of the stratification of Cohen–Macaulay rings, to investigate the question of when the fiber products are almost Gorenstein rings. We show that the fiber product $R \times_T S$ of Cohen–Macaulay local rings R, S of the same dimension $d > 0$ over a regular local ring T with $\dim T = d - 1$ is an almost Gorenstein ring if and only if so are R and S . In addition, the other generalizations of Gorenstein properties are also explored.

1 Introduction

The fiber product of homomorphisms $R \xrightarrow{f} T \xleftarrow{g} S$ of rings is defined by

$$R \times_T S = \{(a, b) \in R \times S \mid f(a) = g(b)\},$$

which forms a subring of $R \times S$. The notion of fiber products, in general, appears not only in commutative algebra but in diverse branches of mathematics, and it often plays a crucial role. Much research has been conducted regarding a relation of ring theoretic properties (e.g., Cohen–Macaulay, Gorenstein) between the fiber products and the base rings. The reader may consult [1, 5, 7, 30, 31, 32, 33, 34] for details. By [33], the fiber product of one-dimensional Cohen–Macaulay local rings over the residue class field is Gorenstein if and only if the base rings are discrete valuation rings (DVRs), and hence the fiber products are rarely Gorenstein rings. Nevertheless, for example, the fiber product of hypersurfaces might have good properties even though it is not Gorenstein, which we will clarify in this paper.

An almost Gorenstein ring is one of the generalizations of a Gorenstein ring, defined by a certain embedding of the rings into their canonical modules. The motivation of the theory derives from the strong desire to stratify Cohen–Macaulay rings, finding new and interesting classes that naturally cover the Gorenstein rings. The class of almost Gorenstein rings could be a very nice candidate for such classes. Originally, the theory of almost Gorenstein rings was established by V. Barucci and R. Fröberg [2] in the case where the local rings are analytically unramified of dimension one. They mainly considered the numerical semigroup rings and gave

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pioneering results. However, the notion given by [2] was not flexible for the analysis of analytically ramified case, so in 2013, the second author, N. Matsuoka, and T. T. Phuong [12] proposed the notion of over one-dimensional Cohen-Macaulay local rings, using the first Hilbert coefficient of canonical ideals. Finally, in 2015, the first and second authors, and R. Takahashi [19] gave the definition of almost Gorenstein graded/local rings of arbitrary dimension, by means of Ulrich modules.

The purpose of this paper is to explore the question of when the fiber product is an almost Gorenstein ring. The main result of this paper is stated as follows.

Theorem 1.1 *Let $(R, \mathfrak{m})$, $(S, \mathfrak{n})$ be Cohen-Macaulay local rings with $d = \dim R = \dim S > 0$, and let T be a regular local ring with $\dim T = d - 1$ possessing an infinite residue class field. Let $f : R \rightarrow T$, $g : S \rightarrow T$ be surjective homomorphisms. Suppose that $A = R \times_T S$ has a canonical module K_A and that $Q(A)$ is a Gorenstein ring. Then the following conditions are equivalent.*

- (i) *The fiber product $A = R \times_T S$ is an almost Gorenstein ring.*
- (ii) *R and S are almost Gorenstein rings.*

In what follows, unless otherwise specified, let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$. Let $Q(R)$ be the total quotient ring of R and $\bar{R}$ the integral closure of R in $Q(R)$. For each finitely generated R -module M , let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators of M (resp. the length of M). For R -submodules X, Y of $Q(R)$, let $X : Y = \{a \in Q(R) \mid aY \subseteq X\}$. We denote by K_R a canonical module of R if one exists. We set $r(R) = \ell_R(\text{Ext}_R^d(R/\mathfrak{m}, R))$ the Cohen-Macaulay type of R , where $d = \dim R$.

2 Basic Facts

The purpose of this section is to summarize some basic properties of fiber products. For a moment, let R, S, T be arbitrary commutative rings, and let $f : R \rightarrow T$, $g : S \rightarrow T$ denote homomorphisms of rings. We set

$$A = R \times_T S = \{(a, b) \in R \times S \mid f(a) = g(b)\}$$

and call it the fiber product of R and S over T with respect to f and g , which forms a subring of $B = R \times S$. We then have a commutative diagram

$$\begin{array}{ccc} & R & \\ p_1 \nearrow & & \searrow f \\ A & & T \\ p_2 \searrow & & \nearrow g \\ & S & \end{array}$$

of rings, where $p_1 : A \rightarrow R$, $(x, y) \mapsto x$, $p_2 : A \rightarrow S$, $(x, y) \mapsto y$ stand for the projections. Hence, we get an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\varphi} T$$

of A -modules, where $\varphi = \begin{bmatrix} f \\ -g \end{bmatrix}$. The map φ is surjective if either f or g is surjective. In addition, if both f and g are surjective, then T is cyclic as an A -module, and B is a module-finite extension over A . This partially justifies the following (see e.g., [1, Lemmas 1.2 and 1.5]).

Lemma 2.1 *Suppose that $f : R \rightarrow T$ and $g : S \rightarrow T$ are surjective. Then the following assertions hold true.*

- (i) A is a Noetherian ring if and only if R and S are Noetherian rings.
- (ii) (A, J) is a local ring if and only if $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are local rings. When this is the case, $J = (\mathfrak{m} \times \mathfrak{n}) \cap A$.
- (iii) If $(R, \mathfrak{m}), (S, \mathfrak{n})$ are Cohen–Macaulay local rings with $\dim R = \dim S = d > 0$ and $\text{depth } T \geq d - 1$, then (A, J) is a Cohen–Macaulay local ring and $\dim A = d$.

In the case of local rings, we have the following. Although Proposition 2.2(iii) follows from the result of Ogoma ([33]), let us give a brief proof for the sake of completeness. For a Noetherian local ring R , we denote by $e(R)$ (resp. $v(R)$) the multiplicity (resp. the embedding dimension) of R .

Proposition 2.2 *Suppose that $(R, \mathfrak{m}), (S, \mathfrak{n})$ are Noetherian local rings with a common residue class field $k = R/\mathfrak{m} = S/\mathfrak{n}$, and $f : R \rightarrow k, g : S \rightarrow k$ denote the canonical surjective maps. Then the following assertions hold true.*

- (i) $v(A) = v(R) + v(S)$.
- (ii) If $\dim R = \dim S > 0$, then $e(A) = e(R) + e(S)$.
- (iii) If R, S are Cohen–Macaulay local rings with $\dim R = \dim S = 1$, then $A = R \times_k S$ is Gorenstein if and only if R and S are DVRs.

Proof Since $J = \mathfrak{m} \times \mathfrak{n}$ is the maximal ideal of A , we get $J^\ell = \mathfrak{m}^{\ell+1} \times \mathfrak{n}^{\ell+1}$ for every $\ell \geq 0$.

- (i) This part follows from the equalities $\ell_A(J/J^2) = \ell_k([\mathfrak{m}/\mathfrak{m}^2] \oplus [\mathfrak{n}/\mathfrak{n}^2]) = \ell_R(\mathfrak{m}/\mathfrak{m}^2) + \ell_S(\mathfrak{n}/\mathfrak{n}^2)$.
- (ii) Remember that f and g are surjective. We then have the equalities

$$\begin{aligned} \ell_A(A/J^{\ell+1}) &= \ell_A(A/J) + \ell_A(J/J^{\ell+1}) \\ &= 1 + [\ell_R(\mathfrak{m}/\mathfrak{m}^{\ell+1}) + \ell_S(\mathfrak{n}/\mathfrak{n}^{\ell+1})] \\ &= 1 + \{[\ell_R(R/\mathfrak{m}^{\ell+1}) - 1] + [\ell_S(S/\mathfrak{n}^{\ell+1}) - 1]\} \\ &= [\ell_R(R/\mathfrak{m}^{\ell+1}) + \ell_S(S/\mathfrak{n}^{\ell+1})] - 1 \end{aligned}$$

for every $\ell \geq 0$, which yield that $e(A) = e(R) + e(S)$, because $\dim A = \dim R = \dim S > 0$.

- (iii) Suppose that A is a Gorenstein ring. By applying the functor $\text{Hom}_A(-, A)$ to the exact sequence $0 \rightarrow A \xrightarrow{i} B \xrightarrow{\varphi} k \rightarrow 0$ of A -modules (here, $\varphi = \begin{bmatrix} f \\ -g \end{bmatrix}$),

we get

$$0 \longrightarrow A : B \xrightarrow{i} A \longrightarrow \operatorname{Ext}_A^1(A/J, A) \longrightarrow 0,$$

which implies $J = A : B$, because $\operatorname{Ext}_A^1(A/J, A) \cong A/J$. As A is Gorenstein, we have $B = A : (A : B)$, so the equalities

$$R \times S = B = A : (A : B) = A : J = J : J = (\mathfrak{m} : \mathfrak{m}) \times (\mathfrak{n} : \mathfrak{n})$$

hold where the fourth equality comes from the fact that A is not a DVR (note that B is a module-finite birational extension of A). Consequently, $R = \mathfrak{m} : \mathfrak{m}$ and $S = \mathfrak{n} : \mathfrak{n}$, whence R and S are DVRs (see e.g., [12, Lemma 3.15]). The converse implication follows from assertion (ii). This completes the proof. ■

3 Survey on Almost Gorenstein Rings

This section is devoted to the definition and some basic properties of almost Gorenstein rings, which we will use throughout this paper. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with $d = \dim R$, possessing a canonical module K_R .

Definition 3.1 ([19, Definition 1.1]) We say that R is an *almost Gorenstein local ring*, if there exists an exact sequence

$$0 \longrightarrow R \longrightarrow K_R \longrightarrow C \longrightarrow 0$$

of R -modules such that $\mu_R(C) = e_{\mathfrak{m}}^0(C)$, where $\mu_R(C)$ denotes the number of elements in a minimal system of generators for C and

$$e_{\mathfrak{m}}^0(C) = \lim_{n \rightarrow \infty} \frac{\ell_R(C/\mathfrak{m}^{n+1}C)}{n^{d-1}} \cdot (d-1)!$$

is the multiplicity of C with respect to $\mathfrak{m}$.

Notice that every Gorenstein ring is an almost Gorenstein ring, and the converse holds if the ring R is Artinian ([19, Lemma 3.1 (3)]). Definition 3.1 requires that if R is an almost Gorenstein ring, then R might not be Gorenstein, but the ring R can be embedded into its canonical module K_R so that the difference K_R/R should have good properties. For any exact sequence

$$0 \longrightarrow R \longrightarrow K_R \longrightarrow C \longrightarrow 0$$

of R -modules, C is a Cohen–Macaulay R -module with $\dim_R C = d - 1$, provided $C \neq (0)$ ([19, Lemma 3.1 (2)]). Suppose that R possesses an infinite residue class field $R/\mathfrak{m}$. Set $R_1 = R/[(0) :_R C]$ and let $\mathfrak{m}_1$ denote the maximal ideal of R_1 . Choose elements $f_1, f_2, \dots, f_{d-1} \in \mathfrak{m}$ such that $(f_1, f_2, \dots, f_{d-1})R_1$ forms a minimal reduction of $\mathfrak{m}_1$. We then have

$$e_{\mathfrak{m}}^0(C) = e_{\mathfrak{m}_1}^0(C) = \ell_R(C/(f_1, f_2, \dots, f_{d-1})C) \geq \ell_R(C/\mathfrak{m}C) = \mu_R(C).$$

Therefore, $e_m^0(C) \geq \mu_R(C)$, and we say that C is an Ulrich R -module if $e_m^0(C) = \mu_R(C)$, since C is a maximally generated maximal Cohen–Macaulay R_1 -module in the sense of [3]. Hence, C is an Ulrich R -module if and only if $mC = (f_1, f_2, \dots, f_{d-1})C$. If $d = 1$, then the Ulrich property for C is equivalent to saying that C is a vector space over R/m . In addition, we have the following fact.

Fact 3.2 ([12, Theorem 3.11], [19, Proposition 3.4], [24, Theorem 2.14]) *Suppose that $d = 1$ and there exists an R -submodule K of the total ring $Q(R)$ of fractions such that $R \subseteq K \subseteq \bar{R}$ and $K \cong K_R$ as an R -module, where $\bar{R}$ stands for the integral closure of R in $Q(R)$. Then the following conditions are equivalent.*

- (i) R is an almost Gorenstein ring.
- (ii) $mK \subseteq R$, that is, $mK = m$.
- (iii) $mK \cong m$ as an R -module.

In this paper, we say that an R -submodule K of $Q(R)$ is a fractional canonical ideal of R , if $R \subseteq K \subseteq \bar{R}$ and $K \cong K_R$ as an R -module.

One can construct many examples of almost Gorenstein rings (e.g., [4, 8, 10, 12, 13, 14, 15, 16, 18, 19, 20, 23, 28, 29, 35]). The significant examples of almost Gorenstein rings are one-dimensional Cohen–Macaulay local rings of finite Cohen–Macaulay representation type and two-dimensional rational singularities. Because the origin of the theory of almost Gorenstein rings are the theory of numerical semigroup rings, there are numerous examples of almost Gorenstein numerical semigroup rings (see [2, 12]).

4 Proof of Theorem 1.1

This section mainly focuses on proving Theorem 1.1. Theorem 1.1 is reduced, by induction on $\dim A$, to the case where $\dim A = 1$. Let us start from the key result of dimension one. First, we fix our notation and assumptions.

Setting 4.1 *Let (R, m) , (S, n) be one-dimensional Cohen–Macaulay local rings with a common residue class field $k = R/m = S/n$, and let $f : R \rightarrow k$, $g : S \rightarrow k$ be the canonical surjective maps. Then $A = R \times_k S$ is a one-dimensional Cohen–Macaulay local ring with maximal ideal $J = m \times n$. Since $B = R \times S$ is a module-finite birational extension over A , $Q(A) = Q(R) \times Q(S)$ and $\bar{A} = \bar{R} \times \bar{S}$.*

Throughout this section, unless otherwise specified, we assume that $Q(A)$ is a Gorenstein ring, A admits a canonical module K_A , and the field $k = A/J$ is infinite. Hence, all the rings A , R , and S possess fractional canonical ideals (see [12, Corollary 2.9]).

We denote by K (resp. L) the fractional canonical ideal of R (resp. S). Thus, K is an R -submodule of $Q(R)$ such that $R \subseteq K \subseteq \bar{R}$, $K \cong K_R$ as an R -module, and L is an S -submodule of $Q(S)$ such that $S \subseteq L \subseteq \bar{S}$, $L \cong K_S$ as an S -module.

To prove Theorem 1.1 in the case where $\dim A = 1$, we can assume that A is not a Gorenstein ring. Hence, either R or S is not a DVR (see Proposition 2.2).

4.1 The Case Where R and S are not DVRs

In this subsection, suppose that both R and S are not DVRs. We then have $K \neq \bar{R}$, since $K : K = R$ ([22, Bemerkung 2.5]). As $\ell_R([K : \mathfrak{m}]/K) = \ell_R(\text{Ext}_R^1(R/\mathfrak{m}, K)) = 1$, we obtain $K : \mathfrak{m} \subseteq \bar{R}$. In addition, we also have $R : \mathfrak{m} \not\subseteq K$. Indeed, if $R : \mathfrak{m} \subseteq K$, then $K \supseteq R : \mathfrak{m} = (K : K) : \mathfrak{m} = K : \mathfrak{m}K$. Thus, $R = K : K \subseteq K : (K : \mathfrak{m}K) = \mathfrak{m}K \subseteq \mathfrak{m}\bar{R}$. This makes a contradiction. Hence, $R : \mathfrak{m} \not\subseteq K$, and therefore

$$K : \mathfrak{m} = K + R \cdot g_1$$

for some $g_1 \in (R : \mathfrak{m}) \setminus K$. Similarly, because S is not a DVR, we can choose $g_2 \in (S : \mathfrak{n}) \setminus L$ such that $L : \mathfrak{n} = L + S \cdot g_2$. We set

$$X = (K \times L) + A \cdot \psi$$

with $\psi = (g_1, g_2) \in \bar{A} = \bar{R} \times \bar{S}$. Then X is an A -submodule of $\bar{A}$, satisfying $A \subseteq B \subseteq X \subseteq \bar{A}$. Furthermore, we have the following lemma, which plays a key role in our argument.

Lemma 4.2 $X \cong K_A$ as an A -module. Hence, X is a fractional canonical ideal of A and $\mathfrak{r}(A) = \mathfrak{r}(R) + \mathfrak{r}(S) + 1$.

Proof First, we notice that $X : J = (K : \mathfrak{m}) \times (L : \mathfrak{n})$. In fact, take an element $\alpha \in X : J$ and write $\alpha = (x, y)$, where $x \in Q(R)$, $y \in Q(S)$. Since $J \cdot (x, y) \subseteq X = (K \times L) + A \cdot \psi$, we have

$$(m, 0)(x, y) = (mx, 0) \in X \quad \text{and} \quad (0, n)(x, y) = (0, ny) \in X$$

for each $m \in \mathfrak{m}$ and $n \in \mathfrak{n}$. Hence, $(mx, 0) = (k, \ell) + (a, b)(g_1, g_2)$ for some $(k, \ell) \in K \times L$, $(a, b) \in A$. Therefore,

$$mx = k + ag_1 \quad \text{and} \quad 0 = \ell + bg_2,$$

which yield that $b \in \mathfrak{n}$. Moreover, because $(a, b) \in A$, we get $a \in \mathfrak{m}$. Consequently, we have $ag_1 \in \mathfrak{m}(R : \mathfrak{m}) \subseteq R$, whence $x \in K : \mathfrak{m}$. Similarly, $y \in L : \mathfrak{n}$, as desired.

Note that X is a faithful maximal Cohen–Macaulay A -module with $\dim_A X = 1$. We now choose the regular elements $a \in \mathfrak{m}$ on R and $b \in \mathfrak{n}$ on S . Then $(a, b) \in A$ is a regular element on A . Therefore,

$$(a, b)X :_X J = (a, b)X :_{Q(A)} J = (a, b)[X :_{Q(A)} J]$$

so that $[(a, b)X :_X J]/(a, b)X \cong [X :_{Q(A)} J]/X$ as an A -module. We are now going to prove that $[X :_{Q(A)} J]/X$ is cyclic as an A -module. Indeed, choose an element $\alpha \in X : J$ and write $\alpha = (x, y)$, where $x \in Q(R)$ and $y \in Q(S)$. Then, since $X : J = (K + Rg_1) \times (L + Sg_2)$, we have

$$x = k + cg_1 \quad \text{and} \quad y = \ell + dg_2,$$

where $k \in K$, $\ell \in L$, $c \in R$, and $d \in S$. Notice that the projection $p_1 : A \rightarrow R$, $(a, b) \mapsto a$ is surjective. There exists $c' \in S$ such that $(c, c') \in A$ and $p_1(c, c') = c$. Similarly, let us

choose $d' \in R$ such that $(d', d - c') \in A$ and $p_2(d', d - c') = d - c'$. Hence, we have the equalities

$$\begin{aligned}\alpha &= (x, y) = (k, \ell) + (cg_1, 0) + (0, dg_2) \\ &= (k, \ell) + (c, c')(g_1, g_2) + (0, (d - c')g_2) \\ &= (k, \ell) + (c, c')(g_1, g_2) + (d', d - c')(0, g_2),\end{aligned}$$

whence $\alpha \in (K \times L) + A \cdot \psi + A \cdot (0, g_2) = X + A \cdot (0, g_2)$. Therefore, $[X :_{Q(A)} J] / X$ is a cyclic A -module. Hence, $X \cong K_A$ as an A -module.

Remember that $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$ and $S : \mathfrak{n} = \mathfrak{n} : \mathfrak{n}$, because R and S are not DVRs (see [12, Lemma 3.15]). We then have $\mathfrak{m}(K : \mathfrak{m}) = \mathfrak{m}K$ and $\mathfrak{n}(L : \mathfrak{n}) = \mathfrak{n}L$. Hence,

$$\begin{aligned}JX &= J(K \times L) + J\psi = (\mathfrak{m}K \times \mathfrak{n}L) + (\mathfrak{m}g_1 \times \mathfrak{n}g_2) \\ &= \mathfrak{m}(K : \mathfrak{m}) \times \mathfrak{n}(L : \mathfrak{n}) = \mathfrak{m}K \times \mathfrak{n}L.\end{aligned}$$

Therefore, we have

$$\begin{aligned}r(A) &= \ell_A(X/JX) = \ell_A(X/[K \times L]) + \ell_A([K \times L]/[\mathfrak{m}K \times \mathfrak{n}L]) \\ &= 1 + \ell_R(K/\mathfrak{m}K) + \ell_S(L/\mathfrak{n}L) = 1 + r(R) + r(S),\end{aligned}$$

which completes the proof. $\blacksquare$

As a consequence of Lemma 4.2, we have the following corollary, which ensures that Theorem 1.1 holds when R and S are not DVRs, and $\dim A = 1$.

Corollary 4.3 *Suppose that R and S are not DVRs. Then the fiber product $A = R \times_k S$ is an almost Gorenstein ring if and only if R and S are almost Gorenstein rings.*

Proof Notice that $JX = (\mathfrak{m} \times \mathfrak{n})((K \times L) + A \cdot \psi) = \mathfrak{m}(K + Rg_1) \times \mathfrak{n}(L + Sg_2) = \mathfrak{m}(K : \mathfrak{m}) \times \mathfrak{n}(L : \mathfrak{n}) = \mathfrak{m}K \times \mathfrak{n}L$, where the last equality follows from the fact that R and S are not DVRs. Therefore, $A = R \times_k S$ is an almost Gorenstein ring if and only if $JX = J$ (Fact 3.2). The latter condition is equivalent to saying that $\mathfrak{m}K = \mathfrak{m}$ and $\mathfrak{n}L = \mathfrak{n}$, that is, R and S are almost Gorenstein rings, as desired. $\blacksquare$

Let us note one example.

Example 4.4 Let k be a field and let $a, b, c, d \geq 2$ be integers. We set $R = k[[X, Y]]/(X^a - Y^b)$, $S = k[[Z, W]]/(Z^c - W^d)$ and consider the canonical surjections $f : R \rightarrow k$, $g : S \rightarrow k$. Then

$$R \times_k S \cong k[[X, Y, Z, W]]/(X, Y)(Z, W) + (X^a - Y^b, Z^c - W^d)$$

is an almost Gorenstein local ring with $r(R) = 3$.

For a one-dimensional case, we will show that the conditions stated in Theorem 1.1 are equivalent to saying that the fiber product $A = R \times_k S$ is a generalized Gorenstein ring, which naturally covers the class of almost Gorenstein rings. Similarly, for almost Gorenstein rings, the notion is defined by a certain specific embedding of the rings into their canonical modules. Let us now recall the definition of generalized

Gorenstein rings, in particular, of dimension one, which was recently proposed by the second author and S. Kumashiro (see [11] for the precise definition).

Definition 4.5 ([11, Definition 1.2]) Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with $\dim R = 1$, possessing a fractional canonical ideal K of R ; that is, K is an R -submodule of $\overline{R}$ such that $R \subseteq K \subseteq \overline{R}$ and $K \cong K_R$ as an R -module. We say that R is a *generalized Gorenstein ring*, if either R is Gorenstein, or R is not a Gorenstein ring, and K/R is a free $R/\mathfrak{a}$ -module, where $\mathfrak{a} = R : K$.

Notice that, by [12, Theorem 3.11], R is a non-Gorenstein almost Gorenstein ring if and only if $\mathfrak{a} = \mathfrak{m}$, so that the generalized Gorenstein condition is weaker than the almost Gorenstein condition.

We begin with the following proposition.

Proposition 4.6 $\ell_A(X/A) = \ell_R(K/R) + \ell_S(L/S) + 2$.

Proof Remember that $X = (K \times L) + A \cdot \psi$ and $JX = J(K \times L)$. Since $JX \subseteq \overline{JA}$, we then have $J = JX \cap A$. Hence, we get the equalities

$$\begin{aligned} \ell_A(X/A) &= \ell_A(X/JX + A) + \ell_A(JX + A/A) \\ &= \mu_A(X/A) + \ell_A(JX/J) \\ &= r(R) + r(S) + \ell_R(\mathfrak{m}K/\mathfrak{m}) + \ell_S(\mathfrak{n}L/\mathfrak{n}) \\ &= r(R) + r(S) + [\ell_R(K/R) + 1 - r(R)] + [\ell_S(L/S) + 1 - r(S)] \\ &= \ell_R(K/R) + \ell_S(L/S) + 2, \end{aligned}$$

as claimed. ■

Lemma 4.7 Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with $\dim R = 1$, possessing a fractional canonical ideal K of R . If R is not a Gorenstein ring, then $\mathfrak{a}(K : \mathfrak{m}) = \mathfrak{a}K$, where $\mathfrak{a} = R : K$.

Proof Since R is not a DVR, $K : \mathfrak{m} = K + Rg$ for some $g \in (R : \mathfrak{m}) \setminus K$. For each $a \in \mathfrak{a}$ and $k \in K$, we obtain $(ag)k = (ak)g \in \mathfrak{m}g \subseteq R$, because $ak \in \mathfrak{a}K \cap R \subseteq \mathfrak{m}K \cap R \subseteq \mathfrak{m}\overline{R} \cap R = \mathfrak{m}$. Hence, $ag \in R : K = \mathfrak{a} \subseteq \mathfrak{a}K$, so that $\mathfrak{a}(K : \mathfrak{m}) = \mathfrak{a}K + \mathfrak{a}g = \mathfrak{a}K$, which completes the proof. ■

By setting $\mathfrak{a}_1 = R : K$ and $\mathfrak{a}_2 = S : L$, we have the following lemma.

Lemma 4.8 The following assertions hold true.

- (i) Suppose that R and S are not Gorenstein rings. Then $A : X = \mathfrak{a}_1 \times \mathfrak{a}_2$.
- (ii) Suppose that R is Gorenstein, but S is not a Gorenstein ring. Then $A : X = \mathfrak{m} \times \mathfrak{a}_2$.

Proof

- (i) By Lemma 4.7, we get $\mathfrak{a}_1(K : \mathfrak{m}) = \mathfrak{a}_1K$ and $\mathfrak{a}_2(L : \mathfrak{n}) = \mathfrak{a}_2L$. Hence,

$$\begin{aligned} X \cdot (\mathfrak{a}_1 \times \mathfrak{a}_2) &= (\mathfrak{a}_1K + \mathfrak{a}_1g) \times (\mathfrak{a}_2L + \mathfrak{a}_2g_2) \\ &= \mathfrak{a}_1(K : \mathfrak{m}) \times \mathfrak{a}_2(L : \mathfrak{n}) = \mathfrak{a}_1K \times \mathfrak{a}_2L \\ &\subseteq (\mathfrak{m}\overline{R} \cap R) \times (\mathfrak{n}\overline{S} \cap S) = \mathfrak{m} \times \mathfrak{n} \subseteq A, \end{aligned}$$

whence $\mathfrak{a}_1 \times \mathfrak{a}_2 \subseteq A : X$. Conversely, for every $(x, y) \in A : X$, we have $(K \times L) \cdot (x, y) \subseteq A$. For each $k \in K$, we get $(k, 0)(x, y) = (kx, 0) \in A$, so that $kx \in \mathfrak{m}$. Hence $x \in R : K = \mathfrak{a}_1$. Similarly, $y \in S : L = \mathfrak{a}_2$, as wanted.

(ii) This part follows from the same argument as in the proof of (i). ■

As a consequence, we have the following corollary.

Corollary 4.9 *Suppose that R and S are not DVRs. Then the fiber product $A = R \times_k S$ is a generalized Gorenstein ring if and only if R and S are almost Gorenstein rings.*

Proof The ‘if’ part is due to Corollary 4.3. We now prove the ‘only if’ part. Suppose that $A = R \times_k S$ is a generalized Gorenstein ring. We can assume that either R or S is not a Gorenstein ring. Notice that there is an isomorphism $(A/A : X)^{\oplus(r+s)} \cong X/A$ of A -modules, where $r = r(R)$, $s = r(S)$ denote the Cohen–Macaulay types of R and S , respectively. Hence, by Proposition 4.6, $(r + s) \cdot \ell_A(A/A : X) = \ell_R(K/R) + \ell_S(L/S) + 2$. Suppose now that both R and S are not Gorenstein rings. Then, because $\ell_A(A/A : X) = \ell_R(R/\mathfrak{a}_1) + \ell_S(S/\mathfrak{a}_2) - 1$, we have

$$\begin{aligned} (r + s) \cdot [\ell_R(R/\mathfrak{a}_1) + \ell_S(S/\mathfrak{a}_2) - 1] \\ = \ell_R(K/R) + \ell_S(L/S) + 2 \\ \leq (r - 1)\ell_R(R/\mathfrak{a}_1) + (s - 1)\ell_S(S/\mathfrak{a}_2) + 2, \end{aligned}$$

which yields that $[\ell_S(S/\mathfrak{a}_2) - 1](r + 1) + [\ell_R(R/\mathfrak{a}_1) - 1](s + 1) \leq 0$. Hence, $\ell_R(R/\mathfrak{a}_1) = \ell_S(S/\mathfrak{a}_2) = 1$, so that $\mathfrak{a}_1 = \mathfrak{m}$, $\mathfrak{a}_2 = \mathfrak{n}$. By Fact 3.2, we conclude that R and S are almost Gorenstein rings. On the other hand, we consider the case where R is Gorenstein, but S is not a Gorenstein ring. We then have $(r + s) \cdot [\ell_S(S/\mathfrak{a}_2)] \leq (s - 1) \cdot \ell_S(S/\mathfrak{a}_2) + 2$, which implies $\ell_S(S/\mathfrak{a}_2) = 1$. Hence, S is an almost Gorenstein ring. ■

4.2 The Case where R is a DVR and S is not a DVR

In this subsection, we assume that R is a DVR and S is not a DVR. Choose an A -submodule X of $Q(A)$ such that $A \subseteq X \subseteq \bar{A}$ and $X \cong K_A$ as an A -module. Then, because $K_B = X : B \cong R \times L$ as a B -module, we have $X : B = \xi \cdot (R \times L)$ for some invertible element $\xi = (\xi_1, \xi_2) \in Q(A)$. Applying the functor $\text{Hom}_A(-, X)$ to the exact sequence $0 \rightarrow A \xrightarrow{\iota} B \xrightarrow{\varphi} k = A/J \rightarrow 0$ where $\varphi = \begin{bmatrix} f \\ -g \end{bmatrix}$, we obtain the sequence

$$0 \longrightarrow X : B \longrightarrow X \longrightarrow A/J \longrightarrow 0$$

of A -modules. Hence, $JX \subseteq X : B \subseteq X$. Thus, we have the following inclusions.

Lemma 4.10 *One has*

$$\begin{aligned} X : B &\subseteq X \subseteq (X : B) : J = (\xi_1 R \times \xi_2 L) : J \\ &= \xi_1(R : \mathfrak{m}) \times \xi_2(L : \mathfrak{n}). \end{aligned}$$

In particular, $J(X : B) \subseteq JX \subseteq J \cdot [\xi_1(R : \mathfrak{m}) \times \xi_2(L : \mathfrak{n})]$.

We are now in a position to prove Theorem 1.1 for the one-dimensional case.

Proposition 4.11 Suppose that R is a DVR and S is not a DVR. Then the fiber product $A = R \times_k S$ is an almost Gorenstein ring if and only if S is an almost Gorenstein ring.

Proof Suppose that A is an almost Gorenstein ring. Then, since $JX = J$, we get

$$\mathfrak{n} \cdot \xi_2 L \subseteq \mathfrak{n} \subseteq \mathfrak{n} \cdot \xi_2 (L : \mathfrak{n}) = \xi_2 \cdot \mathfrak{n} L,$$

because $\mathfrak{n}(L : \mathfrak{n}) = \mathfrak{n}L$. Thus, $\mathfrak{n} = \xi_2 \cdot \mathfrak{n}L \cong \mathfrak{n}L$, and hence S is an almost Gorenstein ring. Conversely, suppose that S is an almost Gorenstein ring. We then have $\xi \in X$, because $\xi = \xi \cdot (1, 1) \in \xi \cdot (K \times L) = X : B \subseteq X$. Since $X \subseteq \overline{A} = R \times \overline{S}$, we get $\xi_1 \in R$. Therefore,

$$\begin{aligned} JX &\subseteq J \cdot [\xi_1(R : \mathfrak{m}) \times \xi_2(L : \mathfrak{n})] \\ &= \xi_1 \cdot \mathfrak{m}(R : \mathfrak{m}) \times \xi_2 \cdot \mathfrak{n} \\ &\subseteq \xi_1 R \times \xi_2 \cdot \mathfrak{n}, \end{aligned}$$

where the second equality comes from the fact that S is an almost Gorenstein ring, but not a DVR. Moreover, we have

$$JX \subseteq A \cdot (1, 0) \cdot \xi + J\xi.$$

Indeed, for each $a \in R$, there exists $\alpha \in A$ such that $p_1(\alpha) = a$. Hence, $\alpha = (a, b)$ for some $b \in S$. Then, for every $n \in \mathfrak{n}$, we get the equalities

$$\begin{aligned} (a\xi_1, \xi_2 n) &= (a, b)(\xi_1, 0) + (0, \xi_2 n) \\ &= (a, b)(1, 0)\xi + (0, n)\xi, \end{aligned}$$

which imply the required inclusion. Therefore, we have

$$JX = [JX \cap A \cdot (1, 0) \cdot \xi] + J\xi.$$

If $JX \cap A \cdot (1, 0) \cdot \xi \subseteq J\xi$, then $JX = J\xi \cong J$. Let us now assume that $JX \cap A \cdot (1, 0) \cdot \xi \not\subseteq J\xi$. Take an element $\varphi \in JX \cap A \cdot (1, 0) \cdot \xi$ such that $\varphi \notin J\xi$. Write $\varphi = (a, b)(1, 0)\xi$, where $(a, b) \in A$. Then, since $\varphi \in JX \subseteq J(R \times \overline{S}) = \mathfrak{m} \times \mathfrak{n}\overline{S}$, we get $a\xi_1 \in \mathfrak{m}$. Furthermore, $a\xi_1 \notin \mathfrak{m}\xi_1$, because $\varphi \notin J\xi$. Hence, $a \notin \mathfrak{m}$ and $\xi_1 \in \mathfrak{m}$. Consequently,

$$JX \subseteq J(R \times \mathfrak{n}) = \xi_1 R \times \xi_2 \mathfrak{n} \subseteq JX$$

yields that $JX = \xi(R \times \mathfrak{n}) \cong \xi(\mathfrak{m} \times \mathfrak{n}) = \xi J \cong J$. In any case, because $JX \cong J$, we conclude that A is an almost Gorenstein ring. ■

To explore the generalized Gorenstein properties of fiber products, we need more auxiliaries.

Proposition 4.12 The following assertions hold true.

- (i) $X : J = R \times \xi_2(S : L)$.
- (ii) $X : B = \mathfrak{m} \times \xi_2 L$ and $\xi_2 \rho = 1$ for some $\rho \in (L : \mathfrak{n}) \setminus L$.
- (iii) $X = (X : B) + A$.
- (iv) $A : X = \mathfrak{m} \times \rho(S : L)$.

Proof (i), (ii) Since $(X : B) : J = X : BJ = X : J \subseteq \overline{A} = R \times \overline{S}$, we have $X : J = \xi_1(R : \mathfrak{m}) \times \xi_2(L : \mathfrak{n})$ and $\xi_1(R : \mathfrak{m}) \subseteq R$, whence $\xi_1 \in R : (R : \mathfrak{m}) = \mathfrak{m}$. Because

$A \subseteq X \subseteq X : J$, we get $R \subseteq \xi_1(R : \mathfrak{m})$. Thus, $\mathfrak{m} = (\xi_1)$. Consequently, $X : B = \xi(R \times L) = \mathfrak{m} \times \xi_2 L$, while $X : J = R \times \xi_2(L : \mathfrak{n})$. Since $1 \in X : J$ and $1 \notin X : B$, there exists $\rho \in (L : \mathfrak{n}) \setminus L$ such that $\xi_2 \rho = 1$.

(iii) Since $\ell_A(X/X : B) = 1$, we get $X = (X : B) + A$.

(iv) Notice that $A : X = A :_A (X : B)$. For each $(a, b) \in A : X$, $am \times b \cdot \xi_2 L \subseteq A$. We then have $b \cdot \xi_2 L \subseteq \mathfrak{n}$, so that $b \cdot \xi_2 \subseteq \mathfrak{n} : L = S : L$. Hence, $b \in \rho(S : L)$. Since $\rho(S : L) \subseteq (L : \mathfrak{n})(S : L) = (S : L)L \subseteq \mathfrak{n}$, we conclude that $a \in \mathfrak{m}$. Thus, $A : X \subseteq \mathfrak{m} \times \rho(S : L) \subseteq A$. Because $(\mathfrak{m} \times \rho(S : L))(X : B) = \mathfrak{m}^2 \times (S : L)L \subseteq \mathfrak{m}^2 \times \mathfrak{n} \subseteq A$, we have that $A : X = \mathfrak{m} \times \rho(S : L)$. ■

We apply Proposition 4.12 to get the following corollary.

Corollary 4.13 $(A : X)X = \mathfrak{m} \times (S : L)L$.

Proof The assertion follows from Proposition 4.12 and $\rho(S : L) \subseteq (S : L)L$. ■

We need one more general lemma.

Lemma 4.14 Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with $\dim R = 1$, possessing a fractional canonical ideal K of R . Suppose that R is not a Gorenstein ring. We set $\mathfrak{a} = R : K$. Then the following assertions hold true.

- (i) $\mathfrak{a} = \mathfrak{a} \cdot (R : \mathfrak{m}) \subseteq \mathfrak{a}K$.
- (ii) $\mathfrak{a} = \mathfrak{a}K$ if and only if $K^2 = K^3$.

Proof

- (i) For each $a \in \mathfrak{a}$ and $x \in (R : \mathfrak{m})$, we have $(ax)k = (ak)x \in \mathfrak{m}x \subseteq R$ for every $k \in K$, because $ak \in \mathfrak{a}K \subseteq \mathfrak{m}K \subseteq \mathfrak{m}\bar{R}$. Hence, $ax \in R : K = \mathfrak{a} \subseteq \mathfrak{a}K$, so that $\mathfrak{a} = \mathfrak{a} \cdot (R : \mathfrak{m}) \subseteq \mathfrak{a}K$, which completes the proof.
- (ii) If $K^2 = K^3$, then $\mathfrak{a} = R : K = R : R[K]$, which forms an ideal of $R[K]$. Hence, $\mathfrak{a}K = \mathfrak{a}$. Conversely, if $\mathfrak{a} = \mathfrak{a}K$, then $\mathfrak{a} = \mathfrak{a}K^n$ for every $n > 0$, so that $\mathfrak{a} = R : R[K]$. Hence, $K^2 = K^3$, because $\mathfrak{a} = R : K = K : K^2$ and $R : R[K] = R : K^2 = K : K^3$. ■

Corollary 4.15 The following conditions are equivalent.

- (i) $X^2 = X^3$.
- (ii) $L^2 = L^3$.

When this is the case, $\ell_A(A/(A : X)) = \ell_S(S/(S : L))$.

Proof

- (i) $\Rightarrow$ (ii) Suppose that $X^2 = X^3$. By Lemma 4.14, we get $(A : X)X = (A : X)$, so that $\mathfrak{m} \times (S : L)L = \mathfrak{m} \times \rho(S : L)$. Hence, $(S : L)L^n = \rho^n(S : L)$ for all $n > 0$. Thus, $(S : L)S[L] = \rho^n(S : L) = (S : L)$ for each $n \gg 0$, because ρ is an invertible element of $S[L]$. Therefore, $L^2 = L^3$.
- (ii) $\Rightarrow$ (i) If $L^2 = L^3$, then $S : L = S : S[L]$, which is an ideal of $S[L]$, so that $\rho(S : L) = (S : L) = (S : L)L$. Hence, $X^2 = X^3$, by Proposition 4.12(iv) and Corollary 4.13. The last assertion follows from the fact that $\rho(S : L) = (S : L)$. ■

Finally, we reach the following theorem.

Theorem 4.16 *Suppose that R is a DVR and S is not a DVR. Then the fiber product $A = R \times_k S$ is a generalized Gorenstein ring if and only if S is an almost Gorenstein ring.*

Proof We only prove the ‘only if’ part. Suppose that A is a generalized Gorenstein ring. By [11, Theorem 4.8], we then have $A[X]/A \cong (A/A : X)^{\oplus(\ell+1)}$ as an A -module, where $\ell = r(S)$. By [11, Theorem 4.8], we get $X^2 = X^3$, that is, $L^2 = L^3$. Hence, we have the equalities

$$\ell_A(A[X]/A) = (\ell + 1)\ell_A(A/(A : X)) = (\ell + 1)\ell_S(S/(S : L)).$$

Since $X \subseteq X : J \subseteq A[X]$ (see [12, Corollary 3.8 (1)]), we have $X^n \subseteq (X : J)^n \subseteq A[X]$ for every $n > 0$, so that $X^n = (X : J)^n = A[X]$ for every $n \gg 0$. Similarly, we have $L^n = (L : n)^n = S[L]$ for every $n \gg 0$. Therefore,

$$A[X] = [R \times \xi_2(L : n)]^n = R \times \xi_2^n \cdot S[L] = R \times \xi_2^{n+1} \cdot S[L] \quad \text{for all } n \gg 0,$$

which yields $S[L] = \xi_2 \cdot S[L]$. Thus, $\xi_2, \rho \in S[L]$, and hence $A[X] = R \times S[L]$. Hence, $\ell_A(A[X]/A) = 1 + \ell_S(S[L]/S)$. On the other hand, by the sequence

$$0 \longrightarrow L/S \longrightarrow S[L]/S \longrightarrow S[L]/L \longrightarrow 0$$

of S -modules, we get

$$\ell_S(S[L]/S) \leq (\ell - 1) \cdot \ell_S(S/(S : L)) + \ell_S(S/(S : L)) = \ell \cdot \ell_S(S/(S : L)),$$

because $S[L]/L \cong K_{S/(S:L)}$ (see [11, Theorem 4.8]). Therefore,

$$(\ell + 1)\ell_S(S/(S : L)) \leq 1 + \ell \cdot \ell_S(S/(S : L)),$$

which implies $\ell_S(S/(S : L)) = 1$, so that $n = S : L$. Hence, S is almost Gorenstein. ■

We now go back to the notation as in Setting 4.1. By combining Corollaries 4.3, 4.9, Proposition 4.11, and Theorem 4.16, we have the following theorem.

Theorem 4.17 *The following conditions are equivalent.*

- (i) *The fiber product $A = R \times_k S$ is an almost Gorenstein ring.*
- (ii) *The fiber product $A = R \times_k S$ is a generalized Gorenstein ring.*
- (iii) *R and S are almost Gorenstein rings.*

Letting $S = R$, we get the following corollary.

Corollary 4.18 *The following conditions are equivalent.*

- (i) *The fiber product $R \times_k R$ is an almost Gorenstein ring.*
- (ii) *The fiber product $R \times_k R$ is a generalized Gorenstein ring.*
- (iii) *The idealization $R \times \mathfrak{m}$ is an almost Gorenstein ring.*
- (iv) *R is an almost Gorenstein ring.*

Proof The proof follows from Theorem 4.17 and [12, Theorem 6.5]. ■

We are now ready to prove Theorem 1.1. Let $(R, \mathfrak{m})$, $(S, \mathfrak{n})$ be Cohen-Macaulay local rings with $d = \dim R = \dim S > 0$, $(T, \mathfrak{m}_T)$ a regular local ring with $\dim T = d - 1$, and let $f: R \rightarrow T$, $g: S \rightarrow T$ be surjective homomorphisms. Suppose that the residue class field $T/\mathfrak{m}_T$ of T is infinite. We then consider the fiber product $A = R \times_T S$, which is a Cohen-Macaulay local ring with $\dim A = d$.

With this notation, we begin with the following proposition.

Proposition 4.19 *The fiber product $A = R \times_T S$ is a Gorenstein ring if and only if R and S are regular local rings.*

Proof By Proposition 2.2, we can assume $d \geq 2$. Choose a regular system $x_1, x_2, \dots, x_{d-1}$ of parameters for T . The surjectivities of f and g show that $x_1, x_2, \dots, x_{d-1}$ forms a regular sequence on R and S . Let us write $x_i = f(a_i) = g(b_i)$ for some $a_i \in \mathfrak{m}$ and $b_i \in \mathfrak{n}$. We then have an isomorphism

$$\begin{aligned} A/(\alpha_1, \alpha_2, \dots, \alpha_{d-1}) &\cong R/(a_1, a_2, \dots, a_{d-1}) \\ &\quad \times_{T/(x_1, x_2, \dots, x_{d-1})} S/(b_1, b_2, \dots, b_{d-1}) \end{aligned}$$

of rings, which yields the required assertion, by induction arguments. $\blacksquare$

We finally reach the proof of Theorem 1.1.

Proof of Theorem 1.1 We set $d = \dim A$. By Theorem 4.17, we can assume that $d \geq 2$ and that the assertion holds for $d - 1$.

(i) $\Rightarrow$ (ii) We can assume A is not a Gorenstein ring. Let us consider an exact sequence

$$0 \longrightarrow A \longrightarrow K_A \longrightarrow C \longrightarrow 0$$

of A -modules with $\mu_A(C) = e_J^0(C)$, where $J = (\mathfrak{m} \times \mathfrak{n}) \cap A$ denotes the maximal ideal of A . Since A has an infinite residue class field, we choose a regular element $\alpha = (a, b) \in J$ on A so that α is a superficial element for C with respect to J , and $(f \circ p_1)(\alpha)$ forms a part of a regular system of parameters of T . We then have that $a \in \mathfrak{m}$ and $b \in \mathfrak{n}$ are regular elements of R and S , respectively. Hence, we have an isomorphism $A/(\alpha) \cong R/(a) \times_{T/aT} S/(b)$ as rings. Then the hypothesis on d shows that $R/(a)$ and $S/(b)$ are almost Gorenstein rings. Therefore, by [19, Theorem 3.7], R and S are almost Gorenstein rings.

(ii) $\Rightarrow$ (i) First, we consider the case where R and S are Gorenstein rings. Choose a regular system of parameters $x_1, x_2, \dots, x_{d-1} \in \mathfrak{m}_T$ of T and write $x_i = f(a_i) = g(b_i)$ with $a_i \in \mathfrak{m}$ and $b_i \in \mathfrak{n}$. For each $1 \leq i \leq d - 1$, we set $\alpha_i = (a_i, b_i) \in J$. Then $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$ forms a regular sequence on A , and there is an isomorphism

$$\begin{aligned} A/(\alpha_1, \alpha_2, \dots, \alpha_{d-1}) &\cong R/(a_1, a_2, \dots, a_{d-1}) \\ &\quad \times_{T/(x_1, x_2, \dots, x_{d-1})} S/(b_1, b_2, \dots, b_{d-1}) \end{aligned}$$

of rings. As $R/(a_1, a_2, \dots, a_{d-1})$, $S/(b_1, b_2, \dots, b_{d-1})$ are Gorenstein and $T/(x_1, x_2, \dots, x_{d-1})$ is a field, we conclude that $A/(\alpha_1, \alpha_2, \dots, \alpha_{d-1})$ is an almost Gorenstein ring, and so is A . Let us assume that R is Gorenstein, and S is an almost Gorenstein ring, but not a Gorenstein ring. Choose an exact sequence

$$0 \longrightarrow S \longrightarrow K_S \longrightarrow D \longrightarrow 0$$

of S -modules such that $\mu_S(D) = e_n^0(D)$. Let us take an S -regular element $b \in n$ such that b is a superficial element for D with respect to n , and $g(b) \in m_T$ is a part of a regular system of parameters of T . Let us choose $a \in m$ such that $f(a) = g(b)$. Then $\alpha = (a, b) \in J$ is a regular element on A and $A/(\alpha) \cong R/(a) \times_{T/aT} S/(b)$. The induction hypothesis shows that $A/(\alpha)$ is an almost Gorenstein ring, whence A is almost Gorenstein. Finally, we are assuming that R and S are non-Gorenstein, almost Gorenstein rings. Consider the exact sequences

$$0 \longrightarrow R \longrightarrow K_R \longrightarrow C \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow S \longrightarrow K_S \longrightarrow D \longrightarrow 0$$

of R -modules, and of S -modules, respectively. We then choose an A -regular element $\alpha = (a, b) \in J$ such that $a \in m$ is a superficial element for C with respect to m , $b \in n$ is a superficial element for D with respect to n , and $(f \circ p_1)(\alpha) \in m_T$ is a part of a regular system of parameters of T . Then $A/(\alpha) \cong R/(a) \times_{T/aT} S/(b)$. Hence, $A/(\alpha)$ is an almost Gorenstein ring, whence A is almost Gorenstein, as desired. ■

5 Further Results in Dimension One

The notion of almost Gorenstein ring in our sense originated from the works [2] of V. Barucci and R. Fröberg in 1997, and [12] of the second author, N. Matsuoka, and T. T. Phuong in 2013, where they dealt with the notion of one-dimensional Cohen–Macaulay local rings. Although the second author and S. Kumashiro have already provided a beautiful generalization of almost Gorenstein rings, say *generalized Gorenstein rings*, there are other directions of the generalization for one-dimensional almost Gorenstein rings, which we call *2-almost Gorenstein* and *nearly Gorenstein* rings.

Let us maintain the notation as in Setting 4.1. The goals of this section are stated as follows.

Theorem 5.1 *The following conditions are equivalent.*

- (i) *The fiber product $A = R \times_k S$ is a 2-almost Gorenstein ring.*
- (ii) *Either R is a 2-almost Gorenstein ring and S is an almost Gorenstein ring, or R is an almost Gorenstein ring and S is a 2-almost Gorenstein ring.*

Theorem 5.2 *The following conditions are equivalent.*

- (i) *The fiber product $A = R \times_K S$ is a nearly Gorenstein ring.*
- (ii) *R and S are nearly Gorenstein rings.*

Before going ahead, we recall the definition of 2-almost Gorenstein rings and nearly Gorenstein rings. For a while, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$, possessing a fractional canonical ideal K of R , i.e., K is an R -submodule of $Q(R)$ such that $R \subseteq K \subseteq \bar{R}$ and $K \cong K_R$ as an R -module.

With this notation, T. D. M. Chau, the second author, S. Kumashiro, and N. Matsuoka proposed the notion of 2-almost Gorenstein rings.

Definition 5.3 ([6]) We say that R is a 2-almost Gorenstein ring, if $K^2 = K^3$ and $\ell_R(K^2/K) = 2$.

By [6, Theorem 3.7], the condition of $K^2 = K^3$ and $\ell_R(K^2/K) = 2$ is equivalent to saying that $\ell_R(R/\mathfrak{c}) = 2$, where $\mathfrak{c} = R : R[K]$. The latter condition is independent of the choice of fractional canonical ideals K (see [6, Theorem 2.5]). Furthermore, because R is an almost Gorenstein ring but not a Gorenstein ring if and only if $\ell_R(R/\mathfrak{c}) = 1$, that is, $\mathfrak{c} = \mathfrak{m}$ ([12, Theorem 3.16]), 2-almost Gorenstein rings could be considered to be one of the successors of almost Gorenstein rings.

Let us now state the definition of nearly Gorenstein rings, which is defined by J. Herzog, T. Hibi, and D. I. Stamate, by using the trace of canonical ideals. The notion is defined for arbitrary dimension; however, let us focus our attention on the case where $\dim R = 1$, because there is no relation between almost Gorenstein and nearly Gorenstein rings for higher dimensions (see [21]). For an R -module X , let

$$\varphi : \operatorname{Hom}_R(X, R) \otimes_R X \longrightarrow R$$

be the R -linear map defined by $\varphi(f \otimes x) = f(x)$ for every $f \in \operatorname{Hom}_R(X, R)$ and $x \in X$. We set $\operatorname{tr}_R(X) = \operatorname{Im} \varphi$ and call it the trace of X . The reader may consult [9, 21, 25, 26, 27] for further information on trace ideals and modules.

Definition 5.4 ([21]) We say that R is a nearly Gorenstein ring, if $\mathfrak{m} \subseteq \operatorname{tr}_R(K)$.

For every fractional ideal X in R , we have $\operatorname{tr}_R(X) = (R : X) \cdot X$ (see [26, Lemma 2.3]). By [21, Proposition 6.1], almost Gorenstein ring is nearly Gorenstein, and the converse is not true in general, but it holds if R has minimal multiplicity, that is, $v(R) = e(R)$.

We close this paper by proving Theorems 5.1 and 5.2.

Proof of Theorem 5.1 Thanks to Proposition 2.2, we can assume that either R or S is not a DVR. Suppose that both R and S are not DVRs. Then $X = (K \times L) + Ag$ forms a fractional canonical ideal of A , where $\psi = (g_1, g_2) \in \bar{A}$, $g_1 \in (R : \mathfrak{m}) \setminus K$, and $g_2 \in (L : \mathfrak{n}) \setminus L$. By Theorem 4.17, we can assume that either R or S is not a Gorenstein ring. Suppose that R and S are not Gorenstein rings. We then have $A : X = (R : K) \times (S : L)$,

so that

$$\ell_A(A/(A : X)) = \ell_A(B/(A : X)) - 1 = \ell_R(R/(R : K)) + \ell_S(S/(S : L)) - 1.$$

Hence, $\ell_A(A/(A : X)) = 2$ holds if and only if either $\ell_R(R/(R : K)) = 1$ and $\ell_S(S/(S : L)) = 2$, or $\ell_R(R/(R : K)) = 2$ and $\ell_S(S/(S : L)) = 1$. If R is Gorenstein and S is a non-Gorenstein ring, then $A : X = \mathfrak{m} \times (S : L)$ so that

$$\ell_A(A/(A : X)) = \ell_R(R/\mathfrak{m}) + \ell_S(S/(S : L)) - 1 = \ell_S(S/(S : L)),$$

which yields the required assertion. $\blacksquare$

Proof of Theorem 5.2 By Proposition 2.2 and Corollary 4.13, we can assume that R and S are not DVRs. If R and S are Gorenstein, then by Corollary 4.3, $A = R \times_k S$ is an almost Gorenstein ring, so that A is nearly Gorenstein ([21, Proposition 6.1]). Thus, we can also assume that either R or S is not a Gorenstein ring. Suppose that R and S are not Gorenstein rings. By Lemma 4.8(i), we have that $A : X = \alpha_1 \times \alpha_2$, where $\alpha_1 = R : K$, $\alpha_2 = S : L$. Since

$$\begin{aligned} (A : X)X &= (\alpha_1 \times \alpha_2)((K \times L) + A \cdot (g_1, g_2)) \\ &= \alpha_1(K : \mathfrak{m}) \times \alpha_2(L : \mathfrak{n}) = \alpha_1 K \times \alpha_2 L, \end{aligned}$$

we see that $A = R \times_k S$ is a nearly Gorenstein ring if and only if $\alpha_1 K = \mathfrak{m}$ and $\alpha_2 L = \mathfrak{n}$, in other words, R and S are nearly Gorenstein. Let us now consider the case where R is Gorenstein, but S is not a Gorenstein ring. In this case, $A : X = \mathfrak{m} \times \alpha_2$, so that

$$(A : X)X = (\mathfrak{m} \times \alpha_2)((R \times L) + A \cdot (g_1, g_2)) = \mathfrak{m}(R : \mathfrak{m}) \times \alpha_2 L.$$

Therefore, condition (ii) is equivalent to saying that $\alpha_2 L = \mathfrak{n}$, that is, S is a nearly Gorenstein ring. This completes the proof. $\blacksquare$

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