Poisson boundary of the discrete quantum group $\hat{A}_u(F)$

Stefaan Vaes and Nikolas Vander Vennet


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Abstract

We identify the Poisson boundary of the dual of the universal compact quantum group $A_u(F)$ with a measurable field of ITPFI (infinite tensor product of finite type I) factors.

1. Introduction and statement of main result

Poisson boundaries of discrete quantum groups were introduced by Izumi [Izu02] in his study of infinite tensor product actions of $SU_q(2)$. Izumi was able to identify the Poisson boundary of the dual of $SU_q(2)$ with the quantum homogeneous space $L^\infty(SU_q(2)/S^1)$, called the Podleś sphere. The generalization to $SU_q(n)$ was established by Izumi et al. [INT06], yielding $L^\infty(SU_q(n)/S^{n-1})$ as the Poisson boundary. A more systematic approach was given by Tomatsu [Tom07] who proved the following very general result: if $G$ is a compact quantum group with commutative fusion rules and amenable dual $\hat{G}$, the Poisson boundary of $\hat{G}$ can be identified with the quantum homogeneous space $L^\infty(G/K)$, where $K$ is the maximal closed quantum subgroup of Kac type inside $G$. Tomatsu’s result provides the Poisson boundary for the duals of all $q$-deformations of classical compact groups.

All examples discussed in the previous paragraph concern amenable discrete quantum groups. In [VV08], we identified the Poisson boundary for the (non-amenable) dual of the compact quantum group $A_o(F)$ with a higher-dimensional Podleś sphere. Although the dual of $A_o(F)$ is non-amenable, the representation category of $A_o(F)$ is monoidally equivalent with the representation category of $SU_q(2)$ for the appropriate value of $q$. The second author and De Rijdt provided in [DV06] a general result explaining the behavior of the Poisson boundary under the passage to monoidally equivalent quantum groups. In particular, a combination of the results of [DV06, Izu02] give a more conceptual approach to our identification in [VV08].

The quantum random walks studied on a discrete quantum group $\hat{G}$ have a semi-classical counterpart, being a Markov chain on the (countable) set $\text{Irred}(G)$ of irreducible representations of $G$ (modulo unitary equivalence). All of the examples above share the feature that the semi-classical random walk on $\text{Irred}(G)$ has trivial Poisson boundary.

In this paper, we identify the Poisson boundary for the dual of $G = A_u(F)$. In that case, $\text{Irred}(G)$ can be identified with the Cayley tree of the monoid $\mathbb{N} \ast \mathbb{N}$ and, by the results of [PW87], has a non-trivial Poisson boundary: the end compactification of the tree with the appropriate harmonic measure. Before discussing our main result in more detail, we introduce...
some terminology and notation. For a more complete introduction to Poisson boundaries of
discrete quantum groups, we refer the reader to [Van08, ch. 4].

Compact quantum groups were originally introduced by Woronowicz in [Wor87] and their
definition finally took the following form.

**Definition 1.1** (Woronowicz [Wor98, Definition 1.1]). A compact quantum group \( G \) is a pair
consisting of a unital C*-algebra \( C(G) \) and a unital *-homomorphism \( \Delta : C(G) \to C(G) \otimes C(G) \),
called comultiplication, satisfying the following two conditions.

- **Co-associativity**: \( (\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta \).
- **span**: \( \Delta(C(G))(1 \otimes C(G)) \) and \( \text{span} \Delta(C(G))((C(G) \otimes 1) \) are dense in \( C(G) \otimes C(G) \).

In the above definition, the symbol \( \otimes \) denotes the minimal (i.e. spatial) tensor product of
C*-algebras.

Let \( G \) be a compact quantum group. By [Wor87, Theorem 1.3], there is a unique state \( h \) on
\( C(G) \) satisfying \( (\text{id} \otimes h) \Delta(a) = h(a)1 = (h \otimes \text{id}) \Delta(a) \) for all \( a \in C(G) \). We call \( h \) the Haar state
of \( G \).

A unitary representation of \( G \) on the finite-dimensional Hilbert space \( H \) is a unitary operator
\( U \in \mathcal{L}(H) \otimes C(G) \) satisfying \( (\text{id} \otimes \Delta)(U) = U_{12}U_{13} \). Given unitary representations \( U_1, U_2 \) on
\( H_1, H_2 \), we put

\[
\text{Mor}(U_2, U_1) := \{ S \in \mathcal{L}(H_1, H_2) \mid (S \otimes 1)U_1 = U_2(S \otimes 1) \}.
\]

Let \( U \) be a unitary representation of \( G \) on the finite-dimensional Hilbert space \( H \). The elements
\( (\xi^* \otimes 1)U(\eta \otimes 1) \in C(G) \) are called the coefficients of \( U \). The linear span of all coefficients of all
finite-dimensional unitary representations of \( G \) forms a dense *-subalgebra of \( C(G) \) (see [Wor98,
Theorem 1.2]). We call \( U \) irreducible if \( \text{Mor}(U, U) = \mathbb{C}1 \). We call \( U_1 \) and \( U_2 \) unitarily equivalent
if \( \text{Mor}(U_2, U_1) \) contains a unitary operator.

Let \( U \) be an irreducible unitary representation of \( G \) on the finite-dimensional Hilbert space \( H \). By [Wor98,
Proposition 5.2], there exists an anti-linear invertible map \( j : H \to \overline{H} \) such that the operator
\( U_c \in \mathcal{L}(\overline{H}) \otimes C(G) \) defined by the formula \( j(\xi^* \otimes 1)U_c(j(\eta \otimes 1) = (\eta^* \otimes 1)U^*(\xi \otimes 1) \) is unitary. One calls \( U_c \) the contragredient of \( U \). Since \( U \) is irreducible, the map \( j \) uniquely determined up to multiplication by a non-zero scalar. We normalize in such a way that \( Q := j^*j \) satisfies
\( \text{Tr}(Q) = \text{Tr}(Q^{-1}) \). Then, \( j \) is determined up to multiplication by \( \lambda \in S^1 \) and \( Q \) is uniquely
determined. We call \( \text{Tr}(Q) \) the quantum dimension of \( U \) and denote it by \( \dim_q(U) \). Note that
\( \dim_q(U) \geq \dim(H) \) with equality holding if and only if \( Q = 1 \).

The tensor product \( U \otimes V \) of two unitary representations is defined as \( U_{13}V_{23} \).

Given a compact quantum group \( G \), we denote by \( \text{Irred}(G) \) the set of irreducible
unitary representations of \( G \) modulo unitary conjugacy. For every \( x \in \text{Irred}(G) \), we choose a
representative \( U^x \) on the Hilbert space \( H^x \). We denote by \( Q_x \in \mathcal{L}(H^x) \) the associated positive
invertible operator and define the state \( \psi_x \) on \( \mathcal{L}(H^x) \) by the formula

\[
\psi_x(A) := \frac{\text{Tr}(Q_xA)}{\text{Tr}(Q_x)}.
\]

The dual, discrete quantum group \( \widehat{G} \) is defined as the \( \ell^\infty \)-direct sum of matrix algebras

\[
\ell^\infty(\widehat{G}) := \prod_{x \in \text{Irred}(G)} \mathcal{L}(H_x).
\]
Poisson boundary of the discrete quantum group $\hat{A}_\mu(F)$

We denote by $p_x$, $x \in \text{Irred}(G)$, the minimal central projections in $\ell^\infty(\hat{G})$. Denote by $\epsilon \in \text{Irred}(G)$ the trivial representation and by $\hat{\epsilon}: \ell^\infty(\hat{G}) \to \mathbb{C}$ the co-unit given by $ap_x = \hat{\epsilon}(a)p_x$.

Whenever $x, y, z \in I$, we use the shorthand notation $\text{Mor}(x \otimes y, z) := \text{Mor}(U^x \otimes U^y, U^z)$ and we write $z \subset x \otimes y$ if $\text{Mor}(x \otimes y, z) \neq \{0\}$.

The von Neumann algebra $\ell^\infty(\hat{G})$ carries a comultiplication $\hat{\Delta}: \ell^\infty(\hat{G}) \to \ell^\infty(\hat{G}) \otimes \ell^\infty(\hat{G})$, uniquely characterized by the formula

$$\hat{\Delta}(a)(p_x \otimes p_y)S = \text{Sap}_x \quad \text{for all } x, y, z \in \text{Irred}(G) \text{ and } S \in \text{Mor}(x \otimes y, z).$$

Denote by $L^\infty(G)$ the weak closure of $C(G)$ in the Gelfand-Naimark-Segal (GNS) representation of the Haar state $h$. One defines the unitary $\mathbb{V} \in \ell^\infty(\hat{G}) \otimes L^\infty(G)$ by the formula

$$\mathbb{V} := \bigoplus_{x \in \text{Irred}(G)} U^x.$$

The unitary $\mathbb{V}$ implements the duality between $G$ and $\hat{G}$, in the sense that it satisfies

$$(\hat{\Delta} \otimes \text{id})(\mathbb{V}) = \mathbb{V}_{13}\mathbb{V}_{23} \quad \text{and} \quad (\text{id} \otimes \Delta)(\mathbb{V}) = \mathbb{V}_{12}\mathbb{V}_{13}.$$

Discrete quantum groups can also be defined intrinsically, see [VanD96].

Whenever $\omega \in \ell^\infty(\hat{G})_\ast$ is a normal state, we consider the Markov operator

$$P_\omega: \ell^\infty(\hat{G}) \to \ell^\infty(\hat{G}): P_\omega(a) = (\text{id} \otimes \omega)\hat{\Delta}(a).$$

By [NT04, Proposition 2.1], the Markov operator $P_\omega$ leaves the center $Z(\ell^\infty(\hat{G}))$ of $\ell^\infty(\hat{G})$ globally invariant if and only if

$$\omega = \psi_\mu := \sum_{x \in \text{Irred}(G)} \mu(x)\psi_x \quad \text{where } \mu \text{ is a probability measure on } \text{Irred}(G).$$

We only consider states $\omega$ of the form $\psi_\mu$ and denote by $P_\mu$ the corresponding Markov operator. Note that we can define a convolution product on the probability measures on $\text{Irred}(G)$ by the formula

$$P_{\mu \ast \eta} = P_\mu \circ P_\eta.$$

Considering the restriction of $P_\mu$ to $\ell^\infty(\text{Irred}(G)) = Z(\ell^\infty(\hat{G}))$, every probability measure $\mu$ on $\text{Irred}(G)$ defines a Markov chain on the countable set $\text{Irred}(G)$ with $n$-step transition probabilities given by

$$p_x^{(n)}(x, y) = p_x P_\mu^n(p_y).$$

Note that the 1-step transition probabilities are given by

$$p_1(x, y) = \sum_{z \in \text{Irred}(G)} \mu(z) \frac{\dim_y(y)}{\dim_y(x) \dim_y(z)}. \quad (1)$$

The probability measure $\mu$ is called generating if, for every $x, y \in \text{Irred}(G)$, there exists an $n \in \mathbb{N} \setminus \{0\}$ such that $p_n(x, y) > 0$.

**Definition 1.2.** Let $G$ be a compact quantum group and $\mu$ a generating probability measure on $\text{Irred}(G)$. The Poisson boundary of $\hat{G}$ with respect to $\mu$ is defined as the space of $P_\mu$-harmonic elements in $\ell^\infty(\hat{G})$:

$$\text{H}^\infty(\hat{G}, \mu) := \{a \in \ell^\infty(\hat{G}) \mid P_\mu(a) = a\}.$$
The weakly closed vector subspace $\mathcal{H}^\infty(\hat{G}, \mu)$ of $\ell^\infty(\hat{G})$ is turned into a von Neumann algebra using the product (cf. [Izu02, Theorem 3.6])

$$a \cdot b := \lim_{n \to \infty} P^a_\mu(ab)$$

and where the sequence on the right-hand side is strongly convergent.

- The restriction of $\hat{\Delta}$ to $\mathcal{H}^\infty(\hat{G}, \mu)$ is a faithful normal state on $\mathcal{H}^\infty(\hat{G}, \mu)$.
- The restriction of $\Delta$ to $\mathcal{H}^\infty(\hat{G}, \mu)$ defines a left action

$$\alpha_G : \mathcal{H}^\infty(\hat{G}, \mu) \to \ell^\infty(\hat{G}) \otimes \mathcal{H}^\infty(\hat{G}, \mu) : a \mapsto \Delta(a)$$

of $\hat{G}$ on $\mathcal{H}^\infty(\hat{G}, \mu)$.
- The restriction of the adjoint action to $\mathcal{H}^\infty(\hat{G}, \mu)$ defines an action

$$\alpha^*_G : \mathcal{H}^\infty(\hat{G}, \mu) \to \mathcal{H}^\infty(\hat{G}) \otimes \mathcal{L}^\infty(\hat{G}) : \alpha \mapsto \psi(\alpha)p_x,$$

we observe that $\mathcal{E}$ also provides a faithful conditional expectation of $\mathcal{H}^\infty(\hat{G}, \mu)$ onto the von Neumann subalgebra $\mathcal{H}^\infty_{\text{centr}}(\hat{G}, \mu)$.

We now turn to the concrete family of compact quantum groups studied in this paper and introduced by Van Daele and Wang in [VW96]. Let $n \in \mathbb{N}\setminus\{0, 1\}$ and let $F \in \text{GL}(n, \mathbb{C})$. One defines the compact quantum group $G = A_n(F)$ such that $C(G)$ is the universal unital $\text{C}^*$-algebra generated by the entries of an $n \times n$ matrix $U$ satisfying the relations

$$U \text{ and } FUF^{-1} \text{ are unitary, with } (U)_{ij} = (U_{ij})^*$$

and such that $\Delta(U_{ij}) = \sum_{k=1}^n U_{ik} \otimes U_{kj}$. By definition, $U$ is an $n$-dimensional unitary representation of $A_n(F)$, called the fundamental representation.

Fix $F \in \text{GL}(n, \mathbb{C})$ and put $G = A_n(F)$. For reasons to become clear later, we assume that $F$ is not a scalar multiple of a unitary $2 \times 2$ matrix.

By [Ban97, Théorème 1], the irreducible unitary representations of $G$ can be labeled by the elements of the free monoid $I := \mathbb{N} \star \mathbb{N}$ generated by $\alpha$ and $\beta$. We represent the elements of $I$ as words in $\alpha$ and $\beta$. The empty word is denoted by $\epsilon$ and corresponds to the trivial representation of $G$, while $\alpha$ corresponds to the fundamental representation and $\beta$ to the contragredient of $\alpha$. We denote by $x \mapsto \overline{x}$ the unique antimultiplicative and involutive map on $I$ satisfying $\overline{\alpha} = \beta$. This involution corresponds to the contragredient on the level of representations. The fusion rules of $G$ are given by

$$x \otimes y \cong \bigoplus_{x \in I, x = x_0 y, y = y_0} x_0 y_0.$$

So, if the last letter of $x$ equals the first letter of $y$, the tensor product $x \otimes y$ is irreducible and given by $xy$. We denote this as $xy = x \otimes y$.

Denote by $\partial I$ the compact space of infinite words in $\alpha$ and $\beta$. For $x \in \partial I$, the expression

$$x = x_1 \otimes x_2 \otimes \cdots$$

means that the infinite word $x$ is the concatenation of the finite words $x_1, x_2, \cdots$ and that the last letter of $x_n$ equals the first letter of $x_{n+1}$ for all $n \in \mathbb{N}$. All elements $x$ of $\partial I$ can be decomposed as in (2), except the countable number of elements of the form $x = y\alpha\beta\alpha\beta\cdots$ for some $y \in I$. 

1076
In the following, we only deal with non-atomic measures on \( \partial I \), so that almost every point of \( \partial I \) has a decomposition as in (2). We denote by \( \partial_0 I \) the subset of \( \partial I \) consisting of the infinite words that have a decomposition of the form (2).

The following is the main result of the paper.

**Theorem 1.3.** Let \( F \in \text{GL}(n, \mathbb{C}) \) such that \( F \) is not a scalar multiple of a unitary \( 2 \times 2 \) matrix. Write \( \mathbb{G} = A_n(F) \) and suppose that \( \mu \) is a finitely supported, generating probability measure on \( I = \text{Irred}(\mathbb{G}) \). Denote by \( \partial I \) the compact space of infinite words in the letters \( \alpha, \beta \). There exists:

- a non-atomic probability measure \( \nu_\epsilon \) on \( \partial I \);
- a measurable field \( M \) of infinite tensor product of finite type I (ITPFI) factors over \( (\partial I, \nu_\epsilon) \) with fibers
  \[
  (M_x, \omega_x) = \bigotimes_{k=1}^{\infty} \left( \mathcal{L}(H_{x_k}), \psi_{x_k} \right)
  \]
  whenever \( x \in \partial_0 I \) is of the form \( x = x_1 x_2 x_3 \cdots = x_1 \otimes x_2 \otimes x_3 \otimes \cdots \);
- an action \( \beta_\mathbb{G} \) of \( \hat{\mathbb{G}} \) on \( M \) concretely given by (3) below;

such that, with \( \omega_\infty = \int \omega_x \, d\nu_\epsilon(x) \), the Poisson integral formula

\[
\Theta_\mu : M \to H^\infty(\hat{\mathbb{G}}, \mu) : \Theta_\mu(a) = (id \otimes \omega_\infty)\beta_\mathbb{G}(a)
\]
defines a \(*\)-isomorphism of \( M \) onto \( H^\infty(\hat{\mathbb{G}}, \mu) \), intertwining the action \( \beta_\mathbb{G} \) on \( M \) with the action \( \alpha_\mathbb{G} \) on \( H^\infty(\hat{\mathbb{G}}, \mu) \).

Moreover, defining the action \( \beta_\mathbb{G} \) of \( \mathbb{G} \) on \( M \) as the infinite tensor product of the inner actions \( a \mapsto U_{x_k}^* (a \otimes 1)(U_{x_k})^* \), we obtain the action \( \beta_\mathbb{G} \) of \( \mathbb{G} \) on \( M \). The \(*\)-isomorphism \( \Theta_\mu \) intertwines \( \beta_\mathbb{G} \) with \( \alpha_\mathbb{G} \).

The comultiplication \( \hat{\Delta} : \ell^\infty(\hat{\mathbb{G}}) \to \ell^\infty(\hat{\mathbb{G}}) \otimes \ell^\infty(\hat{\mathbb{G}}) \) can be uniquely cut down into completely positive maps \( \hat{\Delta}_{x \otimes y, z} : \mathcal{L}(H_z) \to \mathcal{L}(H_x) \otimes \mathcal{L}(H_y) \) in such a way that

\[
\hat{\Delta}(a)(p_x \otimes p_y) = \sum_{z : x \otimes y} \hat{\Delta}_{x \otimes y, z}(a p_z)
\]

for all \( a \in \ell^\infty(\hat{\mathbb{G}}) \).

We denote by \( |x| \) the length of a word \( x \in I \).

If now \( x, y \in I, \ z \in \partial I \) with \( yz = y \otimes z \) and \( |y| > |x| \), we define for all \( s \subset x \otimes y \),

\[
\hat{\Delta}_{x \otimes y, s, z} : M_{sz} \to \mathcal{L}(H_x) \otimes M_{yz}
\]

by composing \( \hat{\Delta}_{x \otimes y, s} \otimes \text{id} \) with the identifications \( M_{sz} \cong \mathcal{L}(H_s) \otimes M_z \) and \( M_{yz} \cong \mathcal{L}(H_y) \otimes M_z \).

The action \( \beta_\mathbb{G} : M \to \ell^\infty(\hat{\mathbb{G}}) \otimes M \) of \( \mathbb{G} \) on \( M \) is now given by

\[
\beta_\mathbb{G}(a)_{x, yz} = \sum_{s \subset x \otimes y} \hat{\Delta}_{x \otimes y, s, z}(a s_z)
\]

whenever \( a \in M, \ x, y \in I, \ z \in \partial I, \ |y| > |x| \) and \( yz = y \otimes z \). Note that we identified \( \ell^\infty(\hat{\mathbb{G}}) \otimes M \) with a measurable field over \( I \times \partial I \) with fiber in \((x, z)\) given by \( \mathcal{L}(H_x) \otimes M_z \).

**Further notation and terminology**

Fix \( F \in \text{GL}(n, \mathbb{C}) \) and put \( \mathbb{G} = A_n(F) \). We identify \( \text{Irred}(\mathbb{G}) \) with \( I := \mathbb{N} * \mathbb{N} \). We assume that \( F \) is not a multiple of a unitary \( 2 \times 2 \) matrix. Equivalently, \( \dim_q(\alpha) > 2 \). The first reason to do so...
is that under this assumption, the random walk defined by any non-trivial probability measure $\mu$ on $I$ (i.e. $\mu(\epsilon) < 1$), is automatically transient, which means that

$$\sum_{n=1}^{\infty} p_n(x, y) < \infty$$

for all $x, y \in I$. This statement can be proven in the same was as [NT04, Theorem 2.6]. For the convenience of the reader, we give the argument. Denote by $\dim_{\min}(y)$ the dimension of the carrier Hilbert space of $y$, when $y$ is viewed as an irreducible representation of $A_u(I_2)$. Since $F$ is not a multiple of a unitary $2 \times 2$ matrix, it follows that $\dim_q(y) > \dim_{\min}(y)$ for all $y \in I \setminus \{\epsilon\}$. Denote by $\mult(z; y_1 \otimes \cdots \otimes y_n)$ the multiplicity of the irreducible representation $z \in I$ in the tensor product of the irreducible representations $y_1, \ldots, y_n$. Since the fusion rules of $A_u(F)$ and $A_u(I_2)$ are identical, it follows that

$$\mult(z; y_1 \otimes \cdots \otimes y_n) \leq \dim_{\min}(y_1) \cdots \dim_{\min}(y_n).$$

One then computes, for all $x, y \in I$, $n \in \mathbb{N}$,

$$p_n(x, y) = \sum_{z \subseteq \pi \otimes y} \mu^n(z) \frac{\dim_q(y)}{\dim_q(x) \dim_q(z)} = \frac{\dim_q(y)}{\dim_q(x)} \sum_{z \subseteq \pi \otimes y} \sum_{y_1, \ldots, y_n \in I} \mult(z; y_1 \otimes \cdots \otimes y_n) \frac{\mu(y_1) \cdots \mu(y_n)}{\dim_q(y_1) \cdots \dim_q(y_n)} \leq \frac{\dim_q(y)}{\dim_q(x)} \dim(\pi \otimes y) \rho^n$$

where $\rho = \sum_{y \in I} \mu(y)(\dim_{\min}(y)/\dim_q(y))$. Since $\mu$ is non-trivial and $F$ is not a multiple of a $2 \times 2$ unitary matrix, we have $0 < \rho < 1$. Transience of the random walk follows immediately.

An element $x \in I$ is said to be indecomposable if $x = y \otimes z$ implies $y = \epsilon$ or $z = \epsilon$. Equivalently, $x$ is an alternating product of the letters $\alpha$ and $\beta$.

For every $x \in I$, we denote by $\dim_q(x)$ the quantum dimension of the irreducible representation labeled by $x$. Since $\dim_q(\alpha) > 2$, take $0 < q < 1$ such that $\dim_q(\alpha) = \dim_q(\beta) = q + 1/q$. An important part of the proof of Theorem 1.3 is based on the technical estimates provided by Lemma A.1 and they require $q < 1$, i.e. $\dim_q(\alpha) > 2$.

Denote the $q$-numbers

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}.$$ 

Writing $x = x_1 \otimes \cdots \otimes x_n$ where the words $x_1, \ldots, x_n$ are indecomposable, we have

$$\dim_q(x) = [|x_1| + 1]_q \cdots [|x_n| + 1]_q.$$ 

For later use, note that it follows that

$$\dim_q(xy) \geq q^{-|y|} \dim_q(x)$$ 

for all $x, y \in I$.

Whenever $x \in I \cup \partial I$, we denote by $[x]_n$ the word consisting of the first $n$ letters of $x$ and by $[x]^n$ the word that arises by removing the first $n$ letters from $x$. So, by definition, $x = [x]_n [x]^n$. 

1078
2. Poisson boundary of the classical random walk on \( \text{Irred}(\mathcal{G}) \)

Given a probability measure \( \mu \) on \( I := \text{Irred}(\mathcal{G}) \), the Markov operator \( P_\mu : \ell^\infty(\widehat{\mathcal{G}}) \to \ell^\infty(\widehat{\mathcal{G}}) \) preserves the center \( Z(\ell^\infty(\widehat{\mathcal{G}})) = \ell^\infty(I) \) and, hence, defines an ordinary random walk on the countable set \( I \) with \( n \)-step transition probabilities

\[
\rho_n(x, y) = P_\mu^n(p_{xy}).
\]

As shown above, this random walk is transient whenever \( \mu(\epsilon) < 1 \). Denote by \( H^\infty_{\text{centr}}(\widehat{\mathcal{G}}, \mu) \) the commutative von Neumann algebra of bounded \( P_\mu \)-harmonic functions in \( \ell^\infty(I) \), with product given by \( a \cdot b = \lim_n P_\mu^n(ab) \) and the sequence being strongly* convergent. Write \( p(x, y) = p_1(x, y) \).

The set \( I \) becomes in a natural way a tree: the Cayley tree of the semi-group \( \mathbb{N} \ast \mathbb{N} \). Let \( \mu \) be a generating probability measure on \( I \) with finite support.

**Lemma 2.1.** There exists a \( \delta > 0 \) such that \( p(x, y) > 0 \) implies that \( p(x, y) \geq \delta \).

**Proof.** Take \( L, \delta_0 > 0 \) such that for all \( z \in \text{supp} \mu \), we have \( |z| \leq L \) and \( \mu(z) \geq \delta_0 \). By (1), if \( p(x, y) > 0 \), we obtain \( z \) with \( |z| \leq L \), \( y \in x \otimes z \) and

\[
p(x, y) \geq \delta_0 \frac{\text{dim}_q(y)}{\text{dim}_q(x) \text{dim}_q(z)}.
\]

Write \( x = x_0v_1 \), \( z = \tau_1z_1 \) and \( y = x_0z_1 \). Put \( \eta = q + 1/q \). Then,

\[
p(x, y) \geq \delta_0 \frac{\text{dim}_q(x_0)}{\text{dim}_q(x_0) \eta^{|x_0|} |z_1|} \geq \delta_0 \eta^{-2L}.
\]

So, we can put \( \delta = \delta_0 \eta^{-2L} \).

The following properties of the random walk on \( I \) can be checked easily.

- Uniform irreducibility: there exists an integer \( M \) such that, for any pair \( x, y \in I \) of neighboring edges of the tree, there exists an integer \( k \leq M \), such that \( p_k(x, y) > 0 \).
- Bounded step-length: there exists an integer \( N \) such that \( p(x, y) > 0 \) implies that \( d(x, y) \leq N \) where \( d(x, y) \) equals the length of the unique geodesic path from \( x \) to \( y \).

Combining these remarks with Lemma 2.1, we can apply [PWS87, Theorem 2] and identify the Poisson boundary of the random walk on \( I \), with the boundary \( \partial I \) of infinite words in \( \alpha, \beta \), equipped with a probability measure in the following way.

**Theorem 2.2** (Picardello and Woess [PWS87, Theorem 2]). Let \( \mu \) be a finitely supported generating measure on \( I = \text{Irred}(A_u(F)) \), where \( F \) is not a scalar multiple of a \( 2 \times 2 \) unitary matrix. Consider the associated random walk on \( I \) with transition probabilities given by (6) and the compactification \( I \cup \partial I \) of \( I \).

- The random walk converges almost surely to a point in \( \partial I \).
- For every \( x \in I \), denote by \( \nu_x \) the hitting probability measure on \( \partial I \), where \( \nu_x(U) \) is defined as the probability that the random walk starting in \( x \) converges to a point in \( U \). Then, the formula

\[
\Upsilon(F)(x) = \int_{\partial I} F(z) \, d\nu_x(z)
\]

defines a *-isomorphism \( \Upsilon : L^\infty(\partial I, \nu) \to H^\infty_{\text{centr}}(\widehat{\mathcal{G}}, \mu) \).

1079
In fact, [PW87, Theorem 2], identifies \( \partial I \) with the *Martin compactification* of the given random walk on \( I \). It is a general fact (see [Woe00, Theorem 24.10]), that a transient random walk converges almost surely to a point of the minimal Martin boundary and that the hitting probability measures provide a realization of the Poisson boundary through the Poisson integral formula (7), see [Woe00, Theorem 24.12].

Since a continuous function on the compact space \( I \cup \partial I \) is entirely determined by its values on \( I \), we can and do view \( C(I \cup \partial I) \) as a \( C^* \)-subalgebra of \( \ell^\infty(I) \).

The rest of this section is devoted to the proof of the non-atomicity of the harmonic measures \( \nu_x \).

**Lemma 2.3.** For all \( x, y \in I \) and \( z \in \partial I \), the sequence

\[
\left( \frac{\dim_q(x|z)_n}{\dim_q(y|z)_n} \right)_n
\]

converges. By a slight abuse of notation, we denote the limit by \( \dim_q(xz/yz) \). The following properties hold.

(i) For all \( x, y \in I \), the map \( \partial I \to \mathbb{R}_+ : z \mapsto \dim_q(xz/yz) \) is continuous.

(ii) For all \( x, y \in I \) and \( w \in \partial I \), the sequence of continuous functions

\[
\partial I \to \mathbb{R}_+ : z \mapsto \dim_q\left( \frac{x[w]|z}{y[w]|z} \right)
\]

converges uniformly on \( \partial I \) to the constant function \( \dim_q(xw/yw) \).

**Proof.** Fix \( x, y \in I \). Whenever \( z \in \partial I \) and \( n \in \mathbb{N} \), denote

\[
f_n(z) = \frac{\dim_q(x|z)_n}{\dim_q(y|z)_n}.
\]

If \( z \notin \{ \alpha \beta \alpha \cdots, \beta \alpha \beta \cdots \} \), write \( z = z_1 \otimes z_2 \) for some \( z_1 \in I \), \( z_1 \neq \epsilon \) and some \( z_2 \in \partial I \). Denote by \( \mathcal{U} \) the neighborhood of \( z \) consisting of words of the form \( z_1 z' = z_1 \otimes z' \). For all \( s \in \mathcal{U} \) and all \( n \geq |z_1| \), we have

\[
f_n(s) = \frac{\dim_q(xz_1)}{\dim_q(yz_1)}.
\]

Hence, for all \( s \in \mathcal{U} \), the sequence \( n \mapsto f_n(s) \) is eventually constant and converges to a limit that is constant on \( \mathcal{U} \).

Also for \( z \in \{ \alpha \beta \alpha \cdots, \beta \alpha \beta \cdots \} \), the sequence \( f_n(z) \) is convergent. Take \( z = \alpha \beta \alpha \cdots \). Write \( x = x_0 \otimes x_1 \) where \( x_1 \) is the longest possible (and maybe empty) indecomposable word ending with \( \beta \). Write \( y = y_0 \otimes y_1 \) similarly. It follows that

\[
f_n(x) = \frac{\dim_q(x_0)}{\dim_q(y_0)} \left[ n + |x_1| + 1 \right]_q \to \frac{\dim_q(x_0)}{\dim_q(y_0)} q^{\left[ y_1 \right] - |x_1|}.
\]

The convergence of \( f_n(z) \) for \( z = \beta \alpha \beta \cdots \) is proven analogously.

Write \( f(z) = \lim_n f_n(z) \). We have seen above that every \( z \in \partial I \), \( z \notin \{ \alpha \beta \alpha \cdots, \beta \alpha \beta \cdots \} \), has a neighborhood on which \( f \) is constant. We now prove that \( f \) is also continuous in \( z = \alpha \beta \alpha \cdots \) and in \( z = \beta \alpha \beta \cdots \). In both cases, define, for every \( n \in \mathbb{N} \), the neighborhood \( \mathcal{U}_n \) of \( z \) consisting of all \( s \in \partial I \) with \( [s]_n = [z]_n \). For every \( s \in \mathcal{U}_n \), \( s \neq z \), there exists \( m \geq n \) such that \( f(s) = f_m(z) \). The continuity of \( f \) in \( z \) follows and we have proven statement (i).
It remains to prove statement (ii). If \( w \) is decomposable, i.e. \( w = w_0 \otimes w_1 \) with \( |w_0| \geq 1 \), then for all \( n > |w_0| \), we have
\[
\dim_q \left( \frac{x[w]_n z}{y[w]_n z} \right) = \dim_q(xw_0) / \dim_q(yw_0)
\]
and hence statement (ii) follows. If \( w \) is indecomposable, let us assume that \( w = \alpha \beta \alpha \cdots \); the case \( w = \beta \alpha \beta \cdots \) is analogous. Write \( x = x_0 \otimes x_1 \) and \( y = y_0 \otimes y_1 \), where \( x_1, y_1 \) are maximal, possibly empty, indecomposable words ending with the letter \( \beta \). If \( z \) is indecomposable, the expression \( \dim_q(x[w]_n z/y[w]_n z) \) is alternatingly equal to
\[
\dim_q(x_0) \left( |x_1| + n + 1 \right)_q \; \dim_q(y_0) \left( |y_1| + n + 1 \right)_q
\]
and
\[
\dim_q(x_0) \left( |x_1| + n + |z_0| + 1 \right)_q \; \dim_q(y_0) \left( |y_1| + n + |z_0| \right)_q.
\]
When \( z = z_0 \otimes z_1 \) where \( z_0 \) is an indecomposable word with length at least 1, the expression \( \dim_q(x[w]_n z/y[w]_n z) \) is alternatingly equal to
\[
\dim_q(x_0) \left( |x_1| + n + 1 \right)_q \; \dim_q(y_0) \left( |y_1| + n + 1 \right)_q
\]
and
\[
\dim_q(x_0) \left( |x_1| + n + |z_0| + 1 \right)_q \; \dim_q(y_0) \left( |y_1| + n + |z_0| \right)_q.
\]
Since the four expressions appearing in (8) and (9) converge uniformly in \( z \), to
\[
\frac{\dim_q(x_0)}{\dim_q(y_0)} q^{|y_1|-|x_1|} = \dim_q \left( \frac{x w}{y w} \right)
\]
when \( n \to \infty \), statement (ii) is proven. \( \Box \)

Whenever \( x, y \in I \cup \partial I \), define \( (x|y) := \max \{n \mid [x]_n = [y]_n \} \).

**Lemma 2.4.** Let \( x, z \in I \) with \( |x| \leq |z| \). Denote by \( U_z \) the subset of \( \partial I \) consisting of infinite words that start with \( z \). For every \( 0 \leq k \leq (x|z) \), define the function \( f_k \in C(\partial I) \) with support \( U_{[x] [z] [z] \cdots} \), given by
\[
f_k([x] [z] y) = \frac{1}{\dim_q(x)} \dim_q \left( \frac{z y}{[x] [z] k} \right).
\]
We then have
\[
\nu_x(U_z) = \sum_{k=0}^{(x|z)} \int_{\partial I} f_k(y) \, d\nu_x(y).
\]
Moreover, for all \( w \in \partial I \), we have
\[
\nu_x(\{w\}) = \frac{1}{\dim_q(x)} \sum_{k=0}^{(x|z)} \dim_q \left( \frac{[w] k [w] k}{[x] [z] k} \right) \nu_x(\{[x] [z] [w] k\}).
\]
**Proof.** By Lemma 2.3, the functions \( f_k \) are well defined and belong to \( C(\partial I) \). By Theorem 2.2, our random walk converges almost surely to a point of \( \partial I \) and we denoted by \( \nu_x \) the hitting probability measure. So, \( (\psi_x \otimes \psi_{\mu^n} \Delta) \to \nu_x \) weakly* in \( C(I \cup \partial I)^* \).

Recall that \( E : \ell^\infty(\hat{G}) \to \ell^\infty(I) \) denotes the conditional expectation defined by \( E(b)p_y = \psi_y(b)p_y \). Whenever \( |z| \geq |x| \), we have
\[
E((\psi_x \otimes \text{id}) \Delta(p_z)) = \sum_{k=0}^{(x|z)} \frac{\dim_q(z)}{\dim_q(x) \dim_q([x] [z] k)} p_{[x] [z] k}.
\]
Denote \( q_z = \sum_{s \in I} p_{zs} \) and observe that \( q_z \in C(I \cup \partial I) \). It follows that for all \( |z| \geq |x| \),
\[
\mathcal{E}((\psi_x \otimes \text{id}) \Delta(q_z)) = \sum_{k=0}^{(x|z)} F_k
\]
where \( F_k \in \ell^\infty(I) \) is defined by \( F_k(y) = 0 \) if \( y \) does not start with \( [x]^k[z]^k \) and
\[
F_k([x]^k[z]^k y) = \frac{1}{\dim_q(x)} \frac{\dim_q(z y)}{\dim_q([x]^k[z]^k y)}.
\]

Note that \( F_k \in C(I \cup \partial I) \subset \ell^\infty(I) \) and that \( F_k \) is a continuous extension of \( f_k \). Hence, it follows that, for \( |z| \geq |x| \),
\[
\nu_x(\mathcal{U}_z) = \sum_{k=0}^{(x|z)} \int_{\partial I} f_k(y) \, d\nu_x(y).
\]

Finally, let \( w \in \partial I \). Write \( w = w_0 w_1 \), where \( |w_0| \geq |x| \). Let \( n \in \mathbb{N} \). We apply the above formula to \( z = w_0[w_1]_n \). Since \( \mathcal{U}_{w_0[w_1]_n} \) decreases to \( \{w\} \), we have
\[
\nu_x(\mathcal{U}_{w_0[w_1]_n}) \to \nu_x(\{w\}).
\]

On the other hand, because \( (x|w_0[w_1]_n) = (x|w_0) \), we have
\[
\nu_x(\mathcal{U}_{w_0[w_1]_n}) = \sum_{k=0}^{(x|w_0)} g_k^n(y) \, d\nu_x(y),
\]
where \( g_k^n \in C(\partial I) \) is supported on the words that start with \( [x]^k[w_0]^k[w_1]_n \) and is given by
\[
g_k^n([x]^k[w_0]^k[w_1]_n y) = \frac{1}{\dim_q(x)} \dim_q \left( \frac{w_0[w_1]_n y}{[x]^k[w_0]^k[w_1]_n y} \right).
\]

By Lemma 2.3(ii), when \( n \to \infty \), the right-hand side of this last expression converges uniformly in \( y \) to
\[
\frac{1}{\dim_q(x)} \dim_q \left( \frac{w_0[w_1]_n}{[x]^k[w_0]^k[w_1]_n} \right) = \frac{1}{\dim_q(x)} \dim_q \left( \frac{[w]^k}{[x]^k} \right).
\]

Since \( \mathcal{U}_{[x]^k[w_0]^k[w_1]_n} \) decreases to \( \{[x]^k[w]^k\} \) and since \( (x|w) = (x|w_0) \), the lemma is proven. \( \Box \)

**Proposition 2.5.** The support of the harmonic measure \( \nu_x \) is the whole of \( \partial I \). The harmonic measure \( \nu_x \) has no atoms in words ending with \( \alpha \beta \alpha \beta \cdots \).

**Remark 2.6.** The same methods as in the proof of Proposition 2.5 given below, but involving more tedious computations, show in fact that \( \nu_x \) is non-atomic. To prove our main theorem, it is only crucial that \( \nu_x \) has no atoms in words ending with \( \alpha \beta \alpha \beta \cdots \). We believe that it should be possible to give a more conceptual proof of the non-atomicity of \( \nu_x \) and refer to [Van08, Proposition 8.3.10] for an ad hoc proof along the lines of the proof of Proposition 2.5.

**Proof of Proposition 2.5.** In order to prove that the support of \( \nu_x \) is the whole of \( \partial I \), it suffices to show that \( \nu_x(\mathcal{U}_z) > 0 \) for all \( z \in I \). Since \( \nu_x \) and \( \nu_z \) are absolutely continuous, it suffices to show that \( \nu_z(\mathcal{U}_z) > 0 \) for all \( z \in I \). By Lemma 2.4, we have
\[
\nu_z(\mathcal{U}_z) \geq \frac{1}{\dim_q(x)} \int_{\partial I} \dim_q \left( \frac{zy}{y} \right) \, d\nu_x(y).
\]

Since the integral of a strictly positive function is strictly positive, it follows that \( \nu_z(\mathcal{U}_z) > 0 \).
Poisson boundary of the discrete quantum group $A_u(F)$

Owing to Lemma 2.4 and the equality
\[ \nu_\epsilon = \sum_{x \in I} \mu^k(x) \nu_x \]
for all $k \geq 1$, we observe that if $w$ is an atom for $\nu_\epsilon$, then all $w'$ with the same tail as $w$ are atoms for all $\nu_x$, $x \in I$. So, we assume that $w := \alpha\beta\alpha\beta \cdots$ is an atom for $\nu_\epsilon$ and derive a contradiction.

Denote by $\delta_w$ the function on $\partial I$ that is equal to one in $w$ and zero elsewhere. Using the $*$-isomorphism in Theorem 2.2, it follows that the bounded function
\[ \xi \in \ell^\infty(\widehat{G}) : \xi(x) := \nu_x(\{w\}) = \int_{\partial I} \delta_w d\nu_x \]
is harmonic.

We prove that $\xi$ attains its maximum and apply the maximum principle for irreducible random walks (see, e.g., [Woe00, Theorem 1.15]) to deduce that $\xi$ must be constant. This will lead to a contradiction.

Denote
\[ w_n^\alpha := \alpha\beta\alpha \cdots \quad \text{and} \quad w_n^\beta := \beta\alpha\beta \cdots. \]
Note that all elements of $I$ are either of the form
\[ w_{2n+1}^\alpha x \quad \text{where} \quad n \in \mathbb{N} \quad \text{and} \quad x \in \{\epsilon\} \cup \alpha I \]
or of the form
\[ w_{2n}^\alpha x \quad \text{where} \quad n \in \mathbb{N} \quad \text{and} \quad x \in \{\epsilon\} \cup \beta I. \]

By Lemma 2.4 and formula (4), we obtain that for $n \in \mathbb{N}$ and $x \in \{\epsilon\} \cup \alpha I$,
\[ \xi(w_{2n+1}^\alpha x) = \sum_{k=0}^{2n+1} \frac{1}{2(n+1)!} q^{-\dim_q(x)^2} \dim_q \left( \frac{w_k^\alpha}{w_{2n+1-k}^\beta \dim_q(x)^2} \nu_x(\pi\alpha\beta\cdots) \right) \]
\[ = \sum_{k=0}^{2n+1} \frac{1}{2(n+1)!} q^{-\dim_q(x)^2} q^{2(n-k)+1} \nu_x(\pi\alpha\beta\cdots) = \frac{\nu_x(\pi\alpha\beta\cdots)}{\dim_q(x)^2}. \]
Since $\nu_\epsilon$ is a probability measure, it follows that $x \mapsto \xi(w_{2n+1}^\alpha x)$ is independent of $n$ and summable over the set $\{\epsilon\} \cup \alpha I$. Analogously, it follows that $x \mapsto \xi(w_{2n}^\alpha x)$ is independent of $n$ and summable over the set $\{\epsilon\} \cup \beta I$. As a result, $\xi$ attains its maximum on $I$. By the maximum principle, $\xi$ is constant. Since $\xi(\epsilon) \neq 0$, this constant is non-zero and we arrive at a contradiction with the summability of $x \mapsto \xi(w_{2n+1}^\alpha x)$ over the infinite set $\{\epsilon\} \cup \alpha I$.

3. Topological boundary and boundary action for the dual of $A_u(F)$

Before proving Theorem 1.3, we construct a compactification for $\widehat{G}$, i.e. a unital C*-algebra $\mathcal{B}$ lying between $c_0(\widehat{G})$ and $\ell^\infty(\widehat{G})$. This C*-algebra $\mathcal{B}$ is a non-commutative version of $C(I \cup \partial I)$. The construction of $\mathcal{B}$ follows word by word the analogous construction given in [VV07, §3] for $G = A_u(F)$. So, we only indicate the necessary modifications.

For all $x, y \in I$ and $z \subset x \otimes y$, we choose an isometry $V(x \otimes y, z) \in \text{Mor}(x \otimes y, z)$. Since $z$ appears with multiplicity one in $x \otimes y$, the isometry $V(x \otimes y, z)$ is uniquely determined up to multiplication by a scalar $\lambda \in S^1$. Therefore, the following unital completely positive maps are uniquely defined (cf. [VV07, Definition 3.1]).
Definition 3.1. Let \( x, y \in I \). We define unital completely positive maps
\[
\psi_{xy,x} : \mathcal{L}(H_x) \to \mathcal{L}(H_{xy}) : \psi_{xy,x}(A) = V(x \otimes y, xy)^*(A \otimes 1)V(x \otimes y, xy).
\]

Theorem 3.2. The maps \( \psi_{xy,x} \) form an inductive system of completely positive maps. Defining
\[
\mathcal{B} = \{ a \in \ell^\infty(\hat{G}) \mid \forall \varepsilon > 0, \ \exists n \in \mathbb{N} \text{ such that } \| ap_{xy} - \psi_{xy,x}(ap_x) \| < \varepsilon
\]
for all \( x, y \in I \) with \( |x| \geq n \),
we get that \( \mathcal{B} \) is a unital \( C^* \)-subalgebra of \( \ell^\infty(\hat{G}) \) containing \( c_0(\hat{G}) \).

- The restriction of the comultiplication \( \Delta \) yields a left action \( \beta_G \) of \( \hat{G} \) on \( \mathcal{B} \):
\[
\beta_G : \mathcal{B} \to M(c_0(\hat{G}) \otimes \mathcal{B}) : a \mapsto \Delta(a).
\]

- The restriction of the adjoint action of \( G \) on \( \ell^\infty(\hat{G}) \) yields a right action of \( G \) on \( \mathcal{B} \):
\[
\beta_G : \mathcal{B} \to \mathcal{B} \otimes C(G) : a \mapsto \forall (a \otimes 1)\forall^*.
\]

Here, \( \forall \in \ell^\infty(\hat{G}) \otimes L^\infty(G) \) is defined as \( \forall \sum_{x \in I} U_x^x \). The action \( \beta_G \) is continuous in the sense that \( \text{span} \beta_G(\mathcal{B})(1 \otimes C(G)) \) is dense in \( \mathcal{B} \otimes C(G) \).

Proof. One can repeat word by word the proofs of [VV07, Propositions 3.4 and 3.6]. The crucial ingredients of these proofs are the approximate commutation formulae provided by [VV07, Lemmas A.1 and A.2] and they have to be replaced by the inequalities provided by Lemma A.1. \( \Box \)

We denote \( \mathcal{B}_\infty := \mathcal{B}/c_0(\hat{G}) \) and call it the topological boundary of \( \hat{G} \). Both actions \( \beta_G \) and \( \beta_G \) preserve the ideal \( c_0(\hat{G}) \) and hence yield actions on \( \mathcal{B}_\infty \) that we still denote by \( \beta_G \) and \( \beta_G \).

As before, we view \( C(I \cup \partial I) \subset \ell^\infty(I) \) by restricting continuous functions on \( I \cup \partial I \) to \( I \). A bounded function on \( I \) extends continuously to \( I \cup \partial I \) if and only if, for every \( \varepsilon > 0 \), there exists an \( n \in \mathbb{N} \) such that \( |f(xy) - f(x)| < \varepsilon \) for all \( x, y \in I \) with \( |x| \geq n \). Hence, when viewing \( C(I \cup \partial I) \) as a \( C^* \)-subalgebra of \( \ell^\infty(I) \), we obtain \( C(I \cup \partial I) = \mathcal{B} \cap \mathcal{Z}(\ell^\infty(\hat{G})) = \mathcal{B} \cap \ell^\infty(I) \). Taking the quotient with \( c_0(I) \), we view \( C(\partial I) \subset \mathcal{B}_\infty \).

We partially order \( I \) by writing \( x \leq y \) if \( y = xz \) for some \( z \in I \). Define
\[
\psi_{\infty,x} : \mathcal{L}(H_x) \to \mathcal{B} : \psi_{\infty,x}(A)p_y = \begin{cases} \psi_{y,x}(A) & \text{if } y \geq x \\ 0 & \text{otherwise}. \end{cases}
\]

We use the same notation for the composition of \( \psi_{\infty,x} \) with the quotient map \( \mathcal{B} \to \mathcal{B}_\infty \), yielding the map \( \psi_{\infty,x} \in \mathcal{L}(H_x) \to \mathcal{B}_\infty \).

Observe that the linear span of all \( \psi_{\infty,x}(\mathcal{L}(H_x)) \) is dense in \( \mathcal{B}_\infty \). Indeed, whenever \( a \in \mathcal{B} \) and \( \varepsilon > 0 \), we can take \( n \in \mathbb{N} \) such that \( \| ap_{xy} - \psi_{xy,x}(ap_x) \| \leq \varepsilon \) whenever \( |x| \geq n \). If \( x_1, \ldots, x_m \) is an enumeration of all elements in \( I \) of length \( n \), it follows that
\[
\left\| \pi(a) - \sum_{k=1}^m \psi_{\infty,x_k}(ap_{x_k}) \right\| \leq \varepsilon.
\]

Lemma 3.3. The inclusion \( C(\partial I) \subset \mathcal{B}_\infty \) defines a continuous field of unital \( C^* \)-algebras. For every \( x \in \partial I \), denote by \( J_x \) the closed two-sided ideal of \( \mathcal{B}_\infty \) generated by the functions in \( C(\partial I) \) vanishing in \( x \).
For every $x = x_1 \otimes x_2 \otimes \cdots$ in $\partial_0 I$, there exists a unique surjective $*$-homomorphism

$$\pi_x : \mathcal{B}_\infty \to \bigotimes_{k=1}^{\infty} \mathcal{L}(H_{x_k})$$

satisfying $\text{Ker} \, \pi_x = J_x$ and $\pi_x(\psi_{\infty,x_1\cdots x_n}(A)) = A \otimes 1$ for all $A \in \bigotimes_{k=1}^{n} \mathcal{L}(H_{x_k}) = \mathcal{L}(H_{x_1\cdots x_n})$.

**Proof.** Given $x \in \partial I$, define the decreasing sequence of projections $e_n \in \mathcal{B}$ given by

$$e_n := \sum_{y \in I} p_{\pi_n}y.$$ 

Denote by $\pi : \mathcal{B} \to \mathcal{B}_\infty$ the quotient map. It follows that

$$\|\pi(a) + J_x\| = \lim_n \|ae_n\|$$

for all $a \in \mathcal{B}$.

To prove that $C(\partial I) \subset \mathcal{B}_\infty$ is a continuous field, let $y \in I, A \in \mathcal{L}(H_y)$ and define $a \in \mathcal{B}$ by $a := \psi_{\infty,y}(A)$. Put $f : \partial I \to \mathbb{R}_+ : f(x) = \|\pi(a) + J_x\|$. We have to prove that $f$ is a continuous function. Define $\mathcal{U} \subset \partial I$ consisting of infinite words starting with $y$. Then, $\mathcal{U}$ is open and closed and $f$ is zero, in particular continuous, on the complement of $\mathcal{U}$. Assume that the last letter of $y$ is $\alpha$ (the other case, of course, being analogous). If $x \in \mathcal{U}$ and $x \neq y\beta\alpha\beta\alpha \cdots$, write $x = yz \otimes u$ for some $z \in I$, $u \in \partial I$. Define $\mathcal{N}$ as the neighborhood of $x$ consisting of infinite words of the form $yzu'$ where $u' \in \partial I$ and $yzu = yz \otimes u'$. Then, $f$ is constantly equal to $\|\psi_{y\beta\alpha\beta\alpha}(A)\|$ on $\mathcal{N}$. It remains to prove that $f$ is continuous in $x := y\beta\alpha\beta\alpha \cdots$. Let

$$w_n = \beta\alpha\beta\alpha \cdots \text{ letters}.$$ 

Then, the sequence $\|\psi_{yw_n\, y}(A)\|$ is decreasing and converges to $f(x)$. If $\mathcal{U}_n$ is the neighborhood of $x$ consisting of words starting with $yw_n$, it follows that

$$f(x) \leq f(u) \leq \|\psi_{yw_n\, y}(A)\|$$

for all $u \in \mathcal{U}_n$. This proves the continuity of $f$ in $x$. So, $C(\partial I) \subset \mathcal{B}_\infty$ is a continuous field of $C^*$-algebras.

Let now $x = x_1 \otimes x_2 \otimes \cdots$ be an element of $\partial_0 I$. Put $y_n = x_1 \cdots \otimes x_n$ and

$$f_n := \sum_{z \in I} p_{y_nz}.$$ 

The map $A \mapsto f_{n+1}\psi_{\infty, y_n}(A)$ defines a unital $*$-homomorphism from $\mathcal{L}(H_{y_n})$ to $f_{n+1}\mathcal{B}$. Since $\pi(1 - f_{n+1}) \in J_x$, we obtain the unital $*$-homomorphism $\theta_n : \mathcal{L}(H_{y_n}) \to \mathcal{B}_\infty / J_x$. The $*$-homomorphisms $\theta_n$ are compatible and combine into the unital $*$-homomorphism

$$\theta : \bigotimes_{k=1}^{\infty} \mathcal{L}(H_{x_k}) \to \mathcal{B}_\infty / J_x.$$ 

By (10), $\theta$ is isometric. Since the union of all $\psi_{\infty, y_n}(\mathcal{L}(H_{y_n}))) + J_x$, $n \in \mathbb{N}$, is dense in $\mathcal{B}_\infty$, it follows that $\theta$ is surjective. The composition of the quotient map $\mathcal{B}_\infty \to \mathcal{B}_\infty / J_x$ and the inverse of $\theta$ provides the required $*$-homomorphism $\pi_x$. $\square$
We prove Theorem 1.3 by performing the following steps.

- Construct on the boundary $\mathcal{B}_\infty$ of $\hat{G}$, a faithful Kubo–Martin–Schwinger (KMS) state $\omega_\infty$, to be considered as the harmonic state and satisfying $(\psi_\mu \otimes \omega_\infty)\beta_G = \omega_\infty$. Extend $\beta_G$ to an action

$$\beta_G : (\mathcal{B}_\infty, \omega_\infty)^{''} \to \ell^\infty(\hat{G}) \otimes (\mathcal{B}_\infty, \omega_\infty)^{''}$$

and denote by $\Theta_\mu := (\text{id} \otimes \omega_\infty)\beta_G$ the Poisson integral.

- Prove a quantum Dirichlet property: for all $a \in \mathcal{B}$, we have $\Theta_\mu(a) - a \in c_0(\hat{G})$. It will follow that $\Theta_\mu$ is a normal and faithful $*$-homomorphism of $(\mathcal{B}_\infty, \omega_\infty)^{''}$ onto a von Neumann subalgebra of $H^\infty(\hat{G}, \mu)$.

- By Theorem 2.2, $\Theta_\mu$ is a $*$-isomorphism of $L^\infty(\partial I, \nu_t) \subset (\mathcal{B}_\infty, \omega_\infty)^{''}$ onto $H^\infty(\hat{G}, \mu)$. Deduce that the image of $\Theta_\mu$ is the whole of $H^\infty(\hat{G}, \mu)$.

- Use Lemma 3.3 to identify $(\mathcal{B}_\infty, \omega_\infty)^{''}$ with a field of ITPFI factors.

**Proposition 4.1.** The sequence $\psi_\mu^{\ast n}$ of states on $\mathcal{B}$ converges weakly* to a KMS state $\omega_\infty$ on $\mathcal{B}$. The state $\omega_\infty$ vanishes on $c_0(\hat{G})$. We still denote by $\omega_\infty$ the resulting KMS state on $\mathcal{B}_\infty$. Then, $\omega_\infty$ is faithful on $\mathcal{B}_\infty$.

We have $(\psi_\mu \otimes \omega_\infty)\beta_G = \omega_\infty$, so that we can uniquely extend $\beta_G$ to an action

$$\beta_G : (\mathcal{B}_\infty, \omega_\infty)^{''} \to \ell^\infty(\hat{G}) \otimes (\mathcal{B}_\infty, \omega_\infty)^{''}$$

that we still denote by $\beta_G$.

The state $\omega_\infty$ is invariant under the action $\beta_G$ of $\hat{G}$ on $\mathcal{B}_\infty$. We extend $\beta_G$ to an action on $(\mathcal{B}_\infty, \omega_\infty)^{''}$ that we still denote by $\beta_G$.

The normal, completely positive map

$$\Theta_\mu : (\mathcal{B}_\infty, \omega_\infty)^{''} \to H^\infty(\hat{G}, \mu) : \Theta_\mu = (\text{id} \otimes \omega_\infty)\beta_G$$

(11)

is called the Poisson integral. It satisfies the following properties (recall that $\alpha_G$ and $\alpha_\hat{G}$ were introduced in Definition 1.2):

- $\hat{c} \circ \Theta_\mu = \omega_\infty$;
- $(\Theta_\mu \otimes \text{id}) \circ \beta_G = \alpha_G \circ \Theta_\mu$;
- $(\text{id} \otimes \Theta_\mu) \circ \beta_G = \alpha_\hat{G} \circ \Theta_\mu$.

For every $x = x_1 \otimes x_2 \otimes \cdots$ in $\partial I$, denote by $\omega_x$ the infinite tensor product state on $\bigotimes_{k=1}^\infty \mathcal{L}(H_{x_k})$, of the states $\psi_{x_k}$ on $\mathcal{L}(H_{x_k})$. Using the notation $\pi_x$ of Lemma 3.3, we have

$$\omega_\infty(a) = \int_{\partial_0 I} \omega_x(\pi_x(a)) \, d\nu_t(x)$$

(12)

for all $a \in \mathcal{B}_\infty$.

**Proof.** Define the one-parameter group of automorphisms $(\sigma_t)_{t \in \mathbb{R}}$ of $\ell^\infty(\hat{G})$ given by

$$\sigma_t(p_x) = Q^t_x a p_x Q_{x^{-t}}^t.$$ 

Since $\sigma_t(\psi_{\infty,x}(A)) = \psi_{\infty,x}(Q^t_x A Q_{x^{-t}}^t)$, it follows that $(\sigma_t)$ is norm-continuous on the $C^*$-algebra $\mathcal{B}$. 

1086
By Theorem 2.2, the sequence of probability measures $\mu^{*n}$ on $I \cup \partial I$ converges weakly* to $\nu$. It follows that $\psi_{\mu^{*n}}(a) \to 0$ whenever $a \in C_0(\hat{G})$. Given $x \in I$ and $A \in \mathcal{L}(H_x)$, put $a := \psi_{x, x}(A)$. As before, denote by $\mathcal{U}_x$ the set of infinite words starting with $x$ and by $\mathcal{U}_x^0$ its intersection with $\partial_0(I)$. Using Proposition 2.5, we obtain

$$\psi_{\mu^{*n}}(a) = \sum_{y \in I} \mu^{*n}(y)\psi_y(\psi_{x, x}(A)) = \sum_{y \in x_I} \mu^{*n}(y)\psi_x(A) \to \psi_x(A)\mu_\nu(\mathcal{U}_x^0) = \psi_x(A)\nu(\mathcal{U}_x^0) = \int_{\partial_0 I} \omega_y(\pi_y(a))\,d\nu(y).$$

So, the sequence $\psi_{\mu^{*n}}$ of states on $\mathcal{B}$ converges weakly* to a state on $\mathcal{B}$ that we denote by $\omega_\infty$ and that satisfies (12). Since all $\psi_{\mu^{*n}}$ satisfy the KMS condition with respect to $(\sigma_t)$, also $\omega_\infty$ is a KMS state. If $a \in B_\infty^+$ and $\omega_\infty(a) = 0$, it follows from (12) that $\omega_\infty(\pi_x(a)) = 0$ for $\nu_\epsilon$-almost every $x \in \partial_0 I$. Since $\omega_\infty$ is faithful, it follows that $\|\pi(a) + J_x\| = 0$ for $\nu_\epsilon$-almost every $x \in \partial_0 I$. By Proposition 2.5, the support of $\nu_\epsilon$ is the whole of $\partial I$ and by Lemma 3.3, $x \mapsto \|\pi(a) + J_x\|$ is a continuous function. It follows that $\|\pi(a) + J_x\| = 0$ for all $x \in \partial I$ and, hence, $a = 0$. So, $\omega_\infty$ is faithful.

Since $(\psi_\mu \otimes \omega_\infty)\beta_{\hat{G}} = \psi_{\mu^{*n}(n+1)}$, it follows that $(\psi_\mu \otimes \omega_\infty)\beta_{\hat{G}} = \omega_\infty$. So, $(\psi_{\mu^{*k}} \otimes \omega_\infty)\beta_{\hat{G}} = \omega_\infty$ for all $k \in \mathbb{N}$. Since $\mu$ is generating, there exists for every $x \in I$, a $C_x > 0$ such that $(\psi_x \otimes \omega_\infty)\beta_{\hat{G}} \leq C_x \omega_\infty$. As a result, we can uniquely extend $\beta_{\hat{G}}$ to a normal $*$-homomorphism

$$(B_\infty, \omega_\infty)^\prime \to \ell^\infty(\hat{G}) \otimes (B_\infty, \omega_\infty)^\prime.$$ 

Since $\beta_{\hat{G}}$ is an action, the same holds for the extension to the von Neumann algebra $(B_\infty, \omega_\infty)^\prime$.

Because $(\psi_\mu \otimes \omega_\infty)\beta_{\hat{G}} = \beta_{\hat{G}}$ and because $\beta_{\hat{G}}$ is an action, the Poisson integral defined by (11) takes values in $H^\infty(\hat{G}, \mu)$. It is straightforward to check that $\Theta_\mu$ intertwines $\beta_{\hat{G}}$ with $\alpha_{\hat{G}}$ and $\beta_{\hat{G}}$ with $\alpha_{\hat{G}}$.

**Theorem 4.2.** The compactification $\hat{\mathcal{B}}$ of $\hat{G}$ satisfies the quantum Dirichlet property, meaning that, for all $a \in \mathcal{B}$,

$$\| (\Theta_\mu(a) - a)p_x \| \to 0$$

if $|x| \to \infty$.

In particular, the Poisson integral $\Theta_\mu$ is a normal and faithful $*$-homomorphism of $(B_\infty, \omega_\infty)^\prime$ onto a von Neumann subalgebra of $H^\infty(\hat{G}, \mu)$.

We deduce Theorem 4.2 from the following lemma.

**Lemma 4.3.** For every $a \in \mathcal{B}$, we have that

$$\sup_{y \in I} \| (\text{id} \otimes \psi_y) \hat{\Delta}(a)p_x - ap_x \| \to 0$$

when $|x| \to \infty$.

**Proof.** Fix $a \in \mathcal{B}$ with $\|a\| \leq 1$. Choose $\varepsilon > 0$. Take $n$ such that $\|ap_{x_0, x_1} - \psi_{x_0, x_1}(ap_{x_0})\| < \varepsilon$ for all $x_0, x_1 \in I$ with $|x_0| = n$.

Denote $d_{S^1}(V, W) = \inf\{\|V - \lambda W\| : \lambda \in S^1\}$. By formula (A.2) in the appendix, take $k$ such that

$$d_{S^1}( (V(x_0 \otimes x_1 x_2, x_0 x_1 x_2) \otimes 1) V(x_0 x_1 x_2 \otimes \mathcal{T}_2 u, x_0 x_1 u) ,$$

$$(1 \otimes V(x_1 x_2 \otimes \mathcal{T}_2 u, x_1 u)) V(x_0 \otimes x_1 u, x_0 x_1 u) ) < \frac{\varepsilon}{2}$$

whenever $|x_1| \geq k$. 

1087
Finally, take \( l \) such that \( q^{2l} < \varepsilon \). We prove that
\[
\| (\text{id} \otimes \psi_y) \hat{\Delta}(a)p_x - ap_x \| < 5\varepsilon
\]
for all \( x, y \in I \) with \( |x| > n + k + l \).

Choose \( x, y \in I \) with \( |x| > n + k + l \) and write \( x = x_0x_1x_2 \) with \( |x_0| = n, |x_1| = k \) and, hence, \( |x_2| > l \). We obtain
\[
(\text{id} \otimes \psi_y) \hat{\Delta}(a) p_x = \sum_{z \in x \otimes y} (\text{id} \otimes \psi_y)(V(x \otimes y, z)ap_zV(x \otimes y, z)^*)
\]
\[
= \sum_{z \in x \otimes y} \frac{\dim_q(z)}{\dim_q(x) \dim_q(y)} V(z \otimes \overline{y}, x)^*(ap_z \otimes 1)V(z \otimes \overline{y}, x)
\]
\[
= \sum_{z \in x_2 \otimes y} \frac{\dim_q(x_0x_1z)}{\dim_q(x) \dim_q(y)} V(x_0x_1z \otimes \overline{y}, x)^*(ap_{x_0x_1z} \otimes 1)V(x_0x_1z \otimes \overline{y}, x)
\]
\[+ \sum \text{remaining terms}. \]

In order to have remaining terms, \( y \) should be of the form \( y = \pi_2y_0 \) and then, using (5) and the assumption \( \|a\| \leq 1 \),
\[
\sum \| \text{remaining terms} \| = \sum_{z \in x_0x_1 \otimes y_0} \frac{\dim_q(z)}{\dim_q(x_0x_1x_2) \dim_q(x_2y_0)}
\leq \sum_{z \in x_0x_1 \otimes y_0} q^{2|x|} \frac{\dim_q(z)}{\dim_q(x_0x_1) \dim_q(y_0)} = q^{2|x_2|} < \varepsilon.
\]

Combining this estimate with the fact that \( \| ap_{x_0x_1z} - \psi_{x_0x_1z,x_0}(ap_{x_0}) \| < \varepsilon \), it follows that
\[
\| (\text{id} \otimes \psi_y) \hat{\Delta}(a)p_x - ap_x \|
\leq 2\varepsilon + \left| ap_x - \sum_{z \in x_2 \otimes y} \frac{\dim_q(x_0x_1z)}{\dim_q(x) \dim_q(y)} V(x_0x_1z \otimes \overline{y}, x)^*ight|
\leq (\psi_{x_0x_1z,x_0}(ap_{x_0}) \otimes 1)V(x_0x_1z \otimes \overline{y}, x).
\]

However, (14) now implies that
\[
\| (\text{id} \otimes \psi_y) \hat{\Delta}(a)p_x - ap_x \| \leq 3\varepsilon + \left| ap_x - \sum_{z \in x_2 \otimes y} \frac{\dim_q(x_0x_1z)}{\dim_q(x) \dim_q(y)} \psi_{x,x_0}(ap_{x_0}) \right|.
\]

Since \( \| \psi_{x,x_0}(ap_{x_0}) - ap_x \| < \varepsilon \) and \( \|a\| \leq 1 \), we obtain
\[
\| (\text{id} \otimes \psi_y) \hat{\Delta}(a)p_x \| \leq 4\varepsilon + \left( 1 - \sum_{z \in x_2 \otimes y} \frac{\dim_q(x_0x_1z)}{\dim_q(x) \dim_q(y)} \right).
\]

The second term on the right-hand side is zero, unless \( y = \pi_2y_0 \), in which case it equals
\[
\sum_{z \in x_0x_1 \otimes y_0} \frac{\dim_q(z)}{\dim_q(x_0x_1x_2) \dim_q(y_0)} \leq \sum_{z \in x_0x_1 \otimes y_0} q^{2|x_2|} \frac{\dim_q(z)}{\dim_q(x_0x_1) \dim_q(y_0)} \leq \varepsilon
\]
because of (5). Finally, (15) follows and the lemma is proven.

**Proof of Theorem 4.2.** Let \( a \in B \). Given \( \varepsilon > 0 \), Lemma 4.3 provides \( k \) such that
\[
\| (\text{id} \otimes \psi_\mu^{<n}) \hat{\Delta}(a)p_x - ap_x \| \leq \varepsilon
\]

Poisson boundary of the discrete quantum group $A_u(F)$

for all $n \in \mathbb{N}$ and all $x$ with $|x| \geq k$. Since $\psi_{\mu^n} \to \omega_\infty$ weakly*, it follows that

$$
\| (\Theta_\mu(a) - a)p_x \| \leq \epsilon
$$

whenever $|x| \geq k$. This proves (13).

It remains to prove the multiplicativity of $\Theta_\mu$. We know that $\Theta_\mu : \mathcal{B}_\infty \to H^\infty(\hat{G}, \mu)$ is a unital, completely positive map. Since $\hat{G} \to H^\infty(\hat{G}, \mu)$ is faithful. Denote by $\pi : H^\infty(\hat{G}, \mu) \to \ell^\infty(\hat{G})/c_0(\hat{G})$ the quotient map, which is also a unital, completely positive map. By (13), we have $\pi \circ \Theta_\mu = id$. So, for all $a \in \mathcal{B}_\infty$, we find

$$
\pi(\Theta_\mu(a)^* \cdot \Theta_\mu(a)) \leq \pi(\Theta_\mu(a^*a)) = \pi(\Theta_\mu(a))^* \pi(\Theta_\mu(a)) \leq \pi(\Theta_\mu(a)^* \cdot \Theta_\mu(a)).
$$

We claim that $\pi$ is faithful. If $a \in H^\infty(\hat{G}, \mu)^+ \cap c_0(\hat{G})$, we have $\hat{G}(a) = \psi_{\mu^n}(a)$ for all $n$ and the transience of $\mu$ combined with the assumption $a \in c_0(\hat{G})$, implies that $\hat{G}(a) = 0$ and, hence, $a = 0$. So, we conclude that $\Theta_\mu(a)^* \cdot \Theta_\mu(a) = \Theta_\mu(a^*a)$ for all $a \in \mathcal{B}_\infty$. Hence, $\Theta_\mu$ is multiplicative on $\mathcal{B}_\infty$ and also on $(\mathcal{B}_\infty, \omega_\infty)$ by normality.

Remark 4.4. We now give a reinterpretation of Theorem 2.2. Denote by $\Omega = I^\mathbb{N}$ the path space of the random walk with transition probabilities (6). Elements of $\Omega$ are denoted by $x, y, z$, etc. For every $x \in I$, one defines the probability measure $\mathbb{P}_x$ on $\Omega$ such that $\mathbb{P}_x(\{x\} \times I \times I \times \cdots) = 1$ and

$$
\mathbb{P}_x(\{(x, x_1, x_2, \ldots, x_n) \times I \times I \times \cdots \} = p(x, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n).
$$

Choose a probability measure $\eta$ on $I$ with $I = \text{supp} \eta$. Write $\mathbb{P} = \sum_{x \in I} \eta(x) \mathbb{P}_x$.

Define on $\Omega$ the following equivalence relation: $x \sim y$ if and only if there exist $k, l \in \mathbb{N}$ such that $x_{n+k} = y_{n+l}$ for all $n \in \mathbb{N}$. Whenever $x \in H^\infty(\hat{G}, \mu)$, the martingale convergence theorem implies that the sequence of measurable functions $\Omega \to \mathbb{C}: x \to F(x_n)$ converges $\mathbb{P}$-almost everywhere to a $\sim$-invariant bounded measurable function on $\Omega$, that we denote by $\pi(\infty)(F)$. Denote by $L^\infty(\Omega/\sim, \mathbb{P})$ the von Neumann subalgebra of $\sim$-invariant functions in $L^\infty(\Omega, \mathbb{P})$. As such, $\pi(\infty) : H^\infty(\hat{G}, \mu) \to L^\infty(\Omega/\sim, \mathbb{P})$ is a $*$-isomorphism.

By Theorem 2.2, we can define the measurable function $\text{bnd} : \Omega \to \partial I$ such that $\text{bnd} x = \lim_n x_n$ for $\mathbb{P}$-almost every $x \in \Omega$ and where the convergence is understood in the compact space $I \cup \partial I$. Recall that, for $x \in I$, we denote by $\nu_x$ the hitting probability measure on $\partial I$. So, $\nu_x(A) = \mathbb{P}_x(\text{bnd}^{-1} (A))$ for all measurable $A \subset \partial I$ and all $x \in I$.

Again by Theorem 2.2, $\pi(\infty) \circ \mathcal{Y}$ is a $*$-isomorphism of $L^\infty(\partial I, \nu_\epsilon)$ into $L^\infty(\Omega/\sim, \mathbb{P})$. We claim that for all $F \in L^\infty(\partial I, \nu_\epsilon)$, we have

$$
((\pi(\infty) \circ \mathcal{Y})(F))(x) = F(\text{bnd} x) \quad \text{for } \mathbb{P}\text{-almost every } x \in \Omega.
$$

Let $A \subset \partial I$ be measurable. Define $F_n : \Omega \to \mathbb{R}: F_n(x) = \nu_{x_n}(A)$. Then, $F_n$ converges almost everywhere with limit equal to $(\pi(\infty) \circ \mathcal{Y})(\chi_A)$. If the measurable function $G : \Omega \to \mathbb{C}$ only depends on $x_0, \ldots, x_k$, one checks that

$$
\int_{\Omega} F_n(x) G(x) d\mathbb{P}(x) = \int_{\text{bnd}^{-1} (A)} G(x) d\mathbb{P}(x)
$$

for all $n > k$.

From this, the claim follows.

Since the $*$-isomorphism $\pi(\infty) \circ \mathcal{Y}$ is given by $\text{bnd}$, it follows that for every $\sim$-invariant bounded measurable function $F$ on $\Omega$, there exists a bounded measurable function $F_1$ on $\partial I$ such that $F(x) = F_1(\text{bnd} x)$ for $\mathbb{P}$-almost every path $x \in \Omega$. 

1089
As before, we view $C(\partial I)$ as a C*-subalgebra of $B_\infty$. The restriction of the state $\omega_\infty$ to $C(\partial I)$ is, by definition, given by integration along $\nu_\varepsilon$. So, we can and do view $L^\infty(\partial I, \nu_\varepsilon)$ as a von Neumann subalgebra of $(B_\infty, \omega_\infty)''$. However, then both $\Upsilon$ and $\Theta_\mu$ are normal *-homomorphisms from $L^\infty(\partial I, \nu_\varepsilon)$ to $H^\infty_{\text{centr}}(\hat{G}, \mu)$. We claim that, viewed in this way, $\Upsilon = \Theta_\mu$ on $L^\infty(\partial I, \nu_\varepsilon)$. Since almost every path $x$ converges to $\text{bnd } x$, Theorem 4.2 implies that $((\pi_\infty \circ \Theta_\mu)(a))(x) = a(\text{bnd } x)$ for all $a \in C(\partial I)$. Since $C(\partial I)$ is weakly dense in $L^\infty(\partial I, \nu_\varepsilon)$ and since $\pi_\infty \circ \Upsilon$ and $\pi_\infty \circ \Theta_\mu$ are both normal, we conclude that $\pi_\infty \circ \Upsilon = \pi_\infty \circ \Theta_\mu$ and, hence, $\Upsilon = \Theta_\mu$ on $L^\infty(\partial I, \nu_\varepsilon)$.

We are now ready to prove the main Theorem 1.3.

Proof of Theorem 1.3. Owing to Theorem 4.2 and Lemma 3.3, it remains to show that

$$\Theta_\mu : (B_\infty, \omega_\infty)'' \rightarrow H^\infty(\hat{G}, \mu)$$

is surjective.

Whenever $\gamma : N \rightarrow N \otimes L^\infty(G)$ is an action of $G$ on the von Neumann algebra $N$, we denote, for $x \in I$, by $N_x^\infty \subset N$ the spectral subspace of the irreducible representation $x$. By definition, $N_x^\infty$ is the linear span of all $S(H_x)$, where $S$ ranges over the linear maps $S : H_x \rightarrow N$ satisfying $\gamma(S(\xi)) = (S \otimes \text{id})(U_x(\xi \otimes 1))$. The linear span of all $N_x^\infty$, $x \in I$, is a weakly dense *-subalgebra of $N$, called the spectral subalgebra of $N$. For $n \in \mathbb{N}$, we denote by $N^n$ the linear span of all $N_x^\infty$, $|x| \leq n$.

Fixing $x, y \in I$, consider the adjoint action $\gamma : \mathcal{L}(H_{xy}) \rightarrow \mathcal{L}(H_{xy}) \otimes C(\hat{G})$ given by $\gamma(A) = U_{xy}(A \otimes 1)U^*_{xy}$. The fusion rules of $G = A_\alpha(F)$ imply that $\mathcal{L}(H_{xy})^{|x|} = \psi_{xy,x}(\mathcal{L}(H_x))$.

For the rest of the proof, put $M := (B_\infty, \omega_\infty)''$. We use the action $\beta_G$ of $G$ on $M$ and the action $\alpha_G$ of $G$ on $H^\infty(\hat{G}, \mu)$. It suffices to prove that $H^\infty(\hat{G}, \mu)^k \subset \Theta_\mu(M)$ for all $k \in \mathbb{N}$.

Define, for all $y \in I$, the subset

$$V_y := \{yz \mid z \in I \text{ and } yz = y \otimes z\}.$$ 

Define the projections

$$q_y = \sum_{z \in V_y} p_z \in \mathcal{B}$$

and consider $q_y$ also as an element of the von Neumann algebra $M$. Define $W_y \subset \partial I$ as the subset of infinite words of the form $yu$, where $u \in \partial I$ and $yu = y \otimes u$.

Fix $y \in I$. Let $F \in C(W_y)$ and $A \in \mathcal{L}(H_y)$. Let $\tilde{F} \in C(I \cup \partial I)$ be a continuous extension of $F$. Define $b \in \ell^\infty(\hat{G})$ by the formula $b p_{yz} = \tilde{F}(yz) \psi_{yz,y}(A)$ when $yz = y \otimes z$ and $b p_r = 0$ elsewhere. Note that $b \in \mathcal{B}$ and that the image $\pi(b)$ of $b$ in $B_\infty$ actually belongs to $Mq_y$. We put $\zeta(F \otimes A) := \pi(b)$. As such, we have defined, for every $y \in I$, the unital *-homomorphism

$$\zeta : C(W_y) \otimes \mathcal{L}(H_y) \rightarrow Mq_y.$$ 

Claim. For all $y \in I$, there exists a linear map

$$T_y : H^\infty(\hat{G}, \mu)^{|y|} \cdot \Theta_\mu(q_y) \subset H^\infty(\hat{G}, \mu) \rightarrow L^\infty(W_y) \otimes \mathcal{L}(H_y)$$

satisfying the following conditions:

- $T_y$ is isometric for the 2-norm on $H^\infty(\hat{G}, \mu)$ given by the state $\hat{\varepsilon}$ and the 2-norm on $L^\infty(W_y) \otimes \mathcal{L}(H_y)$ given by the state $\nu_\varepsilon \otimes \psi_y$;
- $(T_y \circ \Theta_\mu \circ \zeta)(F) = F$ for all $F \in C(W_y) \otimes \mathcal{L}(H_y)$.
Poisson boundary of the discrete quantum group $A(W)$

To prove this claim, we use the notation and results introduced in Remark 4.4. Fix $y \in I$. Consider $a \in H^\infty((\tilde{G}, \mu))$, $\Theta_\mu(q_y)$. If $x \in \Omega$ is such that $bnd(x) \in W_y$, then, for $n$ big enough, $x_n$ will be of the form $x_n = y \otimes z_n$. By the definition of $\alpha_G$, we have that $ap_{x_n} \in \mathcal{L}(H_{x_n})^{2\|y\|}$. So, we can take elements $a_{x_n} \in \mathcal{L}(H_y)$ such that $ap_{x_n} = \psi_{x_n,y}(a_{x_n})$. We prove that, for $\mathbb{P}$-almost every path $x$ with $bnd(x) \in W_y$, the sequence $(a_{x_n})_n$ is convergent. We then define $T_y(a) \in L^\infty(W_y) \otimes \mathcal{L}(H_y)$ such that $T_y(a)(bnd(x)) = \lim_n a_{x_n}$ for $\mathbb{P}$-almost every path $x$ with $bnd(x) \in W_y$.

Take $d \in \mathcal{L}(H_y)$. Then, for $\mathbb{P}$-almost every path $x$ such that $bnd(x) \in W_y$ and $n$ big enough, we obtain that

$$\psi(y da_{x_n}) = \psi_{x_n}(\psi_{x_n,y}(da_{x_n})) = \psi_{x_n}(\psi_{x_n,y}(d)\psi_{x_n,y}(a_{x_n})) = \psi_{x_n}(\psi_{x_n,y}(d)ap_{x_n}).$$

In the second step, we used the multiplicativity of $\psi_{x_n,y} : \mathcal{L}(H_y) \to \mathcal{L}(H_{x_n})$ which follows because $x_n = y \otimes z_n$. Also note that $\|a_{x_n}\| \leqslant \|a\|$. From Theorem 4.2, it follows that

$$\|\Theta_\mu(\zeta(1 \otimes d))p_{x_n} - \psi_{x_n,y}(d)p_{x_n}\| \to 0$$

whenever $x_n$ converges to a point in $W_y$. This implies that

$$\|\psi(y da_{x_n}) - \psi_{x_n}(\Theta_\mu(\zeta(1 \otimes d))ap_{x_n})\| \to 0$$

for $\mathbb{P}$-almost every path $x$ with $bnd(x) \in W_y$.

From [INT06, Proposition 3.3], we know that for $\mathbb{P}$-almost every path $x$,

$$|\psi_{x_n}(\Theta_\mu(\zeta(1 \otimes d))ap_{x_n})p_{x_n} - \mathcal{E}(\Theta_\mu(\zeta(1 \otimes d)) \cdot a)p_{x_n}| \to 0.$$

As before, $\mathcal{E}(b)p_x = \psi_x(b)p_x$. It follows that

$$|\psi(y da_{x_n})p_{x_n} - \mathcal{E}(\Theta_\mu(\zeta(1 \otimes d)) \cdot a)p_{x_n}| \to 0.$$

Note that $\mathcal{E}$ maps $H^\infty((\tilde{G}, \mu))$ onto $H^\infty_{\text{centr}}((\tilde{G}, \mu))$. Whenever $F \in H^\infty_{\text{centr}}((\tilde{G}, \mu))$, the sequence $F(x_n)$ converges for $\mathbb{P}$-almost every path $x$. We conclude that for every $d \in \mathcal{L}(H_y)$, the sequence $\psi_{x_n}(da_{x_n})$ is convergent for $\mathbb{P}$-almost every path $x$ with $bnd(x) \in W_y$. Since $\mathcal{L}(H_y)$ is finite dimensional, it follows that the sequence $(a_{x_n})_n$ in $\mathcal{L}(H_y)$ is convergent for $\mathbb{P}$-almost every path $x$ with $bnd(x) \in W_y$.

By Remark 4.4, we get $T_y(a) \in L^\infty(W_y) \otimes \mathcal{L}(H_y)$ such that $T_y(a)(bnd(x)) = \lim_n a_{x_n}$ for $\mathbb{P}$-almost every path $x$ with $bnd(x) \in W_y$. From the definition of $a_{x_n}$, we obtain that

$$\|\psi_{x_n,y}(T_y(a)(bnd(x)) - ap_{x_n}\| \to 0$$

for $\mathbb{P}$-almost every path $x$ such that $bnd(x) \in W_y$.

The map $T_y$ is isometric. Indeed, by the defining property (16) and again by [INT06, Proposition 3.3], we have, for $\mathbb{P}$-almost every path $x$ with $bnd(x) \in W_y$,

$$\psi_y(T_y(a)(bnd(x)) = \lim_{n \to \infty} \psi_{x_n}(a^* ap_{x_n}) = (\pi_\infty \circ \mathcal{E})(a^* \cdot a)(x).$$

Here, $\pi_\infty$ denotes the $*$-isomorphism $H^\infty_{\text{centr}}((\tilde{G}, \mu)) \to L^\infty(\Omega/\sim, \mathbb{P})$ introduced in Remark 4.4. On the other hand, by Remark 4.4, $((\pi_\infty \circ \Theta_\mu)(q_y))(x) = 0$ for $\mathbb{P}$-almost every path $x$ with $bnd(x) \in W_y$. Since

$$\int_{\Omega} ((\pi_\infty \circ \mathcal{E})(b))(x) d\mathbb{P}_x(x) = \hat{\mathbb{P}}(b)$$

for all $b \in H^\infty((\tilde{G}, \mu))$, it follows that $T_y$ is an isometry in 2-norm.
We next prove that $\langle T_y \circ \Theta_\mu \circ \zeta \rangle(F) = F$ for all $F \in C(W_y) \otimes \mathcal{L}(H_y)$. Let $\tilde{a} \in C(I \cup \partial I) \subset C_c(I)$ and let $a$ be the restriction of $\tilde{a}$ to $\partial I$. Take $A \in \mathcal{L}(H_y)$. It suffices to take $F = a \otimes A$. Theorem 4.2 implies that

$$\| \tilde{a}p_{x_n} \psi_{x_n,y}(A) - (\Theta_\mu \circ \zeta)(a \otimes A)p_{x_n} \| \to 0$$

for $\mathbb{P}$-almost every path $x$. On the other hand, for $\mathbb{P}$-almost every path $x$ with $\text{bnd} x \in W_y$, the scalar $\tilde{a}p_{x_n}$ converges to $a(\text{bnd} x)$. In combination with (16), it follows that $\langle T_y \circ \Theta_\mu \circ \zeta \rangle(a \otimes A) = a \otimes A$, concluding the proof of the claim.

Having proven the claim, we now show that for all $y \in I$, $H^\infty(\hat{G}, \mu)^{2|y|} \cdot \Theta_\mu(q_y) \subset \Theta_\mu(M)$. Take $a \in H^\infty(\hat{G}, \mu)^{2|y|} \cdot \Theta_\mu(q_y)$. Let $d_n$ be a bounded sequence in the C*-algebra $C(W_y) \otimes \mathcal{L}(H_y)$ converging to $T_y(a)$ in 2-norm. Since $T_y \circ \Theta_\mu$ is an isometry in 2-norm, it follows that $\zeta(d_n)$ is a bounded sequence in $M$ that converges in 2-norm. Denoting by $c \in M$ the limit of $\zeta(d_n)$, we conclude that $T_y(\Theta_\mu(c)) = T_y(a)$ and, hence, $\Theta_\mu(c) = a$.

Fix $k \in \mathbb{N}$. A fortiori, $H^\infty(\hat{G}, \mu)^k \cdot \Theta_\mu(q_y) \subset \Theta_\mu(M)$ for all $y \in I$ with $2|y| \geq k$. By Proposition 2.5, the harmonic measure $\nu_\epsilon$ has no atoms in infinite words ending with $\alpha_1 \alpha_2 \alpha_3 \cdots$. As a result, 1 is the smallest projection in $M$ that dominates all $q_y$, $y \in I$, $2|y| \geq k$. So, $H^\infty(\hat{G}, \mu)^k \subset \Theta_\mu(M)$ for all $k \in \mathbb{N}$. This finally implies that $\Theta_\mu$ is surjective. □

5. Solidity and the Akemann–Ostrand property

In §3, we followed the approach of [VV07] to construct the compactification $\mathcal{B}$ of $\hat{G}$. In fact, more of the constructions and results of [VV07] carry over immediately to the case $G = A_u(F)$. We continue to assume that $F$ is not a multiple of a $2 \times 2$ unitary matrix.

Denote by $L^2(G)$ the GNS Hilbert space defined by the Haar state $h$ on $C(G)$. Denote by $\lambda : C(G) \to \mathcal{L}(L^2(G))$ the corresponding GNS representation and define $C_{\text{red}}(G) := \lambda(C(G))$. We can view $\lambda$ as the left-regular representation. We also have a right-regular representation $\rho$ and the operators $\lambda(a)$ and $\rho(b)$ commute for all $a, b \in C(G)$ (see [VV07, Formulae (1.3)])

Repeating the proofs of [VV07, Proposition 3.8 and Theorem 4.5], we arrive at the following result.

**Theorem 5.1.** The boundary action $\beta_\mathbb{G}$ of $\hat{G}$ on $\mathcal{B}$ defined in Theorem 3.2 is:

- amenable in the sense of [VV07, Definition 4.1];
- small at infinity in the sense that the comultiplication $\hat{\Delta}$ restricts as well to a right action of $\hat{G}$ on $\mathcal{B}$; this action leaves $c_0(\hat{G})$ globally invariant and becomes the trivial action on the quotient $\mathcal{B}_\infty$.

By construction, $\mathcal{B}$ is a nuclear C*-algebra and, hence, as in [VV07, Corollary 4.7], we obtain that:

- $G$ satisfies the Akemann–Ostrand property, which is that the homomorphism

$$C_{\text{red}}(G) \otimes_{\text{alg}} C_{\text{red}}(G) \to \frac{\mathcal{L}(L^2(G))}{\mathcal{K}(L^2(G))} : a \otimes b \mapsto \lambda(a)\rho(b) + \mathcal{K}(L^2(G))$$

is continuous for the minimal C*-tensor product $\otimes_{\text{min}}$;
- $C_{\text{red}}(G)$ is an exact C*-algebra.
As before, we denote by $L^\infty(G)$ the von Neumann algebra acting on $L^2(G)$ generated by $\lambda(C(G))$. From [Ban97, Théorème 3], it follows that $L^\infty(G)$ is a factor of type $\text{II}_1$ if $F$ is a multiple of an $n \times n$ unitary matrix and of type $\text{III}$ in the other cases.

Applying [Oza04, Theorem 6] (in fact, its slight generalization provided by [VV07, Théorème 2.5]), we obtain the following corollary of Theorem 5.1. Recall that a $\text{II}_1$ factor $M$ is called solid if for every diffuse von Neumann subalgebra $A \subset M$, the relative commutant $M \cap A'$ is injective. An arbitrary von Neumann algebra $M$ is called generalized solid if the same holds for every diffuse von Neumann subalgebra $A \subset M$ which is the image of a faithful normal conditional expectation.

**Corollary 5.2.** When $n \geq 3$ and $G = A_u(I_n)$, the $\text{II}_1$ factor $L^\infty(G)$ is solid. When $n \geq 2$, $F \in \text{GL}(n, \mathbb{C})$ is not a multiple of an $n \times n$ unitary matrix and $G = A_u(F)$, the type $\text{III}$ factor $L^\infty(G)$ is generalized solid.

**Appendix A. Approximate intertwining relations**

We fix an invertible matrix $F$ and assume that $F$ is not a scalar multiple of a unitary $2 \times 2$ matrix. Define $G = A_u(F)$ and label the irreducible representations of $G$ by the monoid $N \ast N$, freely generated by $\alpha$ and $\beta$. The representation labeled by $\alpha$ is the fundamental representation of $G$ and $\beta$ is its contragredient. Define $0 < q < 1$ such that $\dim_q(\alpha) = \dim_q(\beta) = q + (1/q)$. Recall from §3 that whenever $z \subset x \otimes y$, we choose an isometry $V(x \otimes y, z) \in \text{Mor}(x \otimes y, z)$. Observe that $V(x \otimes y, z)$ is uniquely determined up to multiplication by a scalar $\lambda \in S^1$. We denote by $p_z^x \otimes y$ the projection $V(x \otimes y, z)V(x \otimes y, z)^*$.

**Lemma A.1.** There exists a constant $C > 0$ that only depends on $q$ such that

\[
\begin{align*}
\|(V(xr \otimes yr, xy) \otimes 1_z)p_{xyz}^r &\otimes z - (1_{xr} \otimes p_{ryz}^y)(V(xr \otimes yr, xy) \otimes 1_z)\| \leq Cq^{|r|}, \\
\|(1_x \otimes V(yr \otimes \tau z, yz))p_{xyz}^x &\otimes z - (p_{xyr}^x \otimes 1_{\tau z})(1_x \otimes V(yr \otimes \tau z, yz))\| \leq Cq^{|y|}
\end{align*}
\]  

(A.1)

for all $x, y, z, r \in I$.

One way of proving Lemma A.1 consists of repeating the proof of [VV07, Lemma A.1] step by step. However, as we explain now, Lemma A.1 can also be deduced more directly from [VV07, Lemma A.1].

**Sketch of proof.** Whenever $y = y_1 \otimes y_2$ with $y_1 \neq \epsilon \neq y_2$, the expressions above are easily seen to be 0. Denote

\[v_n = \alpha \otimes \beta \otimes \alpha \otimes \cdots \quad \text{and} \quad w_n = \beta \otimes \alpha \otimes \beta \otimes \cdots\]

The remaining estimates that have to be proven reduce to estimates of norms of operators in $\text{Mor}(v_n, v_m)$ and $\text{Mor}(w_n, w_m)$. Putting these spaces together in an infinite matrix, one defines the C*-algebras

\[A := (\text{Mor}(v_n, v_m))_{n,m} \quad \text{and} \quad B := (\text{Mor}(w_n, w_m))_{n,m}\]

generated by the subspaces $\text{Mor}(v_n, v_m)$ and $\text{Mor}(w_n, w_m)$, respectively. Choose unit vectors $t \in \text{Mor}(\alpha \otimes \beta, \epsilon)$ and $s \in \text{Mor}(\beta \otimes \alpha, \epsilon)$ such that $(t^* \otimes 1)(1 \otimes s) = (q + 1/q)^{-1}$. By [Ban97, Lemme 5], the C*-algebra $A$ is generated by the elements $1^\otimes 2k \otimes t \otimes 1^\otimes l$, $1^\otimes (2k+1) \otimes s \otimes 1^\otimes l$. A similar statement holds for $B$.  

1093
Denote by \( U \) the fundamental representation of the quantum group \( SU_{-q}(2) \) and let \( t_0 \in \text{Mor}(U \otimes U, \epsilon) \) be a unit vector. The proofs of [BDV06, Theorems 5.3 and 6.2] (which heavily rely on the results in [Ban96, Ban97]) imply the existence of \(*\)-isomorphisms

\[
\pi_A : (\text{Mor}(U^{\otimes n}, U^{\otimes m}))_{n,m} \to A \quad \text{and} \quad \pi_B : (\text{Mor}(U^{\otimes n}, U^{\otimes m}))_{n,m} \to B
\]

satisfying

\[
\pi_A(1^{\otimes 2k} \otimes t_0 \otimes 1^{\otimes l}) = 1^{\otimes 2k} \otimes t \otimes 1^{\otimes l} \quad \text{and} \quad \pi_A(1^{\otimes (2k+1)} \otimes t_0 \otimes 1^{\otimes l}) = 1^{\otimes (2k+1)} \otimes s \otimes 1^{\otimes l}
\]

and similarly for \( \pi_B \).

As a result, the estimates to be proven follow directly from the corresponding estimates for \( SU_{-q}(2) \) proven in [VV07, Lemma A.1].

Using the notation

\[
d_{S^1}(V, W) = \inf \{ \| V - \lambda W \| \mid \lambda \in S^1 \},
\]

several approximate commutation relations can be deduced from Lemma A.1. For instance, after a possible increase of the constant \( C \), (A.1) implies that

\[
d_{S^1}((1_x \otimes V(yr \otimes rz, yz))V(x \otimes yz, xyz), (V(x \otimes yr, xyr) \otimes 1_{rz})V(xyr \otimes rz, xyz)) \leq Cq^{\|y\|}
\]

for all \( x, y, z, r \in I \). We again refer to [VV07, Lemma A.1] for a full list of approximate intertwining relations.

References

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Stefaan Vaes stefaan.vaes@wis.kuleuven.be
Department of Mathematics, K.U. Leuven, Celestijnenlaan 200B, B–3001 Leuven, Belgium

Nikolas Vander Vennet nvandervennet@gmail.com
Department of Mathematics, K.U. Leuven, Celestijnenlaan 200B, B–3001 Leuven, Belgium