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Abstract

In this paper we prove that a pure, regular, totally odd, polarizable weakly compatible system of l-adic representations is potentially automorphic. The innovation is that we make no irreducibility assumption, but we make a purity assumption instead. For compatible systems coming from geometry, purity is often easier to check than irreducibility. We use Katz's theory of rigid local systems to construct many examples of motives to which our theorem applies. We also show that if F is a CM or totally real field and if π is a polarizable, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then for a positive Dirichlet density set of rational primes l, the l-adic representations $r_{l,\imath}(\pi)$ associated to π are irreducible.

Introduction

This paper is a postscript to [BGGT14].

Suppose that F and M are number fields, that S is a finite set of primes of F and that n is a positive integer. By a *weakly compatible system* of n-dimensional l-adic representations of G_F defined over M and unramified outside S we shall mean a family of continuous semi-simple representations

$$r_\lambda : G_F \longrightarrow \mathrm{GL}_n(\overline{M}_\lambda),$$

where λ runs over the finite places of M, with the following properties.

(i) If $v \notin S$ is a finite place of F, then for all λ not dividing the residue characteristic of v, the representation r_λ is unramified at v and the characteristic polynomial of $r_\lambda(\mathrm{Frob}_v)$ lies in $M[X]$ and is independent of λ.

(ii) Each representation r_λ is de Rham at all places above the residue characteristic of λ, and in fact crystalline at any place $v \notin S$ which divides the residue characteristic of λ.

(iii) For each embedding $\tau : F \hookrightarrow \overline{M}$ the τ-Hodge–Tate numbers of r_λ are independent of λ.

In this paper we prove the following theorem (see Theorem 2.1).

Theorem A. *Let $\{r_\lambda\}$ be a weakly compatible system of n-dimensional l-adic representations of G_F defined over M and unramified outside S, where for simplicity we assume that M contains the image of each embedding $F \hookrightarrow \overline{M}$. Suppose that $\{r_\lambda\}$ satisfies the following properties.*

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(i) (Purity) *There is an integer w such that, for each prime $v \notin S$ of F, the roots of the common characteristic polynomial of the $r_\lambda(\mathrm{Frob}_v)$ are Weil $(\#k(v))^w$-numbers.*

(ii) (Regularity) *For each embedding $\tau : F \hookrightarrow M$ the representation r_λ has n distinct τ-Hodge–Tate numbers.*

(iii) (Odd essential self-duality) *F is totally real; and either each r_λ factors through a map to $\mathrm{GSp}_n(\overline{M}_\lambda)$ with a totally odd multiplier character; or each r_λ factors through a map to $\mathrm{GO}_n(\overline{M}_\lambda)$ with a totally even multiplier character. Moreover, in either case the multiplier characters form a weakly compatible system.*

Then there is a finite, Galois, totally real extension F'/F over which all of the r_λ become automorphic. In particular, for any embedding $\imath : M \hookrightarrow \mathbb{C}$ the partial L-function $L^S(\imath\{r_\lambda\}, s)$ has meromorphic continuation to the whole complex plane and satisfies the expected functional equation.

A similar result is proved in the case that F is CM.

A similar theorem was proved in [BGGT14] with an irreducibility assumption in place of our purity assumption. For compatible systems arising from geometry irreducibility can, in practice, be hard to check, but purity is often known thanks to Deligne's theorem. For instance, Theorem A has the following consequence. (See Corollary 2.3.)

Corollary B. *Suppose that $m \in \mathbb{Z}_{\geqslant 0}$, that F is a totally real field and that X/F is a smooth projective variety such that for all $\tau : F \hookrightarrow \mathbb{C}$ and all $i = 0, \ldots, m$ we have*

$$\dim H^{i,m-i}((X \times_{F,\tau} \mathbb{C})(\mathbb{C}), \mathbb{C}) \leqslant 1.$$

Then $\{H^m_{\mathrm{et}}(X \times_F \overline{F}, \mathbb{Q}_l)\}$ is a strictly compatible system of l-adic representations and the L-function $L(\{H^m_{\mathrm{et}}(X \times_F \overline{F}, \mathbb{Q}_l)\}, s)$ has meromorphic continuation to the whole complex plane and satisfies the expected functional equation.

Note that Corollary B implies corresponding cases of the weight-monodromy conjecture (see Remark 2.6). There is also a version of this corollary for motives, which is applicable more generally. (See Corollary 2.5.)

Corollary C. *Suppose that F is a totally real field, that M is a number field, and that X and Y are pure motivated motives (in the sense of [And96, § 4]) over F with coefficients in M such that the l-adic realizations of X and Y form weakly compatible systems of l-adic representations $\mathcal{H}(X)$ and $\mathcal{H}(Y)$. Suppose also that:*

(i) (self-duality) *$X \cong X^\vee \otimes Y$;*

(ii) (regularity) *and for all $\tau : F \hookrightarrow \mathbb{C}$, all $\tau' : M \hookrightarrow \mathbb{C}$, and all i and j, we have*

$$\dim_{\mathbb{C}} H^{i,j}(X \times_{F,\tau} \mathbb{C}) \otimes_{M \otimes \mathbb{C}, \tau' \otimes 1} \mathbb{C} \leqslant 1.$$

Then $\mathcal{H}(X)$ is a strictly compatible system of l-adic representations and the L-function $L(\mathcal{H}(X), s)$ has meromorphic continuation to the whole complex plane and satisfies the expected functional equation.

We use Katz's theory of rigid local systems to construct many examples of motivated motives to which this corollary applies. (See Corollary 2.8 and the discussion that follows it.)

Our approach to Theorem A is to apply the potential automorphy result [BGGT14, Theorem 4.5.1] to the irreducible constituents of the r_λ. The arguments of [BGGT14, § 5] can be applied as long as one can show that *there is a positive Dirichlet density set $\mathcal{L}$ of rational primes such that for all $\lambda | l \in \mathcal{L}$ the irreducible constituents of r_λ are all odd, essentially self-dual.*

In [BGGT14] this was proved subject to the very strong condition that $\{r_\lambda\}$ is extremely regular. However, this does not apply in many settings. The main innovation of the present paper is to prove this assertion under the assumption that $\{r_\lambda\}$ is pure and regular. (See Lemma 1.6.) We do this using a technique from [Pat14] (see [Pat14, §§ 2.4 and 4.1.5]), which makes use of the CM nature of the field of coefficients. Curiously our set $\mathcal{L}$ will only contain primes l, such that all primes λ above l in a certain CM field (a souped-up field of definition for $\{r_\lambda\}$) satisfy ${}^c\lambda = \lambda$.

The same technique, combined with the arguments of [BGGT14, § 5] also allow us to prove the following irreducibility result for l-adic representations arising from a cuspidal automorphic representation. (See Theorem 1.7.)

THEOREM D. *Suppose that F is a CM (or totally real) field and that π is a polarizable, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then there is a positive Dirichlet density set $\mathcal{L}$ such that, for all $l \in \mathcal{L}$ and all $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, the representation $r_{l,\imath}(\pi)$ is irreducible.*

In this paper we use, often without comment, the notation and definitions of [BGGT14], particularly of §§ 2.1 and 5.1 of that paper. For instance ϵ_l will denote the l-adic cyclotomic character and $V(m)$ for the Tate twist $V(\epsilon_l^m)$. Also by a CM field we will mean a number field F admitting an automorphism c, which coincides with complex conjugation for every embedding $F \hookrightarrow \mathbb{C}$. If F is a CM field, then F^+ will denote $F^{\{1,c\}}$ the maximal totally real subfield of F. We have $[F : F^+] = 1$ or 2. Finally if $\Gamma \supset \Delta$ are groups, if $\gamma \in \Gamma$ normalizes Δ, and if r is a representation of Δ, then we define a representation r^γ of Δ by

$$r^\gamma(\delta) = r(\gamma\delta\gamma^{-1}).$$

1. Generalities on compatible systems of l-adic representations

We will recall some elementary facts and definitions concerning compatible systems of l-adic representations.

First of all recall that if $F/\mathbb{Q}_l$ is a finite extension, if V is a finite-dimensional $\overline{\mathbb{Q}}_l$-vector space, if

$$r : G_F \longrightarrow \mathrm{GL}(V)$$

is a continuous semi-simple representation, and if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ is a continuous embedding, then we define a multiset

$$\mathrm{HT}_\tau(r)$$

of integers by letting i have multiplicity

$$\dim_{\overline{\mathbb{Q}}_l}(V(\epsilon_l^i) \otimes_{\tau,F} \widehat{\overline{F}})^{G_F},$$

where $\widehat{\overline{F}}$ denotes the completion of the algebraic closure of F. Then $\#\,\mathrm{HT}_\tau(r) \leqslant \dim_{\overline{\mathbb{Q}}_l} V$ with equality if r is de Rham (or even, by definition, Hodge–Tate). For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$ we have

$$\mathrm{HT}_{\sigma\tau}({}^\sigma r) = \mathrm{HT}_\tau(r).$$

In particular, if the trace, $\mathrm{tr}\, r$, is valued in a finite extension $M/\mathbb{Q}_l$ (in $\overline{\mathbb{Q}}_l$) and if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_l/M)$, then

$$\mathrm{HT}_{\sigma\tau}(r) = \mathrm{HT}_\tau(r).$$

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Also if σ is a continuous automorphism of F, then

$$\mathrm{HT}_{\tau\circ\sigma}(r^{\sigma}) = \mathrm{HT}_{\tau}(r).$$

Suppose now that F is a number field, that V is a finite-dimensional $\overline{\mathbb{Q}}_l$-vector space and that

$$r : G_F \longrightarrow \mathrm{GL}(V)$$

is a continuous representation which is de Rham at all primes above l. If $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$, then we define

$$\mathrm{HT}_{\tau}(r) = \mathrm{HT}_{\tau}(r|_{G_{F_{v(\tau)}}}),$$

where $v(\tau)$ is the prime of F induced by τ. Note that if $\sigma \in G_{\mathbb{Q}_l}$, then

$$\mathrm{HT}_{\sigma\tau}({}^{\sigma}r) = \mathrm{HT}_{\tau}(r),$$

and so, if $\mathrm{tr}\, r$ is valued in a closed subfield $M \subset \overline{\mathbb{Q}}_l$ and if $\sigma \in G_M$, then

$$\mathrm{HT}_{\sigma\tau}(r) = \mathrm{HT}_{\tau}(r).$$

If σ is an automorphism of F and if v is the prime of F determined by $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$, then

$$\begin{aligned}(V^{\sigma}(\epsilon_l^i) \otimes_{\tau, F_{v(\tau)}} \widehat{\overline{F}}_{v(\tau)})^{G_{F_{v(\tau)}}} &= (V(\epsilon_l^i) \otimes_{\tau, F_{v(\tau)}} \widehat{\overline{F}}_{v(\tau)})^{G_{F_{\sigma v(\tau)}}} \\ &\xrightarrow{\sim} (V(\epsilon_l^i) \otimes_{\tau\sigma^{-1}, F_{v(\tau\sigma^{-1})}} \widehat{\overline{F}}_{v(\tau\sigma^{-1})})^{G_{F_{v(\tau\sigma^{-1})}}},\end{aligned}$$

where, in the middle space, $G_{F_{\sigma v(\tau)}}$ acts on $\widehat{\overline{F}}_v$ via the conjugation by σ^{-1} map $G_{F_{\sigma v(\tau)}} \to G_{F_{v(\tau)}}$, and where the second map is $1 \otimes \sigma$. Thus, we have

$$\mathrm{HT}_{\tau}(r^{\sigma}) = \mathrm{HT}_{\tau}(r^{\sigma}|_{G_{F_{v(\tau)}}}) = \mathrm{HT}_{\tau\sigma^{-1}}(r|_{G_{F_{v(\tau\sigma^{-1})}}}) = \mathrm{HT}_{\tau\sigma^{-1}}(r).$$

Now let F denote a number field. As in [BGGT14], by a *rank n weakly compatible system of l-adic representations $\mathcal{R}$ of G_F defined over M* we shall mean a 5-tuple

$$(M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

where:

(i) M is a number field;
(ii) S is a finite set of primes of F;
(iii) for each prime $v \notin S$ of F, $Q_v(X)$ is a monic degree n polynomial in $M[X]$;
(iv) for each prime λ of M (with residue characteristic l say)

$$r_\lambda : G_F \longrightarrow \mathrm{GL}_n(\overline{M}_\lambda)$$

is a continuous, semi-simple, representation such that

  (a) if $v \notin S$ and $v \nmid l$ is a prime of F, then r_λ is unramified at v and $r_\lambda(\mathrm{Frob}_v)$ has characteristic polynomial $Q_v(X)$,
  (b) while if $v | l$, then $r_\lambda|_{G_{F_v}}$ is de Rham and in the case $v \notin S$ crystalline;

(v) for $\tau : F \hookrightarrow \overline{M}$, H_τ is a multiset of n integers such that for each prime λ of M and each $i : \overline{M} \hookrightarrow \overline{M}_\lambda$ over M we have $\mathrm{HT}_{i\circ\tau}(r_\lambda) = H_\tau$.

We refer to a rank-one weakly compatible system of l-adic representations as a weakly compatible system of l-adic characters.

Note that if $(M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is a weakly compatible system of l-adic representations of G_F and that if $M' \supset M$ is a finite extension, then the tuple $(M', S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is also a weakly compatible system of l-adic representations of G_F. Also note that if

$$(M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

is a weakly compatible system of l-adic representations of G_F and if $\sigma \in G_M$, then $H_{\sigma\tau} = H_\tau$. (As they both equal $\mathrm{HT}_{i\circ\sigma\circ\tau}(r_\lambda)$, where $i : \overline{M} \hookrightarrow \overline{M}_\lambda$ over M.)

We will say that two weakly compatible systems of l-adic representations of G_F over M, say

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

and

$$\mathcal{R}' = (M, S', \{Q'_v(X)\}, \{r'_\lambda\}, \{H'_\tau\}),$$

are *equivalent* if $Q_v(X) = Q'_v(X)$ for a set of v of Dirichlet density 1. We write $\mathcal{R} \equiv \mathcal{R}'$. In this case we have $Q_v(X) = Q'_v(X)$ for all $v \notin S \cup S'$, we have $r'_\lambda \cong r_\lambda$ for all λ, and we have $H_\tau = H'_\tau$ for all τ.

Recall (from [BGGT14, § 5.1]) that one can apply standard linear algebra operations like direct sum, tensor product and dual to compatible systems of l-adic representations. One can also restrict them from G_F to $G_{F'}$ if $F' \supset F$.

If $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is a weakly compatible system of l-adic representations of G_F and if $\sigma \in \mathrm{Aut}(M)$ then we set

$${}^\sigma\mathcal{R} = (M, S, \{{}^\sigma Q_v(X)\}, \{{}^\sigma r_{\sigma^{-1}\lambda}\}, \{H_{\sigma^{-1}\tau}\}).$$

It is again a weakly compatible system of l-adic representations of G_F. Similarly if $\sigma \in \mathrm{Aut}(F)$, then we set

$$\mathcal{R}^\sigma = (M, \sigma^{-1}S, \{Q_{\sigma v}(X)\}, \{r_\lambda^\sigma\}, \{H_{\tau\sigma^{-1}}\}).$$

It is again a weakly compatible system of l-adic representations of G_F.

Lemma 1.1. *Suppose that $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is a weakly compatible system of l-adic representations of G_F and that $M' \subset M$ is a subfield such that $M'[X]$ contains $Q_v(X)$ for a set of $v \notin S$ of Dirichlet density 1. Then $(M', S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is a weakly compatible system of l-adic representations of G_F.*

Proof. Enlarging M if need be we may assume that $M/\mathbb{Q}$ is Galois. If λ' is a prime of M' we define $r_{\lambda'}$ to be r_λ for any prime $\lambda|\lambda'$ of M. To see that this is well defined we must check that if λ_1, λ_2 are primes of M above λ' and if $\sigma : \overline{M}_{\lambda_1} \xrightarrow{\sim} \overline{M}_{\lambda_2}$ is an isomorphism restricting to the identity on $M'_{\lambda'}$, then ${}^\sigma r_{\lambda_1} \cong r_{\lambda_2}$. This follows because these two representations have the same trace, which fact in turn follows from the Cebotarev density theorem because $\mathrm{tr}\, {}^\sigma r_{\lambda_1}(\mathrm{Frob}_v) = \mathrm{tr}\, r_{\lambda_2}(\mathrm{Frob}_v)$ for a set of primes v of F of Dirichlet density 1.

We must next check that if $v \notin S$, then $Q_v(X) \in M'[X]$. If λ' is a prime of M' and $\sigma \in G_{M'_{\lambda'}}$, then ${}^\sigma r_{\lambda'} \cong r_{\lambda'}$ so that ${}^\sigma Q_v(X) = Q_v(X)$. As $\mathrm{Gal}(M/M')$ is generated by the elements of decomposition groups at finite primes we see that $Q_v(X)$ is fixed by $\mathrm{Gal}(M/M')$ and the claim follows.

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Finally we must check that if $\tau : F \hookrightarrow \overline{M}$ and $i : \overline{M} \hookrightarrow \overline{M}'_{\lambda'}$ over M', then $\mathrm{HT}_{i\circ\tau}(r_{\lambda'}) = H_\tau$. Choose $\sigma \in G_{M'}$ such that $i \circ \sigma^{-1}$ is M-linear. Then we certainly have

$$\mathrm{HT}_{i\circ\tau}(r_{\lambda'}) = H_{\sigma\tau}.$$

Thus, it suffices to check that for all $\sigma \in G_{M'}$ and all $\tau : F \hookrightarrow \overline{M}$ we have

$$H_{\sigma\tau} = H_\tau.$$

In fact, we only need treat the case that $\sigma \in G_{M'_{\lambda'_0}}$ for some prime λ'_0. (As such elements topologically generate $G_{M'}$.)

Let λ_0 be the prime of M above λ'_0 corresponding to a given embedding $G_{M'_{\lambda'_0}} \hookrightarrow G_{M'}$, and let $j : \overline{M} \hookrightarrow \overline{M}_{\lambda_0}$ be M-linear. Then

$$r_{\lambda_0} \cong {}^{\sigma} r_{\sigma^{-1}\lambda_0}$$

and so

$$H_{\sigma\circ\tau} = \mathrm{HT}_{j\circ\sigma\circ\tau}(r_{\lambda_0}) = \mathrm{HT}_{j\circ\sigma\circ\tau}({}^{\sigma} r_{\sigma^{-1}\lambda_0}) = \mathrm{HT}_{j'\circ\tau}(r_{\sigma^{-1}\lambda_0}) = H_\tau,$$

where

$$j' = \sigma^{-1} \circ j \circ \sigma : \overline{M} \hookrightarrow \overline{M}_{\sigma^{-1}\lambda_0}$$

is M-linear. $\square$

If the conclusion of this lemma holds we will say that $\mathcal{R}$ *is rational over* M'. The lemma implies that any weakly compatible system of l-adic representations has a unique minimal field of definition, namely the subfield of M generated by the coefficients of all of the $Q_v(X)$ for $v \notin S$. If $M/\mathbb{Q}$ is Galois, then it is also the fixed field of

$$\{\sigma \in \mathrm{Gal}(M/\mathbb{Q}) : {}^{\sigma}\mathcal{R} \equiv \mathcal{R}\}.$$

We will call $\mathcal{R}$ *pure* of weight w if for each $v \notin S$, for each root α of $Q_v(X)$ in $\overline{M}$ and for each $\imath : \overline{M} \hookrightarrow \mathbb{C}$ we have

$$|\imath\alpha|^2 = (\#k(v))^w.$$

Note that this definition is apparently slightly weaker than the definition given in [BGGT14], but the next lemma, which is essentially due to [Pat14] (see [Pat14, § 4.1.5]), shows that the two definitions are actually equivalent.

Lemma 1.2. *Suppose that $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is a pure weakly compatible system of l-adic representations of G_F of weight w.*

(i) *If c is the restriction to M of any complex conjugation, then ${}^{c}\mathcal{R} \cong \mathcal{R}^\vee \otimes \{\epsilon_l^{-w}\}$.*

(ii) *The system $\mathcal{R}$ is rational over a CM field.*

(iii) *If $c \in \mathrm{Aut}(\overline{M})$ denotes a complex conjugation, then*

$$H_{c\tau} = \{w - h : h \in H_\tau\}.$$

(iv) *Suppose that F_0 is the maximal CM subfield of F. If $\tau|_{F_0} = \tau'|_{F_0}$, then*

$$H_\tau = H_{\tau'}.$$

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Proof. The first part follows as for $v \notin S$ we have

$$^{c}Q_v(X) = X^{\dim \mathcal{R}} Q_v(q_v^w/X)/Q_v(0),$$

where $q_v = \#k(v)$. If $M/\mathbb{Q}$ is Galois and $c, c' \in \mathrm{Gal}(M/\mathbb{Q})$ are two complex conjugations, we deduce that

$$^{cc'}\mathcal{R} \cong \mathcal{R}.$$

The second part follows. The third part also follows from the first.

For the fourth part we may assume that M is a CM field (using the second part), that $M/\mathbb{Q}$ is Galois and that M contains τF_0 for all $\tau : F \hookrightarrow \overline{M}$. Then for any $\tau : F \hookrightarrow \overline{M}$ we see that $M \cap \tau F = \tau F_0$ and that $M(\tau F) \cong M \otimes_{\tau, F_0} F$. Thus, if $\tau|_{F_0} = \tau'|_{F_0}$ we see that $M(\tau F) \cong M(\tau' F)$ as $M \otimes F$-algebras so that there is $\sigma \in G_M$ with $\sigma\tau = \tau'$. The fourth part follows. □

LEMMA 1.3. *Suppose that F is a CM field, that $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is a pure weakly compatible system of l-adic representations of G_F of weight w and that $\mathcal{M} = (M, S^\mu, \{X - \alpha_v\}, \{\mu_\lambda\}, \{H_\tau^\mu\})$ is a weakly compatible system of l-adic characters of G_{F^+} with*

$$\mathcal{R}^c \equiv \mathcal{M}|_{G_F} \otimes \mathcal{R}^\vee.$$

Then for all τ we have $H_\tau^\mu = \{w\}$.

Proof. This follows by noting that $\mathcal{M}$ must be pure of weight $2w$ and using the classification of algebraic l-adic characters of the absolute Galois group of a totally real field. (They must all be of the form $\mu_0 \epsilon_l^{-w_0}$ where μ_0 has finite order and $w_0 \in \mathbb{Z}$.) □

Recall that we call $\mathcal{R}$ *regular* if for each $\tau : F \hookrightarrow \overline{M}$ every element of H_τ has multiplicity 1.

LEMMA 1.4. *Suppose that $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is a regular pure weakly compatible system of l-adic representations of G_F of weight w and that M is a CM field. Then there is a finite CM extension $M' \supset M$ such that for all open subgroups H of G_F and all primes λ of M', all sub-representations of $r_\lambda|_H$ are defined over M'_λ.*

Proof. This lemma is proved in the same way as [BGGT14, Lemma 5.3.1(3)], noting that, by purity, the splitting field over M for the polynomial $Q_v(X)Q_{v'}(X)$ which occurs in the proof of [BGGT14, Lemma 5.3.1(3)] is a CM field. □

When the conclusion of this lemma holds for M we will call M a *full field of definition* for $\mathcal{R}$.

If M is a CM field we will call a prime λ of M *conjugation invariant* if $^c\lambda = \lambda$. Thus, if M is totally real all primes of M are conjugation invariant.

LEMMA 1.5. *Suppose that M is a CM field. Then there is a set of rational primes Σ of positive Dirichlet density such that all primes of M above any element $l \in \Sigma$ are conjugation invariant.*

Proof. One can reduce to the case that $M/\mathbb{Q}$ is Galois. In this case one can for instance take Σ to be the set of rational primes l unramified in M and with $[\mathrm{Frob}_l] = \{c\} \subset \mathrm{Gal}(M/\mathbb{Q})$. □

LEMMA 1.6. *Suppose that F is a CM field, that $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is a regular, pure weakly compatible system of l-adic representations of G_F of weight w and that $\mathcal{M} = (M, S^\mu, \{X - \alpha_v\}, \{\mu_\lambda\}, \{H_\tau^\mu\})$ is a weakly compatible system of characters of G_{F^+} with*

$$\mathcal{R}^c \equiv \mathcal{M}|_{G_F} \otimes \mathcal{R}^\vee.$$

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Suppose further that M is a CM field and is a full field of definition for $\mathcal{R}$. Let c denote a complex conjugation in G_{F^+}. Suppose further that λ is a conjugation invariant prime of M and that $\langle\,,\rangle$ is a bilinear form on the space underlying r_λ such that

$$\langle r_\lambda(\sigma)x, r_\lambda(c\sigma c^{-1})y\rangle = \mu_\lambda(\sigma)\langle x, y\rangle$$

for all $\sigma \in G_F$ and all x, y in the underlying space of r_λ.

Then the irreducible constituents of r_λ are orthogonal with respect to $\langle\,,\rangle$.

Proof. Note that c on M extends to a unique continuous automorphism of M_λ. Let r be a constituent of r_λ. For $v \notin S$ not dividing the residue characteristic of λ, we let $Q_v^r(X) \in M_\lambda[X]$ denote the characteristic polynomial of $r(\mathrm{Frob}_v)$. Then by purity we see that

$${}^cQ_v^r(X) = X^{\dim r}Q_v^r(q_v^w/X)/Q_v(0)$$

where $q_v = \#k(v)$. Using the Cebotarev density theorem we deduce that

$${}^cr \cong \epsilon_l^{-w} r^\vee.$$

Hence,

$$\mu_\lambda(r^c)^\vee \cong (\mu_\lambda\epsilon_l^w)\ {}^cr^c.$$

Using Lemma 1.3 we deduce that for $\tau : F \hookrightarrow \overline{M}_\lambda$ we have

$$\mathrm{HT}_\tau(\mu_\lambda(r^c)^\vee) = \mathrm{HT}_\tau((\mu_\lambda\epsilon_l^w)\ {}^cr^c) = \mathrm{HT}_\tau({}^cr^c) = \mathrm{HT}_{c\tau c}(r) = \mathrm{HT}_\tau(r).$$

By regularity we deduce that

$$r \cong \mu_\lambda(r^c)^\vee,$$

and moreover r_λ has no other sub-representation isomorphic to $\mu_\lambda(r^c)^\vee$. The lemma follows. $\square$

We remark that the lemma is presumably true without the assumption that λ is conjugation invariant, but we do not know how to prove this.

We will call $\mathcal{R}$ *strictly compatible* if for each finite place v of F there is a Weil–Deligne representation $\mathrm{WD}_v(\mathcal{R})$ of W_{F_v} over $\overline{M}$ such that for each place λ of M and every M-linear embedding $\varsigma : \overline{M} \hookrightarrow \overline{M}_\lambda$ the push forward $\varsigma\mathrm{WD}_v(\mathcal{R}) \cong \mathrm{WD}(r_\lambda|_{G_{F_v}})^{\text{F-ss}}$. (This is slightly stronger than the notion we defined in [BGGT14].) Moreover, we will call $\mathcal{R}$ *strictly pure* of weight w if $\mathcal{R}$ is strictly compatible and for each prime v of F the Weil–Deligne representation $\mathrm{WD}_v(\mathcal{R})$ is pure of weight w (see [BGGT14, § 1.3]).

If $\mathcal{R}$ is pure and if $\imath : M \hookrightarrow \mathbb{C}$, then we can define the partial L-function $L^S(\imath\mathcal{R}, s)$, which is a holomorphic function in some right half-plane. If $\mathcal{R}$ is regular and strictly pure, then we can define the completed L-function $\Lambda(\imath\mathcal{R}, s)$ and the epsilon factor $\epsilon(\imath\mathcal{R}, s)$. (See [BGGT14, § 5.1] for details.)

Suppose that F is a number field and that π is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C}),+}$. (See [BGGT14, § 2.1].) We define M_π to be the fixed field of the subgroup of $\mathrm{Aut}(\mathbb{C})$ consisting of elements $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that ${}^\sigma\pi^\infty \cong \pi^\infty$. It follows from [Clo90, Theorem 3.13] that M_π is a number field, and in fact is a CM field (see [APT91, Lemma 1.3] or [Pat14, 6.2.3]). We let S_π denote the set of primes of F where π is ramified and for $v \notin S_\pi$ a prime of F we let $Q_{\pi,v}(X) \in M_\pi[X]$ denote the characteristic polynomial of $\mathrm{rec}_{F_v}(\pi_v|\det|_v^{(1-n)/2})$. If $\overline{M}_\pi$ denotes the algebraic closure of M in $\mathbb{C}$ and if $\tau : F \hookrightarrow \overline{M}_\pi$, then we set

$$H_{\pi,\tau} = \{a_{\tau,1} + n - 1, a_{\tau,2} + n - 2, \ldots, a_{\tau,n}\}.$$

In the case that F is a CM field and π is polarizable (in the sense of [BGGT14, § 2.1]), it is known that for each prime λ of M_π there is a continuous semi-simple representation

$$r_{\pi,\lambda} : G_F \longrightarrow \mathrm{GL}_n(\overline{M}_{\pi,\lambda})$$

such that

$$\mathcal{R}_\pi = (M_\pi, S_\pi, \{Q_{\pi,v}(X)\}, \{r_{\pi,\lambda}\}, \{H_\tau\})$$

is a regular, strictly pure compatible system of l-adic representations. (Combine [BGGT14, Theorem 2.1.1], [Car12, Theorem 1.1] and the usual twisting and descent arguments. Note that [BGGT14, Theorem 2.1.1] simply collects together results of many other authors, see that paper for more details.)

(We note that part (2) of [BGGT14, Theorem 2.1.1] asserts that certain representations are pure of weight w without specifying w. This is the same w that occurs in part (3) of that theorem. However, perhaps confusingly, it is not the same w that occurs five paragraphs earlier. Rather it equals $n-1+w'$ where the weight a of π lies in $(\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C})}_{w'}$. The numbers $a_{\imath\tau,j}$ occurring in part (3) of [BGGT14, Theorem 2.1.1] are the components of this weight a. RLT thanks Clozel for drawing this to his attention.)

Recall from [BGGT14, § 5.1] that a weakly compatible system of l-adic representations $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is called *automorphic* if there is a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ and an embedding $\imath : M \hookrightarrow \mathbb{C}$, such that if $v \notin S$, then π_v is unramified and $\mathrm{rec}(\pi_v|\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial $\imath(Q_v(X))$.

Suppose that F is a CM field, that $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is a weakly compatible system of l-adic representations of G_F and that $\mathcal{M} = (M, S^\mu, \{X-\alpha_v\}, \{\mu_\lambda\}, \{H^\mu_\tau\})$ is a weakly compatible system of l-adic characters of G_{F^+}. Recall (from [BGGT14, § 5.1]) that we call $(\mathcal{R},\mathcal{M})$ a *polarized* (respectively *totally odd, polarized*) weakly compatible system if for all primes λ of M the pair (r_λ, μ_λ) is a polarized (respectively totally odd polarized) l-adic representation in the sense of [BGGT14, § 2.1].

If F is a CM field and $(\mathcal{R},\mathcal{M})$ is an automorphic, regular, polarized weakly compatible system of l-adic representations of dimension n, then $\mathcal{R} \equiv \mathcal{R}_\pi$ for some regular algebraic, polarizable, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$.

We finish this section with an application of Lemma 1.6 to irreducibility results.

Theorem 1.7. *Suppose that F is a CM field and that π is a polarizable, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then there is a finite CM extension M/M_π and a Dirichlet density 1 set $\mathcal{L}$ of rational primes, such that for all conjugation-invariant primes λ of M dividing an $\ell \in \mathcal{L}$, $r_{\pi,\lambda|_{M_\pi}}$ is irreducible.*

In particular, there is a positive Dirichlet density set $\mathcal{L}'$ of rational primes such that if a prime λ of M_π divides some $\ell \in \mathcal{L}'$, then $r_{\pi,\lambda}$ is irreducible.

Proof. The proof is the same as the proof of [BGGT14, Theorem 5.5.2 and Proposition 5.4.6], except that instead of appealing to [BGGT14, Lemma 5.4.5] one appeals to Lemma 1.6 above. □

2. Applications

We will first consider applications of Lemma 1.6 to potential automorphy theorems.

Theorem 2.1. *Suppose that F/F_0 is a finite Galois extension of CM fields, and that $F^{(\mathrm{avoid})}/F$ is a finite Galois extension. Suppose also for $i = 1, \ldots, r$ that $(\mathcal{R}_i, \mathcal{M}_i)$ is a totally odd, polarized weakly compatible system of l-adic representations of G_F, with each $\mathcal{R}_i$ pure and regular.*

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Then there is a finite CM extension F'/F such that F'/F_0 is Galois and F' is linearly disjoint from $F^{(\mathrm{avoid})}$ over F, with the following property. For each i we have a decomposition

$$\mathcal{R}_i \equiv \mathcal{R}_{i,1} \oplus \cdots \oplus \mathcal{R}_{i,s_i}$$

into weakly compatible systems $\mathcal{R}_{i,j}$ with each $(\mathcal{R}_{i,j}, \mathcal{M}_i)|_{G_{F'}}$ automorphic.

Proof. Suppose that

$$\mathcal{R}_i = (M_i, S_i, \{Q_{i,v}(X)\}, \{r_{i,\lambda}\}, \{H_{i,\tau}\})$$

and that M_i is a CM field which is a full field of definition for $\mathcal{R}_i$. Write

$$r_{i,\lambda} = \bigoplus_j r_{i,\lambda,j}$$

with each $r_{i,\lambda,j}$ irreducible. It follows from Lemma 1.6 that if λ is a conjugation invariant prime of M_i then $(r_{i,\lambda,j}, \mu_{i,\lambda})$ is a totally odd, polarized l-adic representation.

Let $\mathcal{L}$ be the Dirichlet density 1 set of rational primes obtained by applying [BGGT14, Proposition 5.3.2] to $\mathcal{R}$. Then for $\lambda|l \in \mathcal{L}$ we see that $\overline{r}_{i,\lambda,j}|_{G_{F(\zeta_l)}}$ is irreducible. Removing finitely many primes from $\mathcal{L}$ we may further suppose that:

(i) $l \in \mathcal{L}$ implies $l \geqslant 2(\dim \mathcal{R}_i + 1)$ for all i;

(ii) $l \in \mathcal{L}$ implies l is unramified in F and l lies below no element of any S_i;

(iii) if $\lambda|l \in \mathcal{L}$, then all of the Hodge–Tate numbers of each $r_{i,\lambda}$ lie in a range of the form $[a, a+l-2]$.

From [BGGT14, Proposition 2.1.2], we deduce that for all $\lambda|l \in \mathcal{L}$, the image $\overline{r}_{i,\lambda,j}(G_{F(\zeta_l)})$ is adequate. Moreover, by [BGGT14, Lemma 1.4.3], $r_{i,\lambda,j}$ is potentially diagonalizable.

For each $i = 1, \ldots, r$ choose a prime $l_i \in \mathcal{L}$ and a conjugation invariant prime $\lambda_i|l_i$ of M_i. Replace $F^{(\mathrm{avoid})}$ by its compositum with the $\overline{F}^{\ker \overline{r}_{i,\lambda_i,j}}(\zeta_{l_i})$ for all i, j. Now apply [BGGT14, Theorem 4.5.1] to $\{r_{i,\lambda_i,j}\}$. We obtain a CM extension F'/F with F'/F_0 Galois and F' linearly disjoint from $F^{(\mathrm{avoid})}$ over F; and regular algebraic, cuspidal, polarized automorphic representations $(\pi_{i,j}, \chi_{i,j})$ of $\mathrm{GL}_{n_{i,j}}(\mathbb{A}_{F'})$ such that

$$r_{l_i,\imath_i}(\pi_{i,j}) = r_{i,\lambda_i,j}|_{G_{F'}}$$

and

$$r_{l,\imath_i}(\chi_i)\epsilon_{l_i}^{1-n_{i,j}} = \mu_{i,\lambda_i}$$

for some $\imath_i : \overline{M}_{i,\lambda_i} \xrightarrow{\sim} \mathbb{C}$.

As in the proof of [BGGT14, Theorem 5.5.1] we see that $r_{i,\lambda_i,j}$ is part of a weakly compatible system $\mathcal{R}_{i,j}$ of l-adic representations of G_F. It is immediate that

$$\mathcal{R}_i \cong \bigoplus_j \mathcal{R}_{i,j}$$

and that each $(\mathcal{R}_{i,j}, \mathcal{M}_i)|_{G_{F'}}$ is automorphic. □

The next corollary now follows in the same way as [BGGT14, Corollary 5.4.3].

Corollary 2.2. *Suppose that F is a CM field, and that $(\mathcal{R}, \mathcal{M})$ is a totally odd, polarized weakly compatible system of l-adic representations of G_F with $\mathcal{R}$ pure and regular.*

(i) *If $\imath : M \hookrightarrow \mathbb{C}$, then $L^S(\imath\mathcal{R}, s)$ converges (uniformly absolutely on compact subsets) on some right half-plane and has meromorphic continuation to the whole complex plane.*

(ii) *The compatible system $\mathcal{R}$ is strictly pure. Moreover,*

$$\Lambda(\imath\mathcal{R}, s) = \epsilon(\imath\mathcal{R}, s)\Lambda(\imath\mathcal{R}^\vee, 1 - s).$$

COROLLARY 2.3. *Suppose that $m \in \mathbb{Z}_{\geqslant 0}$, that F is a totally real field and that X/F is a smooth projective variety such that for all $\tau : F \hookrightarrow \mathbb{C}$ and all $i = 0, \ldots, m$ we have*

$$\dim H^{i,m-i}((X \times_{F,\tau} \mathbb{C})(\mathbb{C}), \mathbb{C}) \leqslant 1.$$

Then $\{H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l)\}$ is a strictly pure compatible system of l-adic representations and the L-function $\Lambda(\{H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l)\}, s)$ has meromorphic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(\{H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l)\}, s) = \epsilon(\{H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l)\}, s)\Lambda(\{H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l)\}, 1 + m - s).$$

In particular, X satisfies the l-adic weight-monodromy conjecture at v for every rational prime l and every finite place v of F.

Proof. By hard Lefschetz and Poincaré duality we see that there is a perfect pairing

$$H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l) \times H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l) \longrightarrow \mathbb{Q}_l(-m)$$

of parity $(-1)^m$. Thus, by Deligne's theorem [Del74],

$$(\{H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l)\}, \{\epsilon_l^{-m}\})$$

is a pure, totally odd, polarized weakly compatible system of l-adic representations of G_F. Moreover it is regular. This corollary follows from the previous one and the isomorphism

$$H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l)^\vee \cong H^m_{\mathrm{et}}(X_{/\overline{F}}, \mathbb{Q}_l)(m). \qquad \square$$

We can extend this corollary to pure motives. We choose to do this in André's category of motivated motives. Suppose that F and M are number fields. We will let $\mathcal{M}_{F,M}$ denote the category of motivated motives over F with coefficients in M (see [And96, § 4]). An object of $\mathcal{M}_{F,M}$ will be a triple (X, p, m), where:

(i) X/F is a smooth, projective variety;
(ii) $p \in C^0_{\mathrm{mot}}(X, X)_M$ is an idempotent;
(iii) and m is an integer.

(See [And96, § 4], and [And96, § 2] for the definition of $C^0_{\mathrm{mot}}(X, X)_M$.) If (X, p, m) is an object of $\mathcal{M}_{F,M}$, then we can form the following cohomology groups:

(a) The de Rham realization

$$H_{\mathrm{DR}}(X, p, m) = \bigoplus_i H^i_{\mathrm{DR}}(p)(H^i_{\mathrm{DR}}(X/F) \otimes_{\mathbb{Q}} M),$$

an $(F \otimes_{\mathbb{Q}} M)$-module with an exhaustive, separated filtration

$$\mathrm{Fil}^k(X, p, m) = \bigoplus_i H^i_{\mathrm{DR}}(p)\mathrm{Fil}^{k+m}(H^i_{\mathrm{DR}}(X/F) \otimes_{\mathbb{Q}} M)$$

by sub-$(F \otimes_{\mathbb{Q}} M)$-modules.

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(b) For all $\tau\colon F \hookrightarrow \mathbb{C}$, the τ-Betti realization

$$H_{B,\tau}(X,p,m) = \bigoplus_i H^i_{B,\tau}(p) H^i_B((X \times_{F,\tau} \mathbb{C})^{an}, M)(m),$$

an M-vector space.

(c) The λ-adic realization

$$H_\lambda(X,p,m) = \bigoplus_i H^i_\lambda(p) H^i_{\mathrm{et}}(X \times_F \bar{F}, M_\lambda)(m),$$

an M_λ-vector space with a continuous G_F-action.

There is also a natural M-linear map

$$c : H_{B,\tau}(X,p,m) \longrightarrow H_{B,c\tau}(X,p,m).$$

If $\widetilde{\tau} : \overline{F} \hookrightarrow \mathbb{C}$ extends τ, then we get a comparison isomorphism

$$\alpha_{\lambda,B,\widetilde{\tau}} : H_\lambda(X,p,m) \xrightarrow{\sim} H_{B,\tau}(X,p,m) \otimes_M M_\lambda.$$

For $\sigma \in G_F$ we have $\alpha_{\lambda,B,\widetilde{\tau}\sigma} = \alpha_{\lambda,B,\widetilde{\tau}} \circ \sigma$. We also have $\alpha_{\lambda,B,c\widetilde{\tau}} = c \circ \alpha_{\lambda,B,\widetilde{\tau}}$. We also have a comparison map

$$\alpha_{\mathrm{DR},B,\tau} : H_{\mathrm{DR}}(X,p,m) \otimes_{F,\tau} \mathbb{C} \xrightarrow{\sim} H_{B,\tau}(X,p,m) \otimes_{\mathbb{Q}} \mathbb{C},$$

which satisfies

$$(c \otimes c) \circ \alpha_{\mathrm{DR},B,c\tau} = \alpha_{\mathrm{DR},B,\tau} \circ (1 \otimes c) : H_{\mathrm{DR}}(X,p,m) \otimes_{F,c\tau} \mathbb{C} \xrightarrow{\sim} H_{B,\tau}(X,p,m) \otimes_{\mathbb{Q}} \mathbb{C}.$$

We will write

$$\mathrm{Fil}^i_\tau(X,p,m) = \alpha_{\mathrm{DR},B,\tau}(\mathrm{Fil}^i(X,p,m) \otimes_{F,\tau} \mathbb{C}),$$

a $M \otimes_{\mathbb{Q}} \mathbb{C}$-submodule of $H_B(X,p,m) \otimes_{\mathbb{Q}} \mathbb{C}$, so that

$$(c \otimes c)\mathrm{Fil}^i_\tau(X,p,m) = \mathrm{Fil}^i_{c\tau}(X,p,m).$$

We will refer to $\dim_M H_{B,\tau}(X,p,m)$, which is independent of τ, as the rank of (X,p,m).

As described in [And96, § 4] one can form the dual $X^\vee$ of an object X of $\mathcal{M}_{F,M}$, and the direct sum $X \oplus Y$ and the tensor product $X \otimes Y$ of two objects X, Y of $\mathcal{M}_{F,M}$. This makes $\mathcal{M}_{F,M}$ a Tannakian category. The functors H_{DR}, $H_{B,\tau}$ and H_λ are faithful and exact. If F'/F is a finite extension, one can restrict an object X of $\mathcal{M}_{F,M}$ to an object $X|_{F'}$ of $\mathcal{M}_{F',M}$.

We will call (X,p,m) *compatible* if there is a finite set of primes S of F and, for each $v \notin S$, a polynomial $Q_v \in M[X]$ such that: if $v \notin S$ and λ does not divide the residue characteristic of v, then $H_\lambda(X,p,m)$ is unramified at v and Q_v is the characteristic polynomial of Frob_v on $H_\lambda(X,p,m)$. If (X,p,m) is compatible, then

$$\mathcal{H}(X,p,m) = (M, S, \{Q_v\}, \{H_\lambda(X,p,m)\}, \{H_\tau\})$$

is a weakly compatible system of l-adic representations, where H_τ contains i with multiplicity

$$\dim_{\overline{M}} \mathrm{gr}^i H_{\mathrm{DR}}(X,p,m) \otimes_{F \otimes M, \tau \otimes 1} \overline{M}.$$

(To see that $\mathrm{HT}_\tau(H_\lambda(X,p,m)) = H_\tau$ one can apply the remarks of [And96, § 2.4].)

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We will call (X,p,m) *pure of weight* w if (for instance)

$$H_{\mathrm{DR}}(X,p,m)=H_{\mathrm{DR}}^{w+2m}(p)(H_{\mathrm{DR}}^{w+2m}(X/F)\otimes_{\mathbb{Q}}M).$$

Any object (X,p,m) of $\mathcal{M}_{F,M}$ can be written uniquely as

$$(X,p,m)=\bigoplus_r \mathrm{Gr}_r^W(X,p,m),$$

where $\mathrm{Gr}_r^W(X,p,m)$ is pure of weight r. (See [And96, § 4.4].) The motivated motive (X,p,m) is pure of weight w if and only if $H_\lambda(X,p,m)$ is. In particular, if (X,p,m) is compatible and pure, then $\mathcal{H}(X,p,m)$ is also pure of the same weight. If (X,p,m) is pure of weight w, then we have

$$\begin{aligned}H_{B,\tau}(X,p,m)\otimes_{\mathbb{Q}}\mathbb{C}&=\mathrm{Fil}_\tau^i(X,p,m)\oplus(1\otimes c)\mathrm{Fil}_\tau^{w+1-i}(X,p,m)\\&=\mathrm{Fil}_\tau^i(X,p,m)\oplus(c\otimes 1)\mathrm{Fil}_{c\tau}^{w+1-i}(X,p,m).\end{aligned}$$

In particular, $\mathrm{gr}_\tau^i(X,p,m)$ and $\mathrm{gr}_\tau^{w-i}(X,p,m)\otimes_{\mathbb{C},c}\mathbb{C}$ are non-canonically isomorphic as $(M\otimes_{\mathbb{Q}}\mathbb{C})$-modules. Note that if (X,p,m) is pure of weight w and has rank 1, and if $\tau:F\hookrightarrow\mathbb{R}$, then w must be even and $\mathrm{Fil}_\tau^{w/2}(X,p,m)\neq\mathrm{Fil}_\tau^{w/2+1}(X,p,m)$.

We will call (X,p,m) *regular* if for each embedding $\tau:F\hookrightarrow\overline{M}$ and each $i\in\mathbb{Z}$ we have

$$\dim_{\overline{M}}\mathrm{gr}^i(H_{\mathrm{DR}}(X,p,m)\otimes_{F\otimes_{\mathbb{Q}}M,\tau\otimes 1_M}\overline{M})\leqslant 1.$$

If (X,p,m) is regular and compatible, then $\mathcal{H}(X,p,m)$ is regular.

The group $\mathrm{Aut}(F)$ also acts on the category $\mathcal{M}_{F,M}$. An element $\sigma\in\mathrm{Aut}(F)$ takes an object X of $\mathcal{M}_{F,M}$ to ${}^\sigma X$. We have the following observations.

(i) There is a σ-linear isomorphism $\sigma:H_{\mathrm{DR}}(X)\xrightarrow{\sim}H_{\mathrm{DR}}({}^\sigma X)$ with $\sigma\mathrm{Fil}^i(X)=\mathrm{Fil}^i({}^\sigma X)$ for all i.

(ii) There is an isomorphism $\sigma:H_{B,\tau}(X)\xrightarrow{\sim}H_{B,\tau\sigma^{-1}}({}^\sigma X)$.

(iii) If we choose $\widetilde{\sigma}\in\mathrm{Aut}(\overline{F})$ lifting σ, then we get an isomorphism

$$\widetilde{\sigma}:H_\lambda(X)\xrightarrow{\sim}H_\lambda({}^\sigma X)$$

such that $\widetilde{\sigma}\circ(\widetilde{\sigma}^{-1}\sigma_1\widetilde{\sigma})=\sigma_1\circ\widetilde{\sigma}$ for all $\sigma_1\in G_F$.

(iv) We have

$$\alpha_{\lambda,B,\widetilde{\tau}\widetilde{\sigma}^{-1}}({}^\sigma X)\circ\widetilde{\sigma}=\sigma\circ\alpha_{\lambda,B,\widetilde{\tau}}(X):H_\lambda(X)\longrightarrow H_{B,\tau\sigma^{-1}}({}^\sigma X)\otimes_M M_\lambda,$$

and

$$\alpha_{\mathrm{DR},B,\tau\sigma^{-1}}({}^\sigma X)\circ\sigma=\sigma\circ\alpha_{\mathrm{DR},B,\tau}(X):H_{\mathrm{DR}}(X)\otimes_{F,\tau}\mathbb{C}\longrightarrow H_{B,\tau\sigma^{-1}}({}^\sigma X)\otimes_{\mathbb{Q}}\mathbb{C}.$$

(v) If X is compatible, then $\mathcal{H}({}^\sigma X)\cong\mathcal{H}(X)^{\sigma^{-1}}$.

If F is a CM field, we will call an object X of $\mathcal{M}_{F,M}$ *polarizable* if there is an object Y of $\mathcal{M}_{F^+,M}$ (necessarily of rank 1) such that

$$X\cong Y|_F\otimes{}^cX^\vee.$$

We warn the reader that this is a non-standard use of the term 'polarizable' in the context of the theory of motives. If X is polarizable and Y is as above, then there are non-degenerate pairings

$$\langle\,,\rangle_{\mathrm{DR}}:H_{\mathrm{DR}}(X)\times H_{\mathrm{DR}}(X)\xrightarrow{1\times c}H_{\mathrm{DR}}(X)\times H_{\mathrm{DR}}({}^cX)\longrightarrow H_{\mathrm{DR}}(Y),$$

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and

$$\langle\,,\rangle_{B,\tau} : H_{B,\tau}(X) \times H_{B,\tau}(X) \xrightarrow{1\times d} H_{B,\tau}(X) \times H_{B,\tau}({}^cX) \longrightarrow H_{B,\tau}(Y),$$

where d denotes the composite

$$H_{B,\tau}(X) \xrightarrow{c} H_{B,c\tau}(X) = H_{B,\tau c}(X) \xrightarrow{c} H_{B,\tau}({}^cX).$$

We have

$$\langle \alpha_{\mathrm{DR},B,\tau}(X)x, (1\otimes c)\alpha_{\mathrm{DR},B,\tau}(X)y\rangle_{B,\tau} = \alpha_{\mathrm{DR},B,\tau}(Y)\langle x,(1\otimes c)y\rangle_{\mathrm{DR}}$$

for all $x, y \in H_{\mathrm{DR}}(X) \otimes_{F,\tau} \mathbb{C}$. Thus if Y has weight $2w$ and $i + j > w$, then

$$\langle \mathrm{Fil}^i_\tau(X), (1\otimes c)\mathrm{Fil}^j_\tau(X)\rangle_{B,\tau} = (0).$$

(Because, as F^+ is totally real and Y has rank 1, we have $\mathrm{Fil}^{w+1}(Y) = (0)$.) Moreover, if $\tilde{c} \in G_{F^+}$ lifts $c \in \mathrm{Gal}(F/F^+)$, then there is a non-degenerate pairing

$$\langle\,,\rangle_{\lambda,\tilde{c}} : H_\lambda(X) \times H_\lambda(X) \xrightarrow{1\times\tilde{c}} H_\lambda(X) \times H_\lambda({}^cX) \longrightarrow H_\lambda(Y)$$

such that for all $\sigma \in G_F$ and $x, y \in H_\lambda(X)$ we have

$$\langle \sigma x, (\tilde{c}^{-1}\sigma\tilde{c})y\rangle_{\lambda,\tilde{c}} = \sigma\langle x,y\rangle_{\lambda,\tilde{c}}.$$

If $\tau : F \hookrightarrow \mathbb{C}$ and $c(\tau)$ is the corresponding complex conjugation in G_{F^+} we have

$$\alpha_{\lambda,B,\tau}\langle x,y\rangle_{\lambda,c(\tau)} = \langle \alpha_{\lambda,B,\tau}x, \alpha_{\lambda,B,\tau}y\rangle_{B,\tau}$$

for all $x, y \in H_\lambda(X)$.

Lemma 2.4. *Suppose that F is a CM field, that M is a number field and that X is a compatible, regular, polarizable, pure object of $\mathcal{M}_{F,M}$. Then the weakly compatible system of l-adic representations $\mathcal{H}(X)$ is totally odd, polarizable.*

Proof. It suffices to show that $\langle\,,\rangle_{\lambda,\widetilde{c}}$ is symmetric for all complex conjugations $\widetilde{c} \in G_{F^+}$. Choose an embedding $\tau_1 : \overline{F} \hookrightarrow \mathbb{C}$ such that $\tau_1 \circ \widetilde{c} = c \circ \tau_1$. Then τ_1 gives rise to an isomorphism

$$H_{B,\tau_1|_F}(X) \otimes_M M_\lambda \cong H_\lambda(X).$$

Under this isomorphism $\langle\,,\rangle_{\lambda,\widetilde{c}}$ corresponds to $\langle\,,\rangle_{B,\tau_1|_F}$, so it suffices to show that $\langle\,,\rangle_{B,\tau}$ is symmetric for all $\tau : F \hookrightarrow \mathbb{C}$.

If w denotes the weight of X, then we have

$$\mathrm{Fil}^i_\tau(X) \oplus (1\otimes c)\mathrm{Fil}^{w+1-i}_\tau(X) = H_{B,\tau}(X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Moreover, as Y must have weight $2w$, we see that, if $i + j > w$, then $\mathrm{Fil}^i_\tau(X)$ and $(1\otimes c)\mathrm{Fil}^j_\tau(X)$ must annihilate each other under $\langle\,,\rangle_{B,\tau}$.

Let $\tau' : M \hookrightarrow \mathbb{C}$ and set

$$H_{B,\tau}(X)_{\tau'} = H_{B,\tau}(X) \otimes_{M,\tau'} \mathbb{C}$$

and

$$\mathrm{Fil}^i_\tau(X)_{\tau'} = \mathrm{Fil}^i_\tau(X) \otimes_{M\otimes_{\mathbb{Q}}\mathbb{C},\tau'\otimes 1} \mathbb{C}.$$

Thus

$$\mathrm{Fil}^i_\tau(X)_{\tau'} \oplus (1\otimes c)\mathrm{Fil}^{w+1-i}_\tau(X)_{c\tau'} = H_{B,\tau}(X)_{\tau'}.$$

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Moreover, $\langle\, ,\rangle_{B,\tau}$ gives rise to a pairing

$$\langle\, ,\rangle_{B,\tau,\tau'} : H_{B,\tau}(X)_{\tau'} \times H_{B,\tau}(X)_{\tau'} \longrightarrow H_{B,\tau}(Y)_{\tau'},$$

under which $\mathrm{Fil}^i_\tau(X)_{\tau'}$ and $(1\otimes c)\mathrm{Fil}^j_\tau(X)_{c\tau'}$ annihilate each other whenever $i+j>w$. It suffices to show that for all τ and τ' the pairing $\langle\, ,\rangle_{B,\tau,\tau'}$ is symmetric.

Note that if $\mathrm{gr}^i_\tau(X)_{\tau'} \neq (0)$, then $\mathrm{gr}^{w-i}_\tau(X)_{c\tau'} \neq (0)$. Thus, by regularity,

$$\mathrm{Fil}^i_\tau(X)_{\tau'} \cap (1\otimes c)\mathrm{Fil}^{w-i}_\tau(X)_{c\tau'}$$

is one dimensional over $\mathbb{C}$. Let e_i be a basis vector. As

$$\mathrm{Fil}^{i+1}_\tau(X)_{\tau'} \cap (1\otimes c)\mathrm{Fil}^{w-i}_\tau(X)_{c\tau'} = (0)$$

we see that $e_i \in \mathrm{Fil}^i_\tau(X)_{\tau'} - \mathrm{Fil}^{i+1}_\tau(X)_{\tau'}$. Thus, $\{e_i\}$ is a basis of $H_{B,\tau}(X)_{\tau'}$. If $i>j$, then, since $e_i \in \mathrm{Fil}^i_\tau(X)_{\tau'}$ and $e_j \in (1\otimes c)\mathrm{Fil}^{w-j}_\tau(X)_{c\tau'}$,

$$\langle e_i, e_j\rangle_{B,\tau,\tau'} = 0.$$

Similarly if $j>i$, then, since $e_i \in (1\otimes c)\mathrm{Fil}^{w-i}_\tau(X)_{c\tau'}$ and $e_j \in \mathrm{Fil}^j_\tau(X)_{\tau'}$,

$$\langle e_i, e_j\rangle_{B,\tau,\tau'} = 0.$$

We conclude that the matrix of $\langle\, ,\rangle_{B,\tau,\tau'}$ with respect to the basis $\{e_i\}$ is diagonal, and hence $\langle\, ,\rangle_{B,\tau,\tau'}$ is symmetric, as desired. □

Combining this lemma with Theorem 2.1 we obtain the following corollary.

Corollary 2.5. *Suppose that F is a CM field, that M is a number field and that X is a compatible, regular, polarizable, pure object of $\mathcal{M}_{F,M}$. Then there is a finite Galois CM extension F'/F and a decomposition (perhaps after extending the field M)*

$$\mathcal{H}(X) \equiv \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_s$$

such that each $\mathcal{R}_i|_{G_{F'}}$ is automorphic.

In particular, $\mathcal{H}(X)$ is strictly pure and, if $\imath : M \hookrightarrow \mathbb{C}$, then $\Lambda(\imath\mathcal{H}(X), s)$ has meromorphic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(\imath\mathcal{H}(X), s) = \epsilon(\imath\mathcal{H}(X), s)\Lambda(\imath\mathcal{H}(X)^\vee, 1-s).$$

Remark 2.6. For any X as in the statement of Corollary 2.5 and any finite place v of F, it follows from strict purity that the weight-monodromy conjecture holds for the motive $X_{F_v} \in \mathcal{M}_{F_v,M}$. Namely, for all finite places λ of M, of residue characteristic different from that of v, the monodromy and weight filtrations on the G_{F_v}-representation $H_\lambda(X_{F_v})$ coincide.

For example, this corollary applies to the weight $n-1$ part of the motivated motive

$$\Big(Z_t, (1/\#H') \sum_{h\in H'} h, 0\Big),$$

where:

(i) $t \in \mathbb{Q}$;

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(ii) Z_t is the smooth hypersurface defined by

$$X_0^{n+1} + \cdots + X_n^{n+1} = (n+1)tX_0 \cdots X_n$$

in projective n-space;

(iii) H' is the group $\ker(\mu_{n+1}^{n+1} \xrightarrow{\Pi} \mu_{n+1})$, which acts on Z_t by multiplication on the coordinates.

(In the case $t \notin \mathbb{Z}[1/(n+1)]$ this was already proved in [HST10].)

We next give a much more general example coming from Katz's theory of rigid local systems [Kat96]. Let $S \subset \mathbb{P}^1(\mathbb{Q})$ be a finite set with complement $U = \mathbb{P}^1 - S$. Also let $N \in \mathbb{Z}_{>0}$. By a *rigid local system* $\mathcal{F}$ *on* $U_{\overline{\mathbb{Q}}}$ *with quasi-unipotent monodromy of order dividing* N we shall mean a lisse $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}$ on $U_{\overline{\mathbb{Q}}}$ with the following properties:

(i) $\mathcal{F}$ is irreducible;

(ii) the Nth power of the monodromy of $\mathcal{F}$ at every point $s \in S$ is unipotent; and

(iii) any other $\overline{\mathbb{Q}}_l$-local system on the complex manifold $U(\mathbb{C})$ with monodromy conjugate to that of $\mathcal{F}^{\mathrm{an}}$ at every point $s \in S$ is isomorphic to $\mathcal{F}^{\mathrm{an}}$.

By [Kat96, Theorem 1.1.2] the third condition is equivalent to the 'cohomological rigidity' of $\mathcal{F}$ in the sense of [Kat96, § 5.0]. An abundant supply of such sheaves is supplied by the constructions of [Kat96, § 5.1]. One may keep track of the monodromy at points $s \in S$ using the results of [Kat96, chapter 6]. (See [DR10, § 1.2] for a nice summary of these formulae.)

Suppose that $\mathcal{F}$ is a rigid local system with quasi-unipotent monodromy of order dividing N. According to [Kat96, Theorem 8.4.1] there is a smooth quasi-projective morphism of schemes

$$\pi : \mathrm{Hyp} \longrightarrow U$$

with geometrically connected fibres and an action of μ_N on Hyp over U, and a faithful character $\chi : \mu_N(\overline{\mathbb{Q}}) \to \overline{\mathbb{Q}}_l^\times$, so that

$$\mathcal{F} \cong (\mathrm{Gr}_r^W R^r \pi_! \overline{\mathbb{Q}}_l)^{\chi}_{\overline{\mathbb{Q}}}.$$

Here Gr^W denotes the weight filtration; r denotes the relative dimension of Hyp over U; and the subscript $\overline{\mathbb{Q}}$ indicates base change to $U_{\overline{\mathbb{Q}}}$. More specifically, we assume $\infty \in S$ (as we may up to automorphism of $\mathbb{P}^1$) and let Aff^{r+1} denote affine space of dimension $r+1$ (with coordinates $X_1, \dots, X_{r+1}$). Then Hyp is a hypersurface in $\mathbb{G}_m \times \mathrm{Aff}^{r+1}$ defined by an equation of the form (taking Y to be the coordinate on $\mathbb{G}_m$)

$$Y^N = \prod_{s \in S - \{\infty\}} \prod_{i=1}^{r+1} (X_i - s)^{e_i(s)} \prod_{i=1}^{r} (X_{i+1} - X_i)^{f_i},$$

where $e_i(s)$ and $f_i \in \mathbb{Z}_{\geqslant 0}$ and none of the f_i is divisible by N. Moreover, the action of μ_N is by multiplication on Y and the map to U is projection to the coordinate X_{r+1} Any choice of r, $e_i(s)$, f_i and (faithful) χ can arise for some $\mathcal{F}$.

If K is a number field and if $u \in U(K)$, then there is a μ_N-equivariant projective compactification[1]

$$\mathrm{Hyp}_u \hookrightarrow \overline{\mathrm{Hyp}}_u$$

[1] The existence of such was announced but not written up by Hironaka. For a proof, see [AW97]; namely, consider the product of all compositions with $g \in \mu_N(\overline{\mathbb{Q}})$ of a given $\mathrm{Hyp} \to \mathbb{P}^N$ to produce a μ_N-equivariant inclusion of Y as a quasi-projective sub-variety of some $\mathbb{P}^M$; take the projective closure and apply [AW97, Theorem 0.1] to the result.

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with complement $D = \bigcup_i D_i$ a union of smooth divisors with normal crossings. Then by a standard argument (see, for example, [Del71])

$$(\mathrm{Gr}_r^W R^r \pi_! \overline{\mathbb{Q}}_l)_u^\chi \cong \ker\left(H^r(\overline{\mathrm{Hyp}}_{u,\overline{K}}, \overline{\mathbb{Q}}_l) \longrightarrow \bigoplus_{i\in I} H^r(D_{i,\overline{K}}, \overline{\mathbb{Q}}_l) \right)^\chi.$$

If $K \supset \mathbb{Q}(\zeta_N)$, so that $\chi : \mu_N \to \mathbb{G}_m$ over K, then we define a pure object $M(\mathcal{F}, u)$ of $\mathcal{M}_{K,\mathbb{Q}(\zeta_N)}$ to be the χ component of the kernel of the map of motivated motives

$$\mathrm{Gr}_r^W(\overline{\mathrm{Hyp}}_u, 1, 0) \longrightarrow \bigoplus_{i\in I} \mathrm{Gr}_r^W(D_i, 1, 0).$$

Then for any rational prime l' and any embedding $i' : \mathbb{Q}(\zeta_N) \hookrightarrow \overline{\mathbb{Q}}_{l'}$ we have

$$\begin{aligned} H_{\lambda'}(M(\mathcal{F}, u)) \otimes_{\mathbb{Q}(\chi)_{\lambda'}, i'} \overline{\mathbb{Q}}_{l'} &\cong \mathrm{Gr}_r^W H_c^r(\mathrm{Hyp}_{u,\overline{K}}, \overline{\mathbb{Q}}_{l'})^{i'(\chi)} \\ &\cong \ker(H^r(\overline{\mathrm{Hyp}}_{u,\overline{K}}, \overline{\mathbb{Q}}_{l'}) \longrightarrow \bigoplus_{i\in I} H^r(D_{i,\overline{K}}, \overline{\mathbb{Q}}_{l'}))^{i'(\chi)}, \end{aligned}$$

where λ' denotes the prime of $\mathbb{Q}(\zeta_N)$ induced by i'. We do not claim that $M(\mathcal{F}, u)$ only depends on $\mathcal{F}$ and u. To the best of the authors' knowledge it also depends on the choices of Hyp and χ. It is however independent of the choice of compactification $\overline{\mathrm{Hyp}}_u$.[2] *When we make an assertion about $M(\mathcal{F}, u)$ it should be read as applying whatever auxiliary choices are made.*

It follows from [Pin92] that for all but finitely many places v of K, all j sufficiently large and all $\zeta \in \mu_N(K)$ the alternating sum

$$\sum_s (-1)^s \mathrm{tr}\, (\zeta \mathrm{Frob}_v^j)|_{H_c^s(\mathrm{Hyp}_{u,\overline{K}}, \overline{\mathbb{Q}}_{l'})}$$

lies in $\mathbb{Q}$ and is independent of l'. Thus,

$$\sum_s (-1)^s \mathrm{tr}\ \mathrm{Frob}_v^j|_{H_c^s(\mathrm{Hyp}_{u,\overline{K}}, \overline{\mathbb{Q}}_{l'})^{i'(\chi)}}$$

lies in $\mathbb{Q}(\zeta_N)$ and is independent of l' and i'. By [Kat96, Theorem 8.4.1(2)] we see that this latter sum simply equals

$$(-1)^r \mathrm{tr}\ \mathrm{Frob}_v^j|_{H_c^r(\mathrm{Hyp}_{u,\overline{K}}, \overline{\mathbb{Q}}_{l'})^{i'(\chi)}}.$$

Thus,

$$\mathrm{tr}\ \mathrm{Frob}_v^j|_{\mathrm{Gr}_r^W H_c^r(\mathrm{Hyp}_{u,\overline{K}}, \overline{\mathbb{Q}}_{l'})^{i'(\chi)}}$$

is independent of l' and i'. We deduce that $M(\mathcal{F}, u)$ is compatible.

LEMMA 2.7. *If $\mathbb{Q}(u)$ is totally real, then*

$${}^c M(\mathcal{F}, u) \cong M(\mathcal{F}, u)^\vee(-r).$$

Thus, $M(\mathcal{F}, u)$ is compatible, polarized and pure.

[2] Given two such equivariant projective compactifications we can find a third such compactification mapping to both of them. To see this one just needs to apply [AW97, Theorem 0.1] to the closure of the diagonal embedding of Hyp into the product of the two given compactifications. Then the motivated motive resulting from the third compactification maps to the motivated motives arising from the first two compactifications. Moreover, because they induce isomorphisms in cohomology, these maps are isomorphisms.

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Proof. ${}^cM(\mathcal{F},u)$ is the $\chi\circ c=\chi^{-1}$ component of the kernel of the map of motivated motives

$$\mathrm{Gr}_r^W(\overline{\mathrm{Hyp}}_u,1,0)\longrightarrow\bigoplus_{i\in I}\mathrm{Gr}_r^W(D_i,1,0).$$

Here $M(\mathcal{F},u)^\vee(-r)$ is isomorphic to the χ component of the cokernel of the map

$$\mathrm{Gr}_r^W(\overline{\mathrm{Hyp}}_u,1,0)^\vee(-r)\longleftarrow\bigoplus_{i\in I}\mathrm{Gr}_r^W(D_i,1,0)^\vee(-r).$$

By definition of the dual of an object in $\mathcal{M}_{K,\mathbb{Q}(\chi)}$ this map can be thought of as a map

$$\mathrm{Gr}_r^W(\overline{\mathrm{Hyp}}_u,1,0)\longleftarrow\bigoplus_{i\in I}\mathrm{Gr}_r^W(D_i,1,-1).$$

Moreover, if $\zeta\in\mu_N$, then the action $\zeta^\vee$ of ζ on $(\overline{\mathrm{Hyp}}_u,1,0)^\vee(-r)$ corresponds to the action of ζ^{-1} on $(\overline{\mathrm{Hyp}}_u,1,0)$. (The transpose of the graph of an automorphism is the graph of the inverse automorphism.) Thus, $M(\mathcal{F},u)^\vee(-r)$ is isomorphic to the χ^{-1} component of the cokernel of the map

$$\mathrm{Gr}_r^W(\overline{\mathrm{Hyp}}_u,1,0)\longleftarrow\bigoplus_{i\in I}\mathrm{Gr}_r^W(D_i,1,-1).$$

There is a natural map from the kernel of

$$\mathrm{Gr}_r^W(\overline{\mathrm{Hyp}}_u,1,0)\longrightarrow\bigoplus_{i\in I}\mathrm{Gr}_r^W(D_i,1,0)$$

to the cokernel of the map

$$\mathrm{Gr}_r^W(\overline{\mathrm{Hyp}}_u,1,0)\longleftarrow\bigoplus_{i\in I}\mathrm{Gr}_r^W(D_i,1,-1)$$

induced by the identity on $(\overline{\mathrm{Hyp}}_u,1,0)$. To prove the lemma it suffices to show that after taking χ-components this map is an isomorphism, or even that it is an isomorphism after applying H_l. Applying H_l we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
(0) & \longrightarrow & \mathrm{gr}_r^W H_c^r(\mathrm{Hyp}_{u,\overline{\mathbb{Q}}},\mathbb{Q}_l) & \longrightarrow & H_c^r(\overline{\mathrm{Hyp}}_{u,\overline{\mathbb{Q}}},\mathbb{Q}_l) & \longrightarrow & \bigoplus_{i\in I}H_c^r(D_{i,\overline{\mathbb{Q}}},\mathbb{Q}_l)\\
 & & \downarrow & & \| & & \\
(0) & \longleftarrow & \mathrm{gr}_r^W H^r(\mathrm{Hyp}_{u,\overline{\mathbb{Q}}},\mathbb{Q}_l) & \longleftarrow & H^r(\overline{\mathrm{Hyp}}_{u,\overline{\mathbb{Q}}},\mathbb{Q}_l) & \longleftarrow & \bigoplus_{i\in I}H^{r-2}(D_{i,\overline{\mathbb{Q}}},\mathbb{Q}_l(-1))
\end{array}$$

where the second row is the dual of the first row and where the first vertical map is the natural map. Thus, it suffices to show that

$$\mathrm{gr}_r^W H_c^r(\mathrm{Hyp}_{u,\overline{\mathbb{Q}}},\mathbb{Q}_l)^\chi\longrightarrow H^r(\mathrm{Hyp}_{u,\overline{\mathbb{Q}}},\mathbb{Q}_l)$$

is injective.

To do this we return to the constructions of [Kat96, chapter 8]. Looking at the proof of [Kat96, Theorem 8.4.1] we see that it suffices to check that, in the notation of [Kat96, Theorem 8.3.5], the map

$$R^r(\mathrm{pr}_{r+1})_!\mathcal{L}\longrightarrow R^r(\mathrm{pr}_{r+1})_*\mathcal{L}$$

has image $\mathcal{H}_r$.

To prove this we first note that one can add to the conclusion of [Kat96, Lemma 8.3.2] the assertion that the natural map

$$\mathrm{NC}_\chi(\mathcal{F}) \longrightarrow R^1(\mathrm{pr}_2)_*(\mathrm{pr}_1^*(\mathcal{F}) \otimes \mathcal{L}_{\chi(X_2-X_1)})$$

has image $\mathrm{MC}_\chi(\mathcal{F})$. (In the notation of that lemma.) Indeed this follows from the other conclusions of that lemma and the fact that

$$R^1(\overline{\mathrm{pr}}_2)_*(j_*(\mathrm{pr}_1^*(\mathcal{F}) \otimes \mathcal{L}_{\chi(X_2-X_1)})) \longrightarrow R^1(\mathrm{pr}_2)_*(\mathrm{pr}_1^*(\mathcal{F}) \otimes \mathcal{L}_{\chi(X_2-X_1)})$$

is injective. (We remark that this statement is essentially the fact that for $j\colon V \subset Y$ an inclusion of an open V in a smooth proper curve Y, and $\mathcal{F}$ a lisse sheaf on V, the parabolic cohomology

$$H^1(Y, j_*\mathcal{F}) \cong \mathrm{Im}\,(H^1_c(V,\mathcal{F}) \to H^1(V,\mathcal{F})).)$$

Now we return to our desired strengthening of [Kat96, Theorem 8.3.5]. We argue by induction on r. The case $r=1$ is just our strengthening of [Kat96, Lemma 8.3.2]. In general, by the inductive hypothesis we know that

$$\mathcal{H}_{r-1} \longrightarrow R^{r-1}(\mathrm{pr}_r)_*\mathcal{L}$$

is injective, where we use the notation of the proof of [Kat96, Theorem 8.3.5]. Pulling back to $\mathbb{A}(n,2)$ and again applying our strengthening of [Kat96, Lemma 8.3.2] we see that

$$\mathcal{H}_r \hookrightarrow R^1(\mathrm{pr}_2)_*R^{r-1}(\mathrm{pr}'_{r+1})_*\mathcal{L},$$

where pr'_{r+1} denotes the map $\mathbb{A}(n,r+1) \to \mathbb{A}(n,2)$, which forgets the first $r-1$ X-coordinates. As $\mathrm{pr}_{r+1} = \mathrm{pr}_2 \circ \mathrm{pr}'_{r+1}$, we have a spectral sequence with second page

$$E_2^{i,j} = R^i(\mathrm{pr}_2)_*R^j(\mathrm{pr}'_{r+1})_*\mathcal{L} \Rightarrow R^{i+j}(\mathrm{pr}_{r+1})_*\mathcal{L}.$$

However pr_2 is affine of relative dimension one. Thus, $R^i(\mathrm{pr}_2)_*$ vanishes for $i>1$ and our spectral sequence degenerates at E_2. We deduce that

$$\mathcal{H}_r \hookrightarrow R^1(\mathrm{pr}_2)_*R^{r-1}(\mathrm{pr}'_{r+1})_*\mathcal{L} \hookrightarrow R^r(\mathrm{pr}_{r+1})_*\mathcal{L},$$

as desired. □

If we want to apply Corollary 2.5 to $M(\mathcal{F},u)$ we must calculate

$$\dim_{\overline{\mathbb{Q}(\chi)}} \mathrm{gr}^j H_{\mathrm{DR}}(M(\mathcal{F},u)) \otimes_{K\otimes\mathbb{Q}(\chi),\tau\otimes 1} \overline{\mathbb{Q}(\chi)}$$

for all j and all $\tau : K \hookrightarrow \overline{\mathbb{Q}(\chi)}$. Choosing an embedding $\imath : \overline{\mathbb{Q}(\chi)} \hookrightarrow \mathbb{C}$ we see that this equals the dimension of the $\imath(\chi)$-component of jth-graded piece (for the Hodge filtration) of the kernel of

$$H^r(\overline{\mathrm{Hyp}}_u(\mathbb{C}),\mathbb{C}) \longrightarrow \bigoplus_{i\in I} H^r(D_i(\mathbb{C}),\mathbb{C}),$$

where the $\mathbb{C}$ points of these schemes over K are calculated via $\imath\circ\tau : K \hookrightarrow \mathbb{C}$. This is the same as

$$\dim_{\mathbb{C}} \mathrm{gr}^j_F \mathrm{gr}^W_r H^r_c(\mathrm{Hyp}_u(\mathbb{C}),\mathbb{C})^{\imath(\chi)},$$

where gr^j_F denotes the graded pieces for the Hodge filtration. This in turn equals the rank of

$$\mathrm{gr}^j_F \mathrm{gr}^W_r (R^r\pi_!\mathbb{C})^{\imath(\chi)}.$$

This does not depend on the choice of u. Extend $\imath$ to an isomorphism $\overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and let $\imath'$ be an extension of $\imath \circ \tau$ to an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Then

$$\mathrm{gr}^W_r (R^r \pi_! \mathbb{C})^{\imath(\chi)} \cong \imath'_* (\mathcal{F} \otimes_{\overline{\mathbb{Q}}_l, \imath} \mathbb{C})^{\mathrm{an}}.$$

We see that for any $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and $\imath' : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ the sheaf

$$\mathcal{F}_{\imath,\imath'} = \imath'_* (\mathcal{F} \otimes_{\overline{\mathbb{Q}}_l, \imath} \mathbb{C})^{\mathrm{an}}$$

admits a polarizable variation of $\mathbb{C}$-Hodge structures. (For variations of Hodge structures we will follow the terminology of [Tay12, § 3.4].) According to [Del87, Proposition 1.13] this polarizable variation of $\mathbb{C}$-Hodge structures is unique up to translating the numbering. We will say that $\mathcal{F}$ is *regular* if for all $\imath, \imath'$ and all j we have

$$\dim_{\mathbb{C}} \mathrm{gr}^j_F \mathcal{F}_{\imath,\imath'} \leqslant 1.$$

Then we deduce the following consequence of Corollary 2.5.

Corollary 2.8. *Suppose that S is a finite subset of $\mathbb{P}^1(\mathbb{Q})$ and that $\mathcal{F}$ is a regular rigid local system on $(\mathbb{P}^1 - S)_{\overline{\mathbb{Q}}}$ with quasi-unipotent monodromy of order dividing N. Suppose that F is a CM field containing a primitive Nth root of unity and that $u \in F^+$. Then there is a finite Galois CM extension F'/F and a decomposition (perhaps after extending the field of coefficients)*

$$\mathcal{H}(M(\mathcal{F}, u)) \equiv \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_s$$

such that each $\mathcal{R}_i|_{G_{F'}}$ is automorphic.

In particular, $\mathcal{H}(M(\mathcal{F}, u))$ is strictly pure and, if $\imath : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$, then the completed L-function $\Lambda(\imath\mathcal{H}(M(\mathcal{F}, u)), s)$ has meromorphic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(\imath\mathcal{H}(M(\mathcal{F}, u)), s) = \epsilon(\imath\mathcal{H}(M(\mathcal{F}, u)), s)\Lambda(\imath\mathcal{H}(M(\mathcal{F}, u))^\vee, 1 - s).$$

To apply this corollary one must be able to calculate the

$$\dim_{\mathbb{C}} \mathrm{gr}^j_F \mathcal{F}_{\imath,\imath'}.$$

This is discussed in [DS13], but seems in general to be a complicated question. We will discuss a more explicit condition that implies regularity, but which is probably a much stronger condition.

We will say that $\mathcal{F}$ *has somewhere maximally quasi-unipotent monodromy* if for some $s \in S$ the monodromy of $\mathcal{F}$ at s has only one Jordan block. We have the following lemma.

Lemma 2.9. *Let Δ^* denote a punctured open disk in the complex plane, and let $\mathcal{F}$ denote the local system underlying a pure polarizable complex variation of Hodge structure having maximally quasi-unipotent local monodromy at the puncture. Then $\mathcal{F}$ is regular. In particular, this applies to a rigid local system $\mathcal{F}$ on $\mathbb{P}^1(\mathbb{C}) - S$ with somewhere maximally quasi-unipotent monodromy.*

Proof. Let s be the puncture, so that the monodromy of $\mathcal{F}$ at s has only one Jordan block. Passing to a finite étale cover, we may suppose that $\mathcal{F}$ has unipotent monodromy at s.

There is a limit mixed Hodge structure $\mathcal{F}_{\imath,\imath',s}$ as follows. Here $\mathcal{F}_{\imath,\imath',s}$ is a $\mathbb{C}$-vector space with:

(a) two decreasing exhaustive and separated filtrations Fil^i_F and $\overline{\mathrm{Fil}}^i_F$;
(b) an increasing exhaustive and separated filtration Fil^W_j;
(c) and a nilpotent endomorphism N;

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with the following properties:

(i) $\mathrm{Fil}^W_j = \sum_{i_1+r=1+j+i_2} (\ker N^{i_1}) \cap (\mathrm{Im}\, N^{i_2})$;

(ii) $\mathcal{F}_{\imath,\imath',s}$ with its automorphism $\exp N$ is isomorphic to $\mathcal{F}_{\imath,\imath',z}$ with its monodromy operator for any $z \in \Delta^*$;

(iii) $\dim_{\mathbb{C}} \mathrm{gr}^i_F \mathcal{F}_{\imath,\imath',s} = \dim_{\mathbb{C}} \mathrm{gr}^i_F \mathcal{F}_{\imath,\imath',z}$ and $\dim_{\mathbb{C}} \overline{\mathrm{gr}}^i_F \mathcal{F}_{\imath,\imath',s} = \dim_{\mathbb{C}} \overline{\mathrm{gr}}^i_F \mathcal{F}_{\imath,\imath',z}$ for any $z \in \Delta^*$;

(iv) Fil^i_F and $\overline{\mathrm{Fil}}^i_F$ induce on $\mathrm{gr}^W_j \mathcal{F}_{\imath,\imath',s}$ a pure $\mathbb{C}$-Hodge structure of weight j;

(v) we have equalities
$$N \, \mathrm{Fil}^i_F \mathcal{F}_{\imath,\imath',s} = (\mathrm{Im}\, N) \cap \mathrm{Fil}^{i-1}_F \mathcal{F}_{\imath,\imath',s}$$
and
$$N \, \overline{\mathrm{Fil}}^i_F \mathcal{F}_{\imath,\imath',s} = (\mathrm{Im}\, N) \cap \overline{\mathrm{Fil}}^{i-1}_F \mathcal{F}_{\imath,\imath',s}.$$

This is true for any polarizable variation of pure $\mathbb{C}$-Hodge structures with unipotent monodromy on an open complex disc minus one point. This follows from the corresponding fact for polarizable variations of pure $\mathbb{R}$-Hodge structures, by the dictionary between variations of $\mathbb{C}$-Hodge structures and variations of $\mathbb{R}$-Hodge structures with an action of $\mathbb{C}$. (See for instance [Tay12, § 3.4].) In the case of variations of $\mathbb{R}$-Hodge structures it follows from [Sch73, Theorem 6.16]. Note that [Sch73, Theorem 6.16] seems to make the assumption that the variation of Hodge structures is the cohomology of a family of smooth projective varieties. However, as pointed out in [CKS86], see in particular the first paragraph of page 462 of that paper, this plays no role in the proof.

From assertion (v) above we see that for all i we have
$$N : \mathrm{Fil}^i_F \mathcal{F}_{\imath,\imath',s} / (\mathrm{Fil}^{i+1}_F \mathcal{F}_{\imath,\imath',s} + \ker N) \hookrightarrow \mathrm{gr}^{i-1}_F \mathcal{F}_{\imath,\imath',s}. \tag{2.1}$$

Choose i_0 maximal such that $\ker N \subset \mathrm{Fil}^i_F \mathcal{F}_{\imath,\imath',s}$. For $i < i_0$ we have
$$N : \mathrm{gr}^i_F \mathcal{F}_{\imath,\imath',s} \hookrightarrow \mathrm{gr}^{i-1}_F \mathcal{F}_{\imath,\imath',s}$$
and so in fact $\mathrm{gr}^i_F \mathcal{F}_{\imath,\imath',s} = (0)$. Thus, $\mathrm{Fil}^{i_0}_F \mathcal{F}_{\imath,\imath',s} = \mathcal{F}_{\imath,\imath',s}$. As $\ker N$ is one dimensional we see that
$$\mathrm{Fil}^{i_0+1}_F \cap \ker N = (0).$$

Because of the injection equation (2.1) we see that
$$\mathrm{Fil}^{i_0}_F \mathcal{F}_{\imath,\imath',s} = \mathrm{Fil}^{i_0+1}_F \mathcal{F}_{\imath,\imath',s} \oplus \ker N,$$
and that, for $i \geqslant i_0$,
$$N^{i-i_0} : \mathrm{gr}^i_F \mathcal{F}_{\imath,\imath',s} \hookrightarrow \mathrm{gr}^{i_0}_F \mathcal{F}_{\imath,\imath',s}.$$

Thus, for all i
$$\dim_{\mathbb{C}} \mathrm{gr}^i_F \mathcal{F}_{\imath,\imath',s} \leqslant 1,$$
and the lemma follows. □

One can use the constructions of [Kat96, § 8.3] to construct many examples of rigid local systems $\mathcal{F}$ on $U_{\overline{\mathbb{Q}}}$ with quasi-unipotent monodromy of order dividing N and with somewhere maximally quasi-unipotent monodromy. More classically, Corollary 2.8 applies to any hypergeometric local system on $\mathbb{P}^1 - \{0, 1, \infty\}$ with somewhere maximally unipotent local monodromy. By [BH89, Theorem 3.5], we get such an example for any choice of function $m : \mu_\infty(\overline{\mathbb{Q}}) - \{1\} \to \mathbb{Z}_{\geqslant 0}$ with finite support. It will have rank $\sum_\zeta m(\zeta)$. The local monodromies at

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Table 1. The local monodromies for the sheaves $\mathcal{F}_i$.

	At 0	At 1	At ∞
$i \equiv 0 \mod 4$	$\mathbf{1}^{\oplus i/2} \oplus (-\mathbf{1})^{\oplus((i/2)+1)}$	$-\mathbf{1} \oplus U(2)^{\oplus i/2}$	$U(i+1)$
$i \equiv 1 \mod 4$	$U(2)^{\oplus(i+1)/2}$	$(-\mathbf{1} \otimes U(2)) \oplus (-\mathbf{1})^{\oplus(i-1)/2} \oplus \mathbf{1}^{\oplus(i-1)/2}$	$U(i+1)$
$i \equiv 2 \mod 4$	$\mathbf{1}^{\oplus i/2} \oplus (-\mathbf{1})^{\oplus((i/2)+1)}$	$U(3) \oplus U(2)^{\oplus(i-2)/2}$	$U(i+1)$
$i \equiv 3 \mod 4$	$U(2)^{\oplus(i+1)/2}$	$U(2) \oplus \mathbf{1}^{\oplus(i-3)/2} \oplus (-\mathbf{1})^{\oplus(i+1)/2}$	$U(i+1)$

the three punctures will be a quasi-reflection, a single unipotent Jordan block, and a Jordan form where each $\zeta \in \mu_\infty(\overline{\mathbb{Q}}) - \{1\}$ appears in a single Jordan block with length equal to $m(\zeta)$. Finally, here is a less classical, non-hypergeometric example, generalizing a construction in [DR10]:

If χ is a continuous $\overline{\mathbb{Q}}_l$-valued character of $\pi_1(\mathbb{G}_{m,\overline{\mathbb{Q}}})$ let $\mathcal{L}_\chi$ denote the corresponding lisse sheaf on $\mathbb{G}_{m,\overline{\mathbb{Q}}}$. If χ_1 and χ_2 are continuous $\overline{\mathbb{Q}}_l$-valued characters of $\pi_1(\mathbb{G}_{m,\overline{\mathbb{Q}}})$ let $\mathcal{L}(\chi_1, \chi_2)$ denote the lisse sheaf on $(\mathbb{P}^1 - \{0, 1, \infty\})_{\overline{\mathbb{Q}}}$ which is the tensor product of the pull-back of $\mathcal{L}_{\chi_1}$ under the identity map with the pull-back of $\mathcal{L}_{\chi_2}$ under the map $t \mapsto t - 1$. We will write $\mathbf{1}$ for the trivial character of $\pi_1(\mathbb{G}_{m,\overline{\mathbb{Q}}})$ and $-\mathbf{1}$ for the unique character of $\pi_1(\mathbb{G}_{m,\overline{\mathbb{Q}}})$ of exact order 2. Define lisse sheaves $\mathcal{F}_i$ on $(\mathbb{P}^1 - \{0, 1, \infty\})_{\overline{\mathbb{Q}}}$ recursively by

$$\mathcal{F}_0 = \mathcal{L}(-\mathbf{1}, -\mathbf{1})$$

and

$$\mathcal{F}_{2i-1} = \mathcal{L}(\mathbf{1}, -\mathbf{1}) \otimes \mathrm{MC}_{-\mathbf{1}}(\mathcal{F}_{2i-2})$$

and

$$\mathcal{F}_{2i} = \mathcal{L}(-\mathbf{1}, \mathbf{1}) \otimes \mathrm{MC}_{-\mathbf{1}}(\mathcal{F}_{2i-1}),$$

for $i \in \mathbb{Z}_{>0}$. Here $\mathrm{MC}_{-\mathbf{1}}$ denotes the functor associated to the representation $-\mathbf{1}$ of $\pi_1(\mathbb{G}_{m,\overline{\mathbb{Q}}})$ as described in [Kat96, § 8.3]. (It is closely related to the 'middle convolution'.) It follows inductively, using [Kat96, § 5.1] and [DR10, Proposition 1.2.1], that each $\mathcal{F}_i$ is a rigid local system $\mathcal{F}$ on $(\mathbb{P}^1 - \{0, 1, \infty\})_{\overline{\mathbb{Q}}}$ with quasi-unipotent monodromy of order dividing 2 and with somewhere maximally quasi-unipotent monodromy, and that their local monodromies are given by the following table. (Knowing the monodromy of $\mathcal{F}_i$ everywhere is needed to calculate the monodromy of $\mathcal{F}_{i+1}$ at ∞. This example is a generalization of the special case $\mathcal{F}_6$ which is considered in [DR10], and which is a G_2-local system.)

In particular, Corollary 2.8 applies to each of the sheaves $\mathcal{F}_i$. (Note that because of the regular quasi-unipotent monodromy at ∞, this result would previously have been available for $u \in F^+ - \mathcal{O}_{F^+}[1/2]$, because in this case $H_\lambda(M(\mathcal{F}_i, u))$ is automatically irreducible, by the argument of [HST10, Theorem 4.4].)

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