A study of the representations supported by the orbit closure of the determinant

Shrawan Kumar


doi:10.1112/S0010437X14007660
A study of the representations supported by the orbit closure of the determinant

Shrawan Kumar

Abstract
We show the existence of a large family of representations supported by the orbit closure of the determinant. However, the validity of our result is based on the validity of the celebrated ‘Latin square conjecture’ due to Alon and Tarsi or, more precisely, on the validity of an equivalent ‘column Latin square conjecture’ due to Huang and Rota.

1. Introduction
Let \( v \) be a complex vector space of dimension \( m \) and let \( E := \text{End} \, v \). Consider \( \mathcal{D} \in Q := S^m(E)^* \), where \( \mathcal{D} \) is the function taking the determinant of any \( X \in \text{End} \, v \). Fix a basis \( \{v_1, \ldots, v_m\} \) of \( v \) and a positive integer \( n < m \) and consider the function \( \mathcal{P} \in Q \), defined by \( \mathcal{P}(X) = x_1^{m-n} \text{Perm}(X^n), \) \( X^n \) being the component of \( X \) in the right down \( n \times n \) corner, where any element of \( \text{End} \, v \) is represented by an \( m \times m \) matrix \( X = (x_{i,j})_{1 \leq i,j \leq m} \) in the basis \( \{v_i\} \) and \( \text{Perm} \) denotes the permanent. The group \( G = \text{GL}(E) \) canonically acts on \( Q \). Let \( \mathcal{X} \) (respectively \( \mathcal{Y} \)) be the \( G \)-orbit closure of \( \mathcal{D} \) (respectively \( \mathcal{P} \)) inside \( Q \). Then \( \mathcal{X} \) and \( \mathcal{Y} \) are closed (affine) subvarieties of \( Q \) which are stable under the standard homothety action of \( \mathbb{C}^* \) on \( Q \). Thus, their affine coordinate rings \( \mathbb{C}[\mathcal{X}] \) and \( \mathbb{C}[\mathcal{Y}] \) are nonnegatively graded \( G \)-algebras over the complex numbers \( \mathbb{C} \). Clearly, \( \mathcal{D} \circ \text{End} \, E \subset \mathcal{X} \), where \( \text{End} \, E \) acts on \( Q \) on the right via \( (q \circ A)(X) = q(A \cdot X) \), for \( A \in \text{End} \, E, q \in Q \) and \( X \in E \).

For any positive integer \( n \), let \( \bar{m} = \bar{m}(n) \) be the smallest positive integer such that the permanent of any \( n \times n \) matrix can be realized as a linear projection of the determinant of an \( m \times m \) matrix. This is equivalent to saying that \( \mathcal{P} \in \mathcal{D} \circ \text{End} \, E \) for the pair \( (\bar{m}, n) \). Then Valiant conjectured that the function \( \bar{m}(n) \) grows faster than any polynomial in \( n \) (cf. [Val79]).

Similarly, let \( m = m(n) \) be the smallest integer such that \( \mathcal{P} \in \mathcal{X} \) (for the pair \( (m, n) \)). Clearly, \( m(n) \leq \bar{m}(n) \). Now, Mulmuley and Sohoni strengthened Valiant’s conjecture. They conjectured that, in fact, the function \( m(n) \) grows faster than any polynomial in \( n \) (cf. [MS01, MS08] and the references therein). They further conjectured that if \( \mathcal{P} \notin \mathcal{X} \), then there exists an irreducible \( G \)-module which occurs in \( \mathbb{C}[\mathcal{Y}] \) but does not occur in \( \mathbb{C}[\mathcal{X}] \). (Of course, if \( \mathcal{P} \in \mathcal{X} \), then \( \mathbb{C}[\mathcal{Y}] \) is a \( G \)-module quotient of \( \mathbb{C}[\mathcal{X}] \).) This geometric complexity theory program initiated by Mulmuley and Sohoni provides a significant mathematical approach to solving Valiant’s conjecture (in fact, the strengthened version of Valiant’s conjecture proposed by them).

By [Kum13, Theorem 5.2], if an irreducible \( G \)-module \( V_E(\lambda) \) (with highest weight \( \lambda \)) appears in \( \mathbb{C}[\mathcal{Y}] \), then \( V_E(\lambda) \) is a polynomial representation of \( G \) given by a partition

\[ \lambda : (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n^2+1} \geq 0 \geq \cdots \geq 0) \]

with last \( m^2 - (n^2 + 1) \) zeros.
Representations supported by the orbit closure of the determinant

From now on (in this Introduction), we assume that \( m \) is even. Our principal result in this paper (Corollary 6.2) asserts that for any partition \( \mu : (\mu_1 \geq \cdots \geq \mu_m \geq 0 \geq \cdots \geq 0) \) with last \( m^2 - m \) zeros, the irreducible \( G \)-module \( V_E(m\mu) \) appears in \( \mathbb{C}[\mathcal{X}] \) with nonzero multiplicity, provided the column Latin \((m,m)\)-square conjecture is valid (cf. Conjecture 4.3). In particular, if \( m \geq n^2 + 1 \), for any irreducible representation \( V_E(\lambda) \) appearing in \( \mathbb{C}[\mathcal{Y}] \), \( V_E(m\lambda) \) appears in \( \mathbb{C}[\mathcal{X}] \) (again assuming the validity of the column Latin \((m,m)\)-square conjecture). Thus, finding an irreducible representation in \( \mathbb{C}[\mathcal{Y}] \) which does not occur in \( \mathbb{C}[\mathcal{X}] \) (on which the success of the Mulmuley–Sohoni program relies) for \( m \geq n^2 + 1 \) is not so easy. As a consequence of our Corollary 6.2, we deduce that the symmetric Kronecker coefficient \( sk_{mX,d\delta_m,d\delta_m} > 0 \) for any partition \( \bar{X} : (\bar{X}_1 \geq \bar{X}_2 \geq \cdots \geq \bar{X}_m \geq 0) \) of \( d \), where \( \delta_m \) is the partition \( \delta_m : (1 \geq 1 \geq \cdots \geq 1) \) \((m \text{ factors})\) (cf. Corollary 6.5).

This paper is organized as follows. By a result of Howe (cf. Corollary 2.4), for any fundamental weight \( \delta_i \) \((1 \leq i \leq m^2 = \dim E)\) of \( \text{GL}(E) \), the irreducible \( \text{GL}(E) \)-module \( V_E(d\delta_i) \), for \( 0 < d < m \), does not occur in \( S^d(S^m(E)) \), whereas \( V_E(m\delta_i) \) occurs with multiplicity one in \( S^d(S^m(E)) \). In fact, it occurs in \( S^i(S^m(E)) \). In §2 we give an explicit construction of the highest weight vector \( P_i = \gamma_{m,i} \) in this unique copy of \( V_E(m\delta_i) \) in \( S^i(S^m(E)) \) (cf. Definition 2.5).

In §3, for any \( 1 \leq i \leq m \), we calculate \( \gamma_{m,i} \) on a certain subset \( \theta(M(m,i)) \) of \( \mathcal{X} \) given by a morphism \( \theta : M(m,i) \rightarrow \mathcal{X} \), where \( M(m,i) \) denotes the set of \( m \times i \) matrices. The induced map \( \theta^* \) on the level of affine coordinate rings is identified with a certain very explicit map \( \varphi \). The main result of this section is Proposition 3.4, which asserts that \( \gamma_{m,i} \) restricted to the image \( \theta(M(m,i)) \) is nonzero if and only if the \( \text{GL}(E) \)-module \( V_E(d\delta_i) \) intersects the \( i \)-th isotypical component \( I_{m\delta_i} \) of \( S^i(S^m(E)) \) corresponding to the irreducible \( \text{GL}(E) \)-module \( V_m(m\delta_i) \) nontrivially, where the element \( v_o \) is defined by the identity (9).

In §4, we turn to Latin squares (more generally Latin rectangles) and state the column Latin square conjecture due to Huang and Rota. As shown by Huang and Rota, their conjecture is equivalent to the celebrated Latin square conjecture due to Alon and Tarsi. We recall that the Latin square conjecture is known to be true for \( p - 1 \) as well as \( p + 1 \), for any odd prime \( p \); in particular, it is true for any even integer up to 24 (cf. Remark 4.5).

Section 5 is devoted to proving that the validity of the column Latin square conjecture implies that the isotypical component \( I_{m\delta_i} \) of \( S^i(S^m(E)) \) corresponding to the irreducible \( \text{GL}(E) \)-module \( V_m(m\delta_i) \) intersects the \( i \)-th isotypical component \( I_{m\delta_i} \) of \( S^i(S^m(E)) \) nontrivially (cf. Theorem 5.6). In fact, for \( i = m \), we show that the latter assertion is equivalent to the column Latin square conjecture.

This sets the stage for the proof of our main theorem (cf. Theorem 6.1), which asserts that the irreducible module \( V_E(m\delta_i) \) occurs in \( \mathbb{C}[\mathcal{X}] \) with multiplicity one for any \( 1 \leq i \leq m \) if the column Latin square conjecture is true for \( m \times m \) Latin squares. This is shown by proving that \( P_i \) does not vanish identically on \( \mathcal{X} \). As an immediate corollary (cf. Corollary 6.2), we deduce that for any partition \( \lambda : \lambda_1 \geq \cdots \geq \lambda_m \geq 0 \), \( V_E(m\lambda) \) occurs in \( \mathbb{C}[\mathcal{X}] \) (if the column Latin square conjecture is true for \( m \times m \) Latin squares).

Finally, in Remark 6.6(b), we observe that \( V_E(m\delta_i) \) \((\text{for any } 1 \leq i \leq m)\) occurs in \( \mathbb{C}[[\text{GL}(E) \cdot \Psi]] \) with multiplicity one, where \( \Psi \) is the function \( E \rightarrow \mathbb{C} \) taking any matrix \( A \in E = \text{End} v \) to its permanent. (Of course, as mentioned above, \( V_E(d\delta_i) \), for \( 0 < d < m \) and \( 1 \leq i \leq m^2 \), does not occur in \( S^d(S^m(E)) \), and hence it does not occur in \( \mathbb{C}[[\text{GL}(E) \cdot \Psi]] \) or in \( \mathbb{C}[\mathcal{X}] \).

For any vector space \( W \) over the complex numbers, in this paper, we view \( S^k(W) \) as the subspace of \( \otimes^k W \) consisting of symmetric tensors.
2. An explicit realization of multiples of fundamental $GL(E)$-representations in $S^\bullet(S^\bullet(E))$

Let $E$ be a finite-dimensional complex vector space with basis $\{e_1, \ldots, e_\ell\}$. Let $\delta_i$, $1 \leq i \leq \ell$, be the $i$th fundamental weight of $GL(E) = GL(\ell)$. This corresponds to the partition $1 \geq 1 \geq \cdots \geq 1$ ($i$ factors).

**Lemma 2.1.** For any positive integers $d$, $j$ and $m$, the multiplicity of the irreducible $GL(E)$-module $V_E(d\delta_i)$ (with highest weight $d\delta_i$) in $S_j(S^m(E))$ is the same as the multiplicity of the irreducible $GL(E_i)$-module $V_{E_i}(d\delta_i)$ in $S_j(S^m(E_i))$, where $E_i$ is the subspace of $E$ spanned by $\{e_1, \ldots, e_i\}$.

In fact, the highest weight vectors in $S_j(S^m(E))$ for the irreducible $GL(E)$-module $V_E(d\delta_i)$ coincide with the highest weight vectors in $S_j(S^m(E_i))$ for the irreducible $GL(E_i)$-module $V_{E_i}(d\delta_i)$.

**Proof.** Let $B_E$ be the standard Borel subgroup of $GL(E)$ consisting of all the invertible upper triangular matrices (with respect to the basis $\{e_1, \ldots, e_\ell\}$). Let $v \in S_j(S^m(E))$ be a $B_E$-eigenvector of weight $d\delta_i$. Then clearly $v \in S_j(S^m(E_i))$ and $v$ is a $B_{E_i}$-eigenvector of weight $d\delta_i$. Conversely, let $v' \in S_j(S^m(E_i))$ be a $B_{E_i}$-eigenvector of weight $d\delta_i$. Then the line $\mathbb{C}v'$ is clearly stable under $B_E$. Moreover, the vector $v'$ is a weight vector of weight $d\delta_i$ with respect to the standard maximal torus $T_E$ (consisting of invertible diagonal matrices) of $GL(E)$. This proves the lemma.

**Corollary 2.2.** With the notation as above, the multiplicity $\mu_E(d\delta_i)$ of $V_E(d\delta_i)$ in $S_j(S^m(E))$ is equal to the dimension of the invariant space $[S_j(S^m(E_i))]^{SL(E_i)}$ if $di = jm$. If $di \neq jm$, then $\mu_E(d\delta_i) = 0$.

We recall the following result from [How87, Proposition 4.3].

**Proposition 2.3.** Let $E$ be a vector space of dimension $\ell$ as above. For positive integers $j$, $m$, we have

(a) $[S_j(S^m(E))]^{SL(E)} = (0)$, if $0 < j < \ell$;

(b) $[S_j(S^m(E))]^{SL(E)} \simeq \begin{cases} (0) & \text{if } m \text{ is odd}, \\ \mathbb{C} & \text{if } m \text{ is even.} \end{cases}$

Combining Corollary 2.2 with Proposition 2.3, together with the action of the center of $GL(E)$, we get the following result.

**Corollary 2.4.** Let $E$ be a vector space of dimension $\ell$ as above. Let $m$ be a positive even integer and let $1 \leq i \leq \ell$. Let $d$ be the smallest positive integer such that $V_E(d\delta_i)$ occurs in $S^\bullet(S^m(E))$ as a $GL(E)$-submodule. Then $d = m$. Moreover, $V_E(m\delta_i)$ occurs in $S^\bullet(S^m(E))$ with multiplicity 1 and it occurs in $S_j(S^m(E))$.

From now on in this section, let $m$ be an even positive integer.

We first give an explicit construction of the invariant $[S_j(S^m(E_i))]^{SL(E_i)}$ for any $1 \leq i \leq \ell$. Recall from Proposition 2.3 that it is one-dimensional.

**Definition 2.5 (An explicit construction of $[S_j(S^m(E_i))]^{SL(E_i)}$).** Recall that $E_i$ has a basis $\{e_1, \ldots, e_i\}$. Let $M(i, i)$ be the space of $i \times i$ matrices over $\mathbb{C}$. Define a linear isomorphism

$$\theta : (\otimes^2 E_i)^* \xrightarrow{\sim} M(i, i), \quad \theta(f) = (\theta(f))_{p,q}, \quad 1 \leq p, q \leq i,$$

where $\theta(f)_{p,q} = f(e_p \otimes e_q)$, for any $f \in (\otimes^2 E_i)^*$. 

294
Representations supported by the orbit closure of the determinant

Let $GL(E_i)$ act on $M(i,i)$ via

$$g : A = (g^{-1})^t Ag^{-1} \text{ for } g \in GL(E_i) \text{ and } A \in M(i,i).$$

Then, $\theta$ is $GL(E_i)$-equivariant. Now, define the map (setting $m' = m/2$)

$$\theta^{\otimes m'} : (\otimes^m E_i)^* \xrightarrow{\sim} \otimes^{m'} (M(i,i))$$

by identifying

$$(\otimes^m E_i)^* \simeq ((\otimes^2 E_i)^*) \otimes \cdots \otimes ((\otimes^2 E_i)^*) \ (m' \text{ factors})$$

and setting

$$\theta^{\otimes m'}(f_1 \otimes \cdots \otimes f_{m'}) = \theta(f_1) \otimes \cdots \otimes \theta(f_{m'}) \text{ for } f_k \in (\otimes^2 E_i)^*.$$

For any finite-dimensional vector space $W$ and nonnegative integer $k$, let $\mathcal{P}^k(W)$ be the space of homogeneous polynomials of degree $k$ on $W$, i.e.,

$$\mathcal{P}^k(W) = \{f : W \to \mathbb{C} \text{ polynomial such that } f(\lambda w) = \lambda^k f(w) \ \forall w \in W \text{ and } \lambda \in \mathbb{C}\}.$$

Then there is a linear isomorphism (cf. [GW09, Proposition B.2.4])

$$\xi : S^k(W)^* \xrightarrow{\sim} \mathcal{P}^k(W)$$

defined by $\xi(\theta)(w) = \theta(w^{\otimes k})$, for $\theta \in S^k(W)^*$ and $w \in W$. If an algebraic group $G$ acts linearly on $W$, then $\xi$ is $G$-equivariant.

Define the linear map $\bar{\xi} : \mathcal{P}^k(W) \to (\otimes^k W)^*$ by

$$(\bar{\xi}(f))(w_1 \otimes \cdots \otimes w_k) = \frac{1}{k!} \text{ (the coefficient of } t_1 \ldots t_k \text{ in } f(t_1 w_1 + \cdots + t_k w_k),$$

for $f \in \mathcal{P}^k(W)$ and $w_1, \ldots, w_k \in W$. Then the inverse map

$$\xi^{-1} : \mathcal{P}^k(W) \to S^k(W)^*$$

is given by the composition of $\bar{\xi}$ with the restriction map $(\otimes^k W)^* \to S^k(W)^*$.

Consider the linear projection obtained via the symmetrization

$$\pi : \otimes^m E_i \to S^m(E_i), \quad w_1 \otimes \cdots \otimes w_m \mapsto \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(m)},$$

where $\mathfrak{S}_m$ is the permutation group on the symbols $[m] := \{1, \ldots, m\}$. Thus, we have $GL(E_i)$-equivariant linear maps

$$S^m(E_i)^* \xrightarrow{\pi^*} (\otimes^m E_i)^* \xrightarrow{\sim} \otimes^{m'} (M(i,i)).$$

This gives rise to a linear $GL(E_i)$-equivariant map

$$S^i(S^m(E_i)^*) \to S^i(\otimes^{m'} (M(i,i))). \quad (1)$$

Now consider the map

$$\det^{\otimes m'} : M(i,i) \to \mathbb{C}, \quad A \mapsto (\det A)^{m'}.$$
S. Kumar

It is clearly a homogeneous polynomial of degree \( im' \), which is \( \text{SL}(E_i) \)-invariant. Thus, via the above isomorphism \( \xi^{-1} \), we get an \( \text{SL}(E_i) \)-invariant linear map

\[
\det^{\otimes m'} : S^{im'}(M(i, i)) \to \mathbb{C}.
\]  

(2)

Of course, we have a canonical \( \text{GL}(E_i) \)-equivariant projection

\[
S^i(\otimes^{im'}(M(i, i))) \to S^{im'}(M(i, i)),
\]

(3)

obtained via the inclusion

\[
S^i(\otimes^{im'}(M(i, i))) \subset \otimes^i(\otimes^{im'}(M(i, i))) \simeq \otimes^{im'}(M(i, i))
\]

followed by the symmetrization \( \otimes^{im'}(M(i, i)) \to S^{im'}(M(i, i)) \).

Composing the maps \((2) \circ (3) \circ (1)\), we get an \( \text{SL}(E_i) \)-invariant linear map

\[
\gamma_{m, i} : S^i(S^m(E_i)^*) \to \mathbb{C}.
\]

For any vector space \( W \), we have a canonical \( \text{GL}(W) \)-equivariant identification

\[
S^i(W^*) \simeq S^i(W)^*
\]

(4)

via \( S^i(W^*) \subset \otimes^i(W^*) \simeq (\otimes^iW)^* \to S^i(W)^* \), where the last map is the restriction map. Thus, \( \gamma_{m, i} \) can be thought of as an element of \( [S^i(S^m(E_i))^{\text{SL}(E_i)}] \).

**Lemma 2.6.**

\[
\gamma_{m, i}
\left(
\left(\sum_{j=1}^{i} (e_j^*)^{\otimes m}\right)^{\otimes i}
\right) \neq 0,
\]

where \( \{e_1^*, \ldots, e_i^*\} \) is the basis of \( E_i^* \) dual to the basis \( \{e_1, \ldots, e_i\} \) of \( E_i \).

**Proof.** Let \( E_{j,k} \in M(i, i) \) be the matrix with all entries 0, except \((j, k)\) which is 1. By the definition,

\[
\gamma_{m, i}
\left(
\left(\sum_{j=1}^{i} (e_j^*)^{\otimes m}\right)^{\otimes i}
\right) = \gamma_{m, i}
\left(
\sum_{1 \leq j_1, \ldots, j_i \leq i} ((e_{j_1}^*)^{\otimes m} \otimes \cdots \otimes (e_{j_i}^*)^{\otimes m})
\right)
\]

\[
= \sum_{1 \leq j_1, \ldots, j_i \leq i} \det^{\otimes m'}(E_{j_1,j_1}^{m'} \otimes \cdots \otimes E_{j_i,j_i}^{m'})
\]

\[
= \sum_{1 \leq j_1, \ldots, j_i \leq i} \frac{1}{(im')!} \text{ the coefficient of } (t_1 \cdots t_{im'}) \text{ in }
\]

\[
[\det((t_1 + \cdots + t_{m'}E_{j_1,j_1} + (t_{m'+1} + \cdots + t_{2m'})E_{j_2,j_2}
\]

\[
+ \cdots + (t_{(i-1)m'+1} + \cdots + t_{im'})E_{j_i,j_i})^{m'}]
\]

\[
= \sum_{\sigma \in S_i} \frac{1}{(im')!} \text{ the coefficient of } (t_1t_2 \cdots t_{im'}) \text{ in }
\]

\[
[(t_1 + \cdots + t_{m'})^{m'} \cdots (t_{(i-1)m'+1} + \cdots + t_{im'})^{m'}]
\]

\[
= \frac{i!}{(im')!} (m')^i \neq 0.
\]

This proves the lemma. \( \square \)

We record this in the following.

**Lemma 2.7.** The element \( \gamma_{m, i} \) is the unique (up to a scalar multiple) nonzero element of \([S^i(S^m(E_i))^{\text{SL}(E_i)}] \).
3. Calculation of $\gamma_{m,i}$ on the determinant orbit closure

We continue to assume that $m$ is even and $m' = m/2$.

Let $v$ be a complex vector space of dimension $m$ and let $E := \text{End} v = v \otimes v^*$. Fix a basis $\{v_1, \ldots, v_m\}$ of $v$ and let $\{v_1^*, \ldots, v_m^*\}$ be the dual basis of $v^*$. Take the basis $\{v_i \otimes v_j^*\}_{1 \leq i, j \leq m}$ of $E$ and order the basis elements as $\{e_1, e_2, \ldots, e_m\}$ satisfying

$$e_1 = v_1 \otimes v_1^*, \quad e_2 = v_2 \otimes v_2^*, \ldots, \quad e_m = v_m \otimes v_m^*.$$ 

Fix $1 \leq i \leq m$ and consider the subspace $E_i$ of $E$ spanned by $\{e_1, \ldots, e_i\}$. Take $A \in \text{End} E$ of the form

$$Ae_j = \sum_{p=1}^{m} a_p^j e_p, \quad 1 \leq j \leq i. \quad (5)$$

In the sequel, we will only consider $A \in \text{End} E$ of the above form and the values of $Ae_j$ for $j > i$ will be irrelevant for us. Thus, we can (and will) think of $A = (a_p^j)_{1 \leq p \leq m, 1 \leq j \leq i}$ as an $m \times i$ matrix.

Define a right action of the semigroup $\text{End} E$ on $Q := \mathcal{P}^m(E) \simeq S^m(E)^*$ (cf. Definition 2.5 for the last identification under $\xi$) via

$$(f \circ A)(e) = f(Ae) \quad \text{for } f \in Q, A \in \text{End} E \text{ and } e \in E. \quad (6)$$

Take $f = \mathcal{D} \circ A \in Q$, where $A$ is of the form (5) and $\mathcal{D} \in \mathcal{P}^m(E)$ is the function taking the determinant of any $X \in E$. For $1 \leq l_1, \ldots, l_m \leq i$, let $A^{l_1, \ldots, l_m}$ denote the $m \times m$ matrix with the first column $\begin{bmatrix} a_1^1 \\ \vdots \\ a_m^1 \end{bmatrix}$, etc. For integers $d_1, \ldots, d_i \geq 0$ with $d_1 + \cdots + d_i = m$, $\{1^{d_1}, 2^{d_2}, \ldots, i^{d_i}\}$ means the collection

$$\left\{1, \ldots, 1; 2, \ldots, 2; \ldots; i, \ldots, i\right\},$$

and $A^{(d_1, \ldots, d_i)}$ means the $m \times m$ matrix with columns

$$\underbrace{A^1, \ldots, A^1}_{d_1 \text{ times}}, \underbrace{A^2, \ldots, A^2}_{d_2 \text{ times}}, \ldots, \underbrace{A^i, \ldots, A^i}_{d_i \text{ times}},$$

where $A^j$ is the column

$$\begin{bmatrix} a_1^j \\ \vdots \\ a_m^j \end{bmatrix}.$$ 

**Lemma 3.1.** The image of $f_{|E_i}$ in $\otimes^m(M(i, i))$ under $\theta^\otimes m \circ \pi^*$ (cf. Definition 2.5) is given by

$$\frac{1}{m!} \sum_{d_1 + \cdots + d_i = m} \text{Perm} A^{(d_1, \ldots, d_i)} \sum_{1 \leq j_1, k_1, \ldots, j_{m'}, k_{m'} \leq i} E_{j_1, k_1} \otimes \cdots \otimes E_{j_{m'}, k_{m'}},$$

where the last summation runs over those ordered $m$-tuples $(j_1, k_1, \ldots, j_{m'}, k_{m'})$ such that the collection (without regard to the order)

$$\{j_1, k_1, \ldots, j_{m'}, k_{m'}\} = \{1^{d_1}, 2^{d_2}, \ldots, i^{d_i}\},$$

and $\text{Perm}$ denotes the permanent of the matrix.
Proof. For any \( 1 \leq j_p, k_p \leq i \),
\[
  f(\pi((e_{j_1} \otimes e_{k_1}) \otimes \cdots \otimes (e_{j_m} \otimes e_{k_m})))
  = \mathcal{D}((Ae_{j_1} \otimes Ae_{k_1}) \otimes \cdots \otimes (Ae_{j_m} \otimes Ae_{k_m}))
  = \mathcal{D}\left( \sum_{1 \leq p_1, \ldots, p_m \leq m} (a_{j_1}^{p_1} e_{p_1} \otimes a_{k_1}^{p_1} e_{p_2}) \otimes \cdots \otimes (a_{j_m}^{p_{m-1}} e_{p_{m-1}} \otimes a_{k_m}^{p_m} e_{p_m}) \right)
  = \sum_{1 \leq p_1, \ldots, p_m \leq m} (a_{j_1}^{p_1} a_{k_1}^{p_2}) \cdots (a_{j_m}^{p_{m-1}} a_{k_m}^{p_m}) \mathcal{D}(e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_{m-1}} \otimes e_{p_m})
  = \sum_{\sigma \in \mathfrak{S}_m} (a_{\sigma(1)}^{p_1} a_{\sigma(2)}^{p_2}) \cdots (a_{\sigma(m-1)}^{p_{m-1}} a_{\sigma(m)}^{p_m}) \mathcal{D}(e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(m-1)} \otimes e_{\sigma(m)})
  = \frac{1}{m!} \text{Perm}(A^j_{i_1, k_1, \ldots, i_m, k_m}).
\] (7)

Now the image of \( f|_{E_i} \) in \( \otimes^{m'}(M(i, i)) \) under \( \theta \otimes^{m'} \circ \pi^* \) is given by
\[
  \sum_{1 \leq j_p, k_p \leq i} f((e_{j_1} \otimes e_{k_1}) \otimes \cdots \otimes (e_{j_m} \otimes e_{k_m})) E_{j_1, k_1} \otimes \cdots \otimes E_{j_m, k_m}
  = \frac{1}{m!} \sum_{1 \leq j_p, k_p \leq i} \text{Perm}(A^j_{i_1, \ldots, i_m, k_m}) E_{j_1, k_1} \otimes \cdots \otimes E_{j_m, k_m}, \text{ by (7)}
  = \frac{1}{m!} \sum_{d_1 + \cdots + d_i = m} \text{Perm} A^{(d_1, \ldots, d_i)} \sum_{1 \leq j_p, k_p \leq i} E_{j_1, k_1} \otimes \cdots \otimes E_{j_m, k_m},
\]
where the last summation runs over those ordered \( m \)-tuples \( (j_1, k_1, \ldots, j_m, k_m) \) such that the collection (without regard to the order)
\[
  \{j_1, k_1, \ldots, j_m, k_m\} = \{1^{d_1}, 2^{d_2}, \ldots, i^{d_i}\}.
\]
This proves the lemma. \( \square \)

On the vector space \( M(m, i) \) of \( m \times i \) matrices, \( \text{GL}(m) \times \text{GL}(i) \) acts via
\[
  (g, h) \cdot X = gXh^{-1}, \text{ for } g \in \text{GL}(m), h \in \text{GL}(i), X \in M(m, i).
\]
In particular, the permutation group \( \mathfrak{S}_m \), thought of as the subgroup of permutation matrices in \( \text{GL}(m) \), acts on \( M(m, i) \) and hence on any \( \mathcal{P}^k(M(m, i)) \). For any \( \varnothing = (d_1, \ldots, d_i) \), \( |\varnothing| := d_1 + \cdots + d_i = m \) and \( d_j \geq 0 \), set (for any \( A \) of the form (5))
\[
  a_{\varnothing}(A) = \text{Perm} A^{(d_1, \ldots, d_i)}.
\]
Then, clearly, for any \( \varnothing \) as above,
\[
  a_{\varnothing} \in \mathcal{P}^m(M(m, i))^{-(\epsilon_1 + \cdots + \epsilon_m)} \mathfrak{S}_m,
\]
where the superscript ‘\(-(\epsilon_1 + \cdots + \epsilon_m)\), \( \mathfrak{S}_m \)’ denotes the \( \mathfrak{S}_m \)-invariants of weight \(-(\epsilon_1 + \cdots + \epsilon_m)\) with respect to the action of \( \text{GL}(m) \), i.e., the invertible diagonal matrices \((t_1, \ldots, t_m)\) act via \((t_1 \cdots t_m)^{-1}\). Recall from [GW09, Theorem 5.6.7] that, as \( \text{GL}(m) \times \text{GL}(i) \)-modules, for any \( j \geq 0 \),
\[
  \mathcal{P}^j(M(m, i)) \simeq \sum_{\mu_1 \geq 0, \mu_2 \geq 0, \ldots, \mu_i \geq 0} V_m(\mu)^* \otimes E_i(\mu), \quad (8)
\]
298
where $|\mu| := \sum \mu_p$ and $V_m(\mu)$ (respectively $E_i(\mu)$) denotes the irreducible GL($m$)-module (respectively GL($i$)-module) corresponding to the partition $\mu$.

Let $V_m := \mathbb{C}^m$ with the standard basis $\{v_1, \ldots, v_m\}$. Define the elements

$$v_o := v_1^* \otimes \cdots \otimes v_m^* \in \otimes^m(V_m^*)$$

$$v_o := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma \cdot v_o \in S^m(V_m),$$

and

$$v_o := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma \cdot (v_1 \otimes \cdots \otimes v_m) \in S^m(V_m).$$

From (8), by a classical result due to Kostant [Kos76, Remark 4.1] (which asserts that for any irreducible SL($V_m$)-module $V_m(\lambda)$ corresponding to the partition $\lambda$ with $|\lambda| = m$, its zero weight space is an irreducible representation $W_\lambda$ of $\mathfrak{S}_m$ corresponding to the partition $\lambda$), we get

$$\mathcal{P}^m(M(m, i))^{-(\epsilon_1 + \cdots + \epsilon_m), \mathfrak{S}_m} \simeq (S^m(V_m)^{\epsilon_1 + \cdots + \epsilon_m, \mathfrak{S}_m})^* \otimes S^m(E_i),$$

$$S^m(V_m)^{\epsilon_1 + \cdots + \epsilon_m, \mathfrak{S}_m} \simeq \mathbb{C} v_o.$$

Thus,

$$\mathcal{P}^m(M(m, i))^{-(\epsilon_1 + \cdots + \epsilon_m), \mathfrak{S}_m} \simeq S^m(E_i),$$

as GL($i$)-modules. It is easy to see that $\{a_0\}_{0=(d_1, \ldots, d_i)}$ with $|0| = m$ are linearly independent (by taking, for example, $a_p^0 = a_1^1$ for all $1 \leq p \leq m$). Hence, by the dimensional consideration, $\{a_0\}_{|0|=m}$ provides a basis of $S^m(E_i)$ under the above identification (11). The GL($i$)-module isomorphism (11) clearly induces a GL($i$)-algebra homomorphism:

$$\varphi : S^*(S^m(E_i)) \rightarrow \mathcal{P}^{m*}(M(m, i))^{-(\epsilon_1 + \cdots + \epsilon_m), \mathfrak{S}_m} \simeq \oplus(V_m(\mu)^{\epsilon_1 + \cdots + \epsilon_m, \mathfrak{S}_m})^* \otimes E_i(\mu),$$

where the above sum runs over $\mu : \mu_1 \geq \cdots \geq \mu_i \geq 0$, $|\mu| = m\bullet$; the last isomorphism follows by the identity (8).

We now give an alternative description of the map

$$\varphi : S^*(S^m(E_i)) \rightarrow \mathcal{P}^{m*}(M(m, i))^{-(\epsilon_1 + \cdots + \epsilon_m), \mathfrak{S}_m}.$$

First of all, as GL($m$) × GL($i$)-modules,

$$\mathcal{P}^{mj}(M(m, i)) \simeq \mathcal{P}^{mj}(V_m \otimes E_i^*) \simeq S^{mj}(V_m^* \otimes E_i),$$

where the last identification is obtained from the isomorphism $\xi^{-1}$ of Definition 2.5 followed by the identification (4). Define the map

$$\varphi : \otimes^m E_i \rightarrow \otimes^m(V_m^* \otimes E_i) = (\otimes^m(V_m^*)) \otimes (\otimes^m E_i), \quad \varphi(v) = v_o \otimes v.$$

Clearly, the map $\varphi$ is a GL($E_i$)-module map. Moreover, it restricts to the map $\varphi_1$:

$$\xymatrix{ S^m(E_i) \ar[r]^-{\varphi_1} & S^m(V^*_m \otimes E_i) \ar[d] \ar[r]^-{\otimes^m(E_i)} & \otimes^m(E_i) \ar[d] \ar[r]^-{\varphi} & \otimes^m(V^*_m \otimes E_i),}$$

where the vertical maps are the canonical inclusions.
It is easy to see that \( \varphi_1 \) is a GL\((E_i)\)-module map with image inside \( S^m(V_m^* \otimes E_i)^{-(\epsilon_1+\ldots+\epsilon_m), S_m} \). Thus, from the irreducibility of \( S^m E_i \) as a GL\((E_i)\)-module, applying Schur’s lemma, we can choose the identification \((11)\) so that \( \varphi_1 \) coincides with the map \( \varphi|_{S^m(E_i)} \) under the identification \((13)\).

The map \( \varphi_1: S^m(E_i) \to S^m(V_m^* \otimes E_i) \) extends to an algebra homomorphism (still denoted by)
\[
\varphi_1: S^*(S^m(E_i)) \to S^*(S^m(V_m^* \otimes E_i)).
\]
The isomorphism \((13)\) for \( j = 1 \),
\[
P^m(M(m,i)) \cong S^m(V_m^* \otimes E_i), \tag{14}
\]
induces an algebra homomorphism \( \beta: S^*(S^m(V_m^* \otimes E_i)) \to P^m(M(m,i)) \). Let \( \varphi'_1: S^*(S^m(E_i)) \to P^m(M(m,i)) \) be the GL\((E_i)\)-algebra homomorphism which is the composite \( \beta \circ \varphi_1 \).

Since \( \varphi'_1 \) coincides with \( \varphi \) on \( S^m(E_i) \), and both \( \varphi \) and \( \varphi'_1 \) are algebra homomorphisms, we get that
\[
\varphi'_1 = \varphi. \tag{15}
\]

Consider the function (for \( i \leq m \))
\[
\theta: M(m,i) \to P^m(E_i) \cong S^m(E_i)^*, \quad A \mapsto (\emptyset \otimes A)|_{E_i}.
\]
Explicitly,
\[
\theta(A) \left( \sum_{j=1}^i \lambda_j e_j \right) = \prod_{p=1}^m \left( \sum_{j=1}^i \lambda_j a_{pj}^j \right) \quad \text{for} \quad A = (a_{pj}^j)_{1 \leq p \leq m, 1 \leq j \leq i}.
\]
Clearly, \( \theta \) is a polynomial function of homogeneous degree \( m \). Moreover, it is GL\((E_i)\)-equivariant:
\[
\theta(A \cdot g^{-1}) = (\emptyset \otimes (Ag^{-1})|_{E_i} = g \cdot ((\emptyset \otimes A)|_{E_i}) = g \cdot \theta(A).
\]

Of course, \( \theta \) gives rise to a GL\((E_i)\)-algebra homomorphism
\[
\theta^*: S^*(S^m(E_i)) \to P^m(M(m,i)).
\]

**Lemma 3.2.** \( \text{Im}(\theta^*|_{S^m(E_i)}) \subset P^m(M(m,i))^{-(\epsilon_1+\ldots+\epsilon_m), S_m} \).

**Proof.** Let \( t \) be the diagonal matrix \( (t_1, \ldots, t_m) \in \text{GL}(m) \). For any \( f \in S^m(E_i), A \in M(m,i), \)
\[
(\theta^* f)(t^{-1} A) = f((\emptyset \otimes t^{-1} A)|_{E_i}) = t_1^{-1} \cdots t_m^{-1} f((\emptyset \otimes A)|_{E_i}).
\]
This shows that
\[
\text{Im}(\theta^*|_{S^m(E_i)}) \subset P^m(M(m,i))^{-(\epsilon_1+\ldots+\epsilon_m)}.
\]
We next show that for any \( f \in S^m(E_i), \theta^* f \) is \( S_m \)-invariant. Take \( \sigma \in S_m \) (considered as a permutation matrix), then
\[
(\theta^* f)(\sigma A) = f((\emptyset \otimes \sigma A)|_{E_i}) = f((\emptyset \otimes A)|_{E_i}).
\]
This proves the lemma. \( \square \)
Representations supported by the orbit closure of the determinant

Lemma 3.3. The function \( \theta^* \) coincides (up to a nonzero scalar multiple in any degree) with the function \( \varphi : S^*(S^m(E_i)) \to P^m(M(m, i))^{-\bullet (e_1+...+e_m), \mathfrak{S}_m} \) defined earlier (cf. (12)).

Proof. First of all the function \( \theta \) is clearly nonzero. Further, both \( \theta^* \) and \( \varphi \) are \( \text{GL}(i) \)-algebra homomorphisms. Moreover, \( S^m(E_i) \) and \( P^m(M(m, i))^{-\bullet (e_1+...+e_m), \mathfrak{S}_m} \) are both irreducible \( \text{GL}(i) \)-modules (use the isomorphism (11) for the latter). Combining these, the lemma follows using Schur’s lemma.

Now, \( S^i(S^m(E_i)) \) has a unique (up to a scalar multiple) \( \text{SL}(E_i) \)-invariant (by Proposition 2.3). We want to determine if \( \varphi|[S^i(S^m(E_i))]^{\text{SL}(E_i)} \neq 0 \).

By the definition, \( S^i(S^m(E_i)) = [\otimes^i(S^m E_i)]^{H_j} \), where \( H_j \subset \mathfrak{S}_{mj} \) is the subgroup \( \mathfrak{S}_m^X \times \mathfrak{S}_j \) acting on the right as

\[
\left( (v_1^i \otimes \cdots \otimes v_m^i) \otimes \cdots \otimes (v_1^j \otimes \cdots \otimes v_m^j) \right) \cdot ((\sigma_1, \ldots, \sigma_j), \mu) = (\sigma_{\mu(1)}^i(1) \otimes \cdots \otimes \sigma_{\mu(j)}^i(m_j)) \otimes \cdots \otimes (\sigma_{\mu(j)}^j(1) \otimes \cdots \otimes \sigma_{\mu(j)}^j(m_j)),
\]

for \( \sigma_p \in \mathfrak{S}_m \) and \( \mu \in \mathfrak{S}_j \).

Proposition 3.4. The map \( \varphi|[S^i(S^m(E_i))]^{\text{SL}(E_i)} \neq 0 \) if and only if the \( \text{GL}(V_m) \)-submodule \( U_i \) generated by \( v_o^i \in S^i(S^m(V_m^*)) = [\otimes^i(S^m E_i)]^{H_i} \), intersects the isotypic component \( \mathcal{I}_{m\delta_i} \) of \( S^i(S^m(V_m^*)) \) corresponding to the irreducible \( \text{GL}(V_m) \)-module \( V_m(m\delta_i) \) nontrivially.

Proof. Take \( 0 \neq v \in [S^i(S^m(E_i))]^{\text{SL}(E_i)} = [\otimes^i(S^m E_i)]^{H_i \times \text{SL}(E_i)} \).

Recall that for any partition \( \lambda : \lambda_1 \geq \cdots \geq \lambda_d > 0 \), \( d \) is called the height \( ht \lambda \) of \( \lambda \). We set \( |\lambda| := \sum \lambda_j \). Let \( W_\lambda \) be the corresponding irreducible \( \mathfrak{S}_m^\lambda \)-module and let \( E_i(\lambda) \) be the corresponding irreducible \( \text{GL}(i) \)-module for any \( i \geq d \). By the Schur–Weyl duality (cf. [GW09, Theorem 9.1.2]),

\[
S^i(S^m(E_i)) \simeq \bigoplus_{ht \lambda \leq i, |\lambda| = mi} [W_\lambda]^{H_i} \otimes E_i(\lambda). \tag{16}
\]

Thus, we get

\[
[S^i(S^m(E_i))]^{\text{SL}(E_i)} \simeq [W_{m\delta_i}]^{H_i} \otimes E_i(m\delta_i). \tag{17}
\]

In particular, \( [W_{m\delta_i}]^{H_i} \) is one-dimensional. Also, consider the analogous decomposition,

\[
S^i(S^m(V_m^*)) \simeq \bigoplus_{ht \mu \leq m, |\mu| = mi} [W_\mu]^{H_i} \otimes V_m(\mu)^*, \tag{18}
\]

and write

\[
v_o^{\otimes i} = \sum_\mu v_\mu, \tag{19}
\]

under the above decomposition.

Let \( M \subset [\otimes^i(S^m E_i)]^{\text{SL}(E_i)} \) be the \( \mathfrak{S}_{mi} \)-submodule generated by \( v \) and, for any \( \mu \) with \( ht \mu \leq m \) and \( |\mu| = mi \), let \( M_\mu \subset \otimes^i(S^m V_m^*) \) be the \( \mathfrak{S}_{mi} \)-submodule generated by \( v_\mu \). Then, by the
Schur–Weyl duality, $M \simeq W_{m\delta}$ and $M_{\mu}$ (if nonzero) is isotypic of type $W_{\mu}$. By the definition,

$$\bar{\varphi}'_i(v) = \frac{1}{im!} \sum_{\sigma \in \mathfrak{S}_m} \sigma \cdot (v_{\sigma} \otimes v) \in S^{mi}(V_m^* \otimes E_i) \subset (\otimes^i(\otimes^m V_m^*)) \otimes (\otimes^i(\otimes^m E_i))$$

$$= \frac{1}{im!} \sum_{ht\mu \leq m, |\mu| = mi} \sum_{\sigma \in \mathfrak{S}_m} \sigma \cdot (v_{\mu} \otimes v). \quad (20)$$

Now, $W_{\mu}$ being self-dual, we get that for $\mu \neq m\delta$,

$$\sum_{\sigma \in \mathfrak{S}_m} \sigma \cdot (v_{\mu} \otimes v) = 0. \quad (21)$$

Moreover, if $v_{m\delta} \neq 0$, we claim that

$$\sum_{\sigma \in \mathfrak{S}_m} \sigma \cdot (v_{m\delta} \otimes v) \neq 0. \quad (22)$$

By projecting to an irreducible component, we can assume that $M_{m\delta} \simeq W_{m\delta}$. Now, take a $\mathfrak{S}_m$-invariant nondegenerate bilinear form $\alpha : M_{m\delta} \otimes M \to \mathbb{C}$. Since $\alpha$ is $\mathfrak{S}_m$-invariant, $\alpha |_{M_{m\delta} \otimes M_{H_i}}$ remains nondegenerate. Since both of $v$ and $v_{m\delta}$ are $H_i$-invariant, and $[W_{m\delta}]_{H_i}$ is one-dimensional as observed above, we get $\alpha(v_{m\delta} \otimes v) \neq 0$. Thus,

$$\alpha \left( \sum_{\sigma \in \mathfrak{S}_m} \sigma \cdot (v_{m\delta} \otimes v) \right) = \sum_{\sigma \in \mathfrak{S}_m} \alpha(v_{m\delta} \otimes v) \neq 0.$$

This proves (22). Now, as it is easy to see, $v_{m\delta} \neq 0$ if and only if $U_i$ intersects $I_{m\delta}$ nontrivially. Hence the proposition is proved by combining the identities (20)–(22) since $\varphi = \bar{\varphi}'_i$ (by the identity (15)).

4. Latin squares

**Definition 4.1.** Let $1 \leq i \leq m$. By a Latin $(i, m)$-rectangle $A$, one means an $i \times m$ matrix

$$A = (a_{p}^q)_{1 \leq p \leq i, 1 \leq q \leq m}$$

such that each row $A_p := \{a_{1}^p, \ldots, a_{m}^p\}$ is a permutation $\sigma_p$ of $[m]$ (i.e. $\sigma_p(q) = a_{q}^p$) and each column $A^q := \{a_{1}^q, \ldots, a_{i}^q\}$ consists of distinct numbers. We define the sign $\epsilon(A^q)$ of $A^q$ as follows:

$$\epsilon(A^q) := \text{sign of } \prod_{1 \leq p < p' \leq i} (a_{p'}^q - a_{p}^q).$$

We call a Latin rectangle $A$ column-even if $\epsilon_c(A) := \prod_{q=1}^m \epsilon(A^q)$ is $+1$ and column-odd otherwise.

Let $A^{q}$ denote the set $A^q$ without regard to order. Then we call the $m$-tuple $A = (A^1, \ldots, A^m)$ the pattern of $A$. Let $L_A$ denote the set of Latin $(i, m)$-rectangles $A$ with pattern $A$. Let $S(i, m)$ be the set of all patterns of size $(i, m)$, where by a pattern $A$ of size $(i, m)$ (or an $(i, m)$-pattern) we mean we mean a $m$-tuple $A = (A^1, \ldots, A^m)$ of subsets of $[m]$ each of cardinality exactly $i$ such that any integer $q \in [m]$ occurs in exactly $i$ sets $A^q$.

For $A \in S(i, m)$, let $L_A^+$ (respectively $L_A^-$) denote the subset of $L_A$ consisting of column even (respectively odd) Latin rectangles.

302
We have the following simple lemma.

**Lemma 4.2.** Fix any $1 \leq i \leq m$. Assume that there exists a pattern $A$ of size $(i, m)$ such that

$$\sharp L_A^+ \neq \sharp L_A^-.$$

Then, for any $1 \leq i' \leq i$, there exists a pattern $B$ of size $(i', m)$ such that

$$\sharp L_B^+ \neq \sharp L_B^-.$$

**Proof.** It suffices to prove the lemma for $i' = i - 1$. Define the map $\varphi : L_A \to \bigsqcup_{B \in S_A(i-1, m)} L_B$ by removing the last row of any Latin rectangle $A$ in $L_A$. The map $\varphi$ is clearly injective. Moreover, the image of $\varphi$ consists exactly of the union $\bigsqcup_{B \in S_A(i-1, m)} L_B$, where

$$S_A(i-1, m) := \{ (i-1,m) \text{-patterns } B \text{ such that there exists } A \in L_A \text{ with its top } (i-1) \text{ rows having pattern } B \}.$$

By the definition of $L_A^\pm$, it is clear that for any $B \in S_A(i-1, m)$, there exists a sign $\epsilon(B) \in \{ \pm 1 \}$ such that

$$\varphi^{-1}(L_B^\pm) \subset L_A^{\pm \epsilon(B)}.$$

(23)

Assume, if possible, that the lemma is false, that is,

$$\sharp L_B^+ = \sharp L_B^- \text{ for every } (i-1,m) \text{-pattern } B; \quad (24)$$

in particular, for any $B \in S_A(i-1, m)$.

Combining (23) and (24), we get (since $\varphi$ is a bijection) $\sharp L_A^+ = \sharp L_A^-$. This contradicts the assumption and hence proves the lemma.

We recall the following celebrated *column Latin $(m,m)$-square conjecture* due to Huang and Rota [HR94, Conjecture 3].

**Conjecture 4.3.** For any positive even integer $m$,

$$\sharp \text{CELS}(m) \neq \sharp \text{COLS}(m),$$

where CELS$(m)$ (respectively COLS$(m)$) denotes the set of column-even (respectively column-odd) Latin $(m,m)$-squares. (Observe that for Latin $(m,m)$-squares, there is a unique pattern: $([m], [m], \ldots, [m])$.)

Combining the above conjecture with Lemma 4.2, we get the following proposition.

**Proposition 4.4.** Let $m$ be a positive even integer. Assume that Conjecture 4.3 is true for $m$. Then, for any $1 \leq i \leq m$, there exists a pattern $A$ of size $(i, m)$ such that

$$\sharp L_A^+ \neq \sharp L_A^-.$$

**Remark 4.5.** As proved by Huang and Rota [HR94, §3], their column Latin $(m,m)$-square conjecture is equivalent to the (full) Latin $(m,m)$-square conjecture given by Alon and Tarsi [AT92]. Now the (full) Latin $(m,m)$-square conjecture is known to be valid in the following cases:

(a) $m = p - 1$, for any odd prime $p$, due to Glynn [Gly10, Theorem 3.2];
(b) $m = p + 1$, for any odd prime $p$, due to Drisko [Dri97, Theorem 9].
We have the following very simple lemma.

**Lemma 4.6.** Let \( A \) (respectively \( B \)) be a pattern of type \((i, m)\) (respectively \((i, m')\)) such that \( \sharp L_A^+ \neq \sharp L_A^- \) and \( \sharp L_B^+ \neq \sharp L_B^- \).

Then,
\[
\sharp L_{(A,B)}^+ \neq \sharp L_{(A,B)}^-,
\]
where each entry in \( B \) is shifted by \( m \).

**Proof.** Clearly, under the concatenation,
\[
L_A \times L_B \sim \rightarrow L_{(A,B)}.
\]
Moreover, under the above bijection,
\[
L_{\epsilon_1}^i \times L_{\epsilon_2}^j \rightarrow L_{\epsilon_1 \cdot \epsilon_2}^i \cdot \epsilon_2
\]
where \( \epsilon_i = \pm 1 \). From this the lemma follows. \( \square \)

5. **Existence of a certain isotypic component in the module generated by** \( v_o \otimes i \)

Recall from §3 that \( V_m = \mathbb{C}^m \) has standard basis \( \{v_1, \ldots, v_m\} \). Recall from the identity (10) that
\[
v_o := \frac{1}{m!} \sum_{\sigma_1 \in \mathfrak{S}_m} v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(m)} \in S^m(V_m),
\]
so that, as elements of \( S^i(S^m(V_m)) \),
\[
v_o \otimes i = \frac{1}{(m!)^i} \sum_{\sigma=\sigma_1, \ldots, \sigma_i} (v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(m)}) \otimes \cdots \otimes (v_{\sigma_i(1)} \otimes \cdots \otimes v_{\sigma_i(m)}).
\]

Let \( \lambda \) be a partition of \( k \) into at most \( m \) parts and let \( A \) be a tableau of shape \( \lambda \). As in [GW09, Proposition 9.3.7], define
\[
\begin{align*}
\text{Row } A &= \{ \sigma \in \mathfrak{S}_k : \sigma \text{ preserves the rows of } A \}, \\
\text{Col } A &= \{ \mu \in \mathfrak{S}_k : \mu \text{ preserves the columns of } A \}, \\
S(A) &= \left( \sum_{\mu \in \text{Col } A} \epsilon(\mu) \mu \right) \cdot \sum_{\sigma \in \text{Row } A} \sigma, \\
v_A &= v_{i_1} \otimes \cdots \otimes v_{i_k} \in \otimes^k(V_m),
\end{align*}
\]
where \( i_j = r \) if \( j \) occurs in the \( r \)th row of \( A \). (Here \( \epsilon(\mu) \) denotes the sign of \( \mu \).)

**Example 5.1.**
\[
A = \begin{bmatrix}
1 & 5 & 9 \\
2 & 6 & 10 \\
3 & 7 \\
4 & 8
\end{bmatrix}
\]
\[
v_A = v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_1 \otimes v_2.
\]

304
Consider the tableau $B_o = B_o(i, m)$ of shape $m \geq m \geq \cdots \geq m$ ($i$ factors):

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\ldots$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m + 1$</td>
<td>$m + 2$</td>
<td>$m + 3$</td>
<td>$\ldots$</td>
<td>$2m$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$(i - 1)m + 1$</td>
<td>$(i - 1)m + 2$</td>
<td>$(i - 1)m + 3$</td>
<td>$\ldots$</td>
<td>$im$</td>
</tr>
</tbody>
</table>

**Proposition 5.2.** For any $1 \leq i \leq m$ and even $m$,

$$\langle v_o^{\otimes i}, S(B_o) \cdot v_o^{\otimes i} \rangle = \left( \frac{1}{m!} \right)^i \sum_{A \in S(i, m)} \left( \sharp L_A^+ - \sharp L_A^- \right)^2,$$

where $\langle \cdot, \cdot \rangle$ is the standard pairing between $\otimes^i (\otimes^m (V^*_m)), \otimes^i (\otimes^m V_m)$ and $v_o \in S^m (V^*_m) \subset \otimes^m (V^*_m)$ is defined by the identity (9).

**Proof.** First of all, by the definition of $S(B_o)$,

$$S(B_o) \cdot v_o^{\otimes i} = (m!)^i \sum_{\mu \in \mathrm{Col} B_o} \epsilon(\mu) \mu \cdot v_o^{\otimes i}$$

$$= \sum_{\sigma = (\sigma_1, \ldots, \sigma_i) \in \mathfrak{S}_i \mu \in (\mu_1, \ldots, \mu_m) \in \mathfrak{S}_m} \epsilon(\mu) \langle v_{\sigma \mu_1(1)} \otimes \cdots \otimes v_{\sigma \mu_m(i)}(m) \rangle$$

$$\otimes (v_{\sigma \mu_1(1)} \otimes \cdots \otimes v_{\sigma \mu_m(2)}(m)) \otimes \cdots \otimes (v_{\sigma \mu_1(i)} \otimes \cdots \otimes v_{\sigma \mu_m(i)}(m)), \quad (26)$$

where $\epsilon(\mu) := \epsilon(\mu_1) \cdots \epsilon(\mu_m)$ and $\mu_j$ is embedded in $\mathfrak{S}_m$ as permuting $\{j, j+m, \ldots, j+(i-1)m\}$ only.

For any $i \times m$ matrix

$$A = (a_{p,q})_{1 \leq p \leq i, 1 \leq q \leq m}$$

of integers $a_{p,q} \in [m]$, let

$$V_A := (v_{a_{1,1}} \otimes v_{a_{1,2}} \otimes \cdots \otimes v_{a_{1,m}}) \otimes \cdots \otimes (v_{a_{i,1}} \otimes v_{a_{i,2}} \otimes \cdots \otimes v_{a_{i,m}}) \in \otimes^i (\otimes^m V_m).$$

Thus, we can rewrite the identity (26) as

$$S(B_o) \cdot v_o^{\otimes i} = \sum_{\sigma \in \mathfrak{S}_i \mu \in \mathfrak{S}_m} \epsilon(\mu)V_A(\sigma, \mu),$$

where $A(\sigma, \mu)$ is the $i \times m$ matrix

$$A(\sigma, \mu) = \begin{pmatrix}
\sigma_{\mu_1(1)}(1) & \cdots & \sigma_{\mu_1(i)}(m) \\
\vdots & \ddots & \vdots \\
\sigma_{\mu_1(i)}(1) & \cdots & \sigma_{\mu_m(i)}(m)
\end{pmatrix}.$$

We claim that

$$\langle v_o^{\otimes i}, S(B_o) \cdot v_o^{\otimes i} \rangle = \left\langle \sum_{(\sigma, \mu) \in R} \epsilon(\mu)V_A(\sigma, \mu), \sum_{(\sigma, \mu) \in R} \epsilon(\mu)V_A(\sigma, \mu) \right\rangle,$$

(27)
where the last summation runs over $R$ consisting of those $\sigma = (\sigma_1, \ldots, \sigma_i) \in \mathfrak{S}_m^i$ and $\mu = (\mu_1, \ldots, \mu_m) \in \mathfrak{S}_m^m$ such that $A(\sigma, \mu)$ is a Latin $(i, m)$-rectangle.

Since $v_0$ is, by definition, $(1/m)! \sum_{\sigma_1 \in \mathfrak{S}_m^i} v_*^{\sigma_1(1)} \otimes \cdots \otimes v_*^{\sigma_1(m)}$, unless each row of $A(\sigma, \mu)$ is a permutation of $[m]$, we have $\langle v_0^{\otimes i}, V_A(\sigma, \mu) \rangle = 0$. Further, assume that the entries in some column of $A(\sigma, \mu)$ are nondistinct, say

$$\sigma_{\mu_q(p)}(q) = \sigma_{\mu_q(p')}(q)$$

for some $1 \leq q \leq m$ and some $1 \leq p \neq p' \leq i$.

Let $\tau \in \mathfrak{S}_i$ be the transposition $(p, p')$. Then

$$V_A(\sigma, \mu) = V_A(\sigma, \mu'),$$

where $\mu' := (\mu_1, \ldots, \mu_q \circ \tau, \ldots, \mu_m)$.

Hence,

$$\epsilon(\mu)V_A(\sigma, \mu) + \epsilon(\mu')V_A(\sigma, \mu') = 0.$$ 

This proves the identity (27).

Let $R' \subset \mathfrak{S}_m^i$ be the subset consisting of $\sigma = (\sigma_1, \ldots, \sigma_i)$ such that $A(\sigma)$ is a Latin $(i, m)$-rectangle. For any $\sigma \in R'$, let $\hat{\sigma}$ be the pattern $(\hat{\sigma}^1, \ldots, \hat{\sigma}^m)$, where

$$\hat{\sigma}^q := \{\sigma_1(q), \ldots, \sigma_i(q)\}.$$ 

Define an equivalence relation on $R'$ by $\sigma \sim \sigma'$ if the patterns $\hat{\sigma} = \hat{\sigma'}$. Denote the equivalence class containing $\sigma$ by $[\sigma]$. Then the sum $\sum_{(\sigma, \mu) \in R} \epsilon(\mu)V_A(\sigma, \mu)$ can clearly be written as

$$\sum_{\sigma \in R'} \sum_{\mu \in \mathfrak{S}_m^m: (\sigma, \mu) \in R} \epsilon(\mu)V_A(\sigma, \mu) = \sum_{\sigma \in R'} \epsilon_c(A(\sigma)) \sum_{B \in L_\hat{\sigma}} \epsilon(\mu)V_B$$

$$= \sum_{[\sigma] \in R' \sim} \sum_{A \in L_\hat{\sigma}} \epsilon_c(A) \sum_{B \in L_\hat{\sigma}} \epsilon(\mu)V_B$$

$$= \sum_{A \in \mathfrak{S}(i, m)} \sum_{A \in L_A} \epsilon_c(A) \sum_{B \in L_A} \epsilon(\mu)V_B.$$ 

Thus, by the identity (27),

$$\langle v_0^{\otimes i}, S(B_0) \cdot v_0^{\otimes i} \rangle = \left(1/m!\right)^i \sum_{A \in \mathfrak{S}(i, m)} \sum_{A \in L_A} \epsilon_c(A) \sum_{B \in L_A} \epsilon(\mu)$$

$$= \left(1/m!\right)^i \sum_{A \in \mathfrak{S}(i, m)} \left(\sum_{A \in L_A} \epsilon_c(A)\right)^2$$

$$= \left(1/m!\right)^i \sum_{A \in \mathfrak{S}(i, m)} \#L_A^+ \#L_A^-.$$ 

This proves the proposition. \(\square\)
Representations supported by the orbit closure of the determinant

As an immediate consequence of the above proposition, we get the following corollary.

**Corollary 5.3.** \( \langle v_o^{\otimes i} \cdot S(B_o) \cdot v_o^{\otimes i} \rangle \neq 0 \) if and only if for some pattern \( A \in S(i, m) \), \( \sharp L_A^+ \neq \sharp L_A^- \).

For any partition \( \lambda \) of \( k \) into at most \( m \) parts, let \( G^\lambda \) be the highest weight space in \( \otimes^k (V_m) \) for \( GL(V_m) \) corresponding to the highest weight \( \lambda \). Then we have the following lemma (cf. [GW09, Lemma 9.3.2]).

**Lemma 5.4.** Let \( \lambda \) be a tableau of shape \( \lambda \). Then, \( S(A) \cdot v_A \) is a nonzero element of \( G^\lambda \). Thus,

\[
G^\lambda = \sum_{\tau \in \mathcal{S}_k} \mathbb{C} \tau \cdot (S(A) \cdot v_A).
\]

We specialize the above lemma to \( k = m^2 \) and \( \lambda \) the partition:

\[
m \delta_m : \frac{m \geq m \geq \cdots \geq m}{m \text{ factors}}.
\]

In this case \( V_m(m \delta_m) \) is a one-dimensional representation of \( GL(V_m) \).

Consider the tableau \( B_o = B_o(m, m) \) (with \( i = m \)) given just above Proposition 5.2. Then

\[
S(B_o) \cdot v_{B_o} = (m!)^m \sum_{\mu = (\mu_1, \ldots, \mu_m) \in \mathcal{S}_m} \epsilon(\mu)(v_{\mu_1(1)} \otimes v_{\mu_2(1)} \otimes \cdots \otimes v_{\mu_m(1)})
\]

\[
\otimes (v_{\mu_1(2)} \otimes v_{\mu_2(2)} \otimes \cdots \otimes v_{\mu_m(2)}) \otimes \cdots \otimes (v_{\mu_1(m)} \otimes v_{\mu_2(m)} \otimes \cdots \otimes v_{\mu_m(m)}).
\]

By the above lemma, the isotypic component \( G^\lambda \) of \( \otimes^m (\otimes^m V_m) \) for the partition \( \lambda = m \delta_m \) is the span of

\[
\{ \tau \cdot (S(B_o) \cdot v_{B_o}) : \tau \in \mathcal{S}_m \}.
\]

I thank J. Landsberg for the part (b) of the following proposition.

**Proposition 5.5.** (a) For CELS and COLS as defined in Conjecture 4.3,

\[
\langle v_o^{\otimes m} \cdot S(B_o) \cdot v_{B_o} \rangle = \sharp \text{CELS}(m) - \sharp \text{COLS}(m).
\]

(b) For any \( \tau \in \mathcal{S}_m \),

\[
\langle v_o^{\otimes m} \cdot \tau \cdot (S(B_o) \cdot v_{B_o}) \rangle = \alpha \langle v_o^{\otimes m} \cdot S(B_o) \cdot v_{B_o} \rangle \text{ for some } \alpha \in \{0, \pm 1\}.
\]

**Proof.** By the identity (28),

\[
\langle v_o^{\otimes m} \cdot S(B_o) \cdot v_{B_o} \rangle = \sum_{\mu \in \mathcal{S}_m} \epsilon(\mu),
\]

where the summation runs over those \( \mu = (\mu_1, \ldots, \mu_m) \in \mathcal{S}_m \) such that

\[
B(\mu) := \begin{pmatrix}
\mu_1(1) & \mu_2(1) & \cdots & \mu_m(1) \\
\mu_1(2) & \mu_2(2) & \cdots & \mu_m(2) \\
\vdots & \vdots & \ddots & \vdots \\
\mu_1(m) & \mu_2(m) & \cdots & \mu_m(m)
\end{pmatrix}
\]

is a Latin square (i.e., each row and each column of the above matrix is a permutation of \([m]\)), and

\[
\epsilon(\mu) := \epsilon(\mu_1) \cdots \epsilon(\mu_m).
\]

From this, part (a) follows.
(b) By the expression of $S(B_o) \cdot v_{B_o}$ as in the identity (28), clearly 
\[
\tau \cdot (S(B_o) \cdot v_{B_o}) = (m!)^m \sum_{\mu = (\mu_1, \ldots, \mu_m) \in \mathfrak{S}_m^m} \epsilon(\mu)(v_{\mu_1}(j_1) \otimes \cdots \otimes v_{\mu_m}(j_m))
\]
\[
\times \cdots \otimes (v_{\mu_1}(j_1) \otimes \cdots \otimes v_{\mu_m}(j_m)),
\]
for some fixed $i_q^p$ and $j_q^p \in [m]$ (depending only upon $\tau$).

We claim that if for any $1 \leq p \leq m$, $i_a^p = \neq i_b^p$ for some $a \neq b$, then $D_\tau = 0$, where $D_\tau := \langle v_{\sigma}^{\otimes m}, \tau \cdot (S(B_o) \cdot v_{B_o}) \rangle$. Observe that $j_a^p \neq j_b^p$ since the element
\[
(v_{\mu_1}(j_1) \otimes \cdots \otimes v_{\mu_m}(j_m)) \otimes \cdots \otimes (v_{\mu_1}(j_1) \otimes \cdots \otimes v_{\mu_m}(j_m))
\]
is a permutation of
\[
(v_{\mu_1}(j_1) \otimes v_{\mu_2}(j_2) \otimes \cdots \otimes v_{\mu_m}(j_m)) \otimes (v_{\mu_1}(j_1) \otimes v_{\mu_2}(j_2) \otimes v_{\mu_3}(j_3) \otimes v_{\mu_4}(j_4) \otimes v_{\mu_5}(j_5)) \otimes \cdots \otimes (v_{\mu_1}(j_1) \otimes v_{\mu_2}(j_2) \otimes v_{\mu_3}(j_3) \otimes v_{\mu_4}(j_4) \otimes v_{\mu_5}(j_5)).
\]

Consider the element $\theta = (j_a^p, j_b^p) \in \mathfrak{S}_m$. Then, replacing $\mu_q$ by $\mu_q \theta$ in the above expression for $\tau \cdot (S(B_o) \cdot v_{B_o})$, we clearly get
\[
D_\tau = \epsilon(\theta)D_\tau.
\]
Thus, $D_\tau = 0$.

Let us assume that for any $1 \leq p \leq m$, $i_a^p \neq i_b^p$ for $a \neq b$. Since $v_{\sigma}^{\otimes m}$ is $H_m$-invariant (where $H_m$ is defined above Proposition 3.4), to calculate $D_\tau$, we can assume that
\[
\tau \cdot (S(B_o) \cdot v_{B_o}) = (m!)^m \sum_{\mu \in \mathfrak{S}_m^m} \epsilon(\mu)(v_{\mu_1}(j_1) \otimes \cdots \otimes v_{\mu_m}(j_m)) \otimes \cdots \otimes (v_{\mu_1}(j_1) \otimes \cdots \otimes v_{\mu_m}(j_m)),
\]
where, for any $1 \leq q \leq m$, $\{j_q^1, \ldots, j_q^m\}$ is a permutation $\sigma_q$ of $[m]$. Now, replacing $\mu_q$ by $\mu_q \circ \sigma_q$, we get (setting $\sigma = (\sigma_1, \ldots, \sigma_m)$)
\[
\tau \cdot (S(B_o) \cdot v_{B_o}) = \epsilon(\sigma)(m!)^m \sum_{\mu \in \mathfrak{S}_m^m} \epsilon(\mu)(v_{\mu_1}(1) \otimes \cdots \otimes v_{\mu_m}(1)) \otimes \cdots \otimes (v_{\mu_1}(m) \otimes \cdots \otimes v_{\mu_m}(m))
\]
\[
= \epsilon(\sigma)S(B_o) \cdot v_{B_o}.
\]
This proves the proposition. \(\square\)

**Theorem 5.6.** Let $m$ be an even positive integer and let $1 \leq i \leq m$. If there exists a pattern $B$ of size $(i, m)$ such that
\[
\mathfrak{I}^+L_{B} \neq \mathfrak{I}^-L_{B},
\]
then the $\text{GL}(V_m)$-submodule $U_i$ generated by $v_{\sigma}^{\otimes i} \in S^i(S^m(V^*)_m) = [\otimes^i(\otimes^m V^*)_m]^H_i$ intersects the isotypic component $I_{m\delta_i}$ of $S^i(S^m(V^*)_m)$ corresponding to the irreducible $\text{GL}(V_m)$-module $V_m(m\delta_i)^*$ nontrivially, where $H_i$ is defined over Proposition 3.4.

In particular, if Conjecture 4.3 is true for $m$, then $U_i \cap I_{m\delta_i} \neq (0)$, for all $1 \leq i \leq m$.

For $i = m$, $U_m \cap I_{m\delta_m} \neq (0)$ if and only if $y_o \neq 0$. By [GW09, Theorem 9.3.10],

308
Thus, \( y_o \neq 0 \), proving the first part of the theorem.

The second ‘in particular’ part of the theorem, of course, follows from Proposition 4.4.

For the last part, by Lemma 5.4, \( S(B_o) \cdot v_{B_o} \) is a nonzero highest weight vector of \( \otimes^m(\otimes^m \mathbb{V}_m) \) with highest weight \( m \delta_m \) and the isotypic component of \( \otimes^m(\otimes^m \mathbb{V}_m) \) corresponding to the highest weight \( m \delta_m \) is given by \( \sum_{\tau \in e_{m^2}} \mathbb{C} \cdot \tau \cdot (S(B_o) \cdot v_{B_o}) \) (since \( \mathbb{V}_m(m \delta_m) \) is a one-dimensional representation).

Thus, \( y_o \in \mathcal{I}_{m \delta_m} \) is nonzero if and only if

\[
\langle v^{\otimes m}_o, x \rangle = \langle y_o, x \rangle \neq 0,
\]

for some \( x \in \sum_{\tau \in e_{m^2}} \mathbb{C} \cdot \tau \cdot (S(B_o) \cdot v_{B_o}) \). The above condition is equivalent to the nonvanishing of \( \langle v^{\otimes m}_o, S(B_o) \cdot v_{B_o} \rangle \) by Proposition 5.5(b); which, in turn, is equivalent to the validity of Conjecture 4.3 by Proposition 5.5(a). This proves the theorem. \( \square \)

Remark 5.7. It is quite possible that for any \( 1 \leq i \leq m \), \( U_i \cap \mathcal{I}_{m \delta_i} \neq 0 \) if and only if (29) is satisfied for some pattern \( B \) of size \( (i, m) \).

6. Statement of the main theorem and its consequences

Let \( \mathfrak{v} \) be a complex vector space of dimension \( m \) and let \( E := \mathfrak{v} \otimes \mathfrak{v}^* = \text{End} \mathfrak{v} \), \( Q := \mathcal{P}^m(E) \simeq S^m(E)^\ast \) (under the isomorphism \( \xi \) of Definition 2.5). Consider \( \mathcal{D} \in Q \), where \( \mathcal{D} \) is the function taking determinant of any \( A \in E = \text{End} \mathfrak{v} \). The group \( G = \text{GL}(E) \) acts canonically on \( Q \). Let \( \mathcal{X} \) be the \( G \)-orbit closure of \( \mathcal{D} \) inside \( Q \).

Fix a basis \( \{v_1, \ldots, v_m\} \) of \( \mathfrak{v} \) and let \( \{v_1^\ast, \ldots, v_m^\ast\} \) be the dual basis of \( \mathfrak{v}^\ast \). Take the basis \( \{v_i \otimes v_j^\ast\}_{1 \leq i, j \leq m} \) of \( E \) and order the basis elements as \( \{e_1, e_2, \ldots, e_{m^2}\} \) satisfying

\[
e_1 = v_1 \otimes v_1^\ast, \quad e_2 = v_2 \otimes v_2^\ast, \ldots, e_m = v_m \otimes v_m^\ast.
\]

Assume that \( m \) is even. Recall from Corollary 2.4 that for any \( 1 \leq i \leq m^2 \), the irreducible \( \text{GL}(E) \)-module \( \mathcal{V}_E(m \delta_i) \) occurs in \( \mathcal{S}^i(\mathcal{S}^m(E)) \) with multiplicity one (and \( \mathcal{V}_E(m \delta_i) \) does not occur in any \( \mathcal{S}^j(\mathcal{S}^m(E)) \), for \( j \neq i \)). Let \( P_i = \gamma_{m,i} \in \mathcal{S}^i(\mathcal{S}^m(E)) \) be the highest weight vector of \( \mathcal{V}_E(m \delta_i) \subset \mathcal{S}^i(\mathcal{S}^m(E)) \) (which is unique up to a nonzero scalar multiple) with respect to the standard Borel subgroup \( B = B_E \) of \( G \) consisting of upper triangular invertible matrices, where \( \text{GL}(E) \) is identified with \( \text{GL}(m^2) \) with respect to the basis \( \{e_1, \ldots, e_{m^2}\} \) of \( E \) given above. By Lemma 2.1, in fact \( P_i \in \mathcal{S}^i(\mathcal{S}^m(E)) \mathcal{E}_i \), where \( E_i \) is the subspace of \( E \) spanned by \( \{e_1, \ldots, e_i\} \).

Recall an explicit construction of \( P_i \) from Lemma 2.7. Since \( P_i \in \mathcal{S}^i(\mathcal{S}^m(E)) \), we can think of \( P_i \) as a homogeneous polynomial of degree \( i \) on the vector space \( Q = \mathcal{S}^m(E)^\ast \).

The following is our main result.
Theorem 6.1. Assume, as above, that \( m \) is even. Assume further that Conjecture 4.3 is true for \( m \). Then, with the above notation, for any \( 1 \leq i \leq m \), the polynomial \( P_i \) does not vanish identically on \( X \).

In particular, the irreducible \( GL(E) \)-module \( V_E(m \delta_i) \) occurs with multiplicity one in the afffine coordinate ring \( \mathbb{C}[X] \). Moreover, by Corollary 2.4, \( V_E(d \delta_i) \), for any \( d < m \) and any \( 1 \leq i \leq m^2 \), does not occur in \( S^* (S^m(E)) \); in particular, it does not occur in \( \mathbb{C}[X] \).

Proof. Recall the definition of the right action of the semigroup \( \text{End} E \) on \( Q = \mathcal{P}^m(E) \) from the identity (6). Consider the map

\[
\hat{\theta} : M(m, i) \to Q, \quad A \mapsto \mathcal{D} \circ \hat{A},
\]

where \( \hat{A} \in \text{End} E \) is defined by

\[
\hat{A} e_j = \sum_{p=1}^{m} a_{j}^{p} e_p \quad \text{for} \quad 1 \leq j \leq i, \quad \text{and} \quad \hat{A} e_j = 0 \quad \text{for} \quad j > i,
\]

where \( A = (a_{j}^{p})_{1 \leq p \leq m, 1 \leq j \leq i} \). Clearly,

\[
\text{Im} \hat{\theta} \subset X.
\]

To prove that \( P_i \in \mathcal{P}^i(Q) \simeq S^i(S^m(E)) \) restricts to a nonzero function on \( X \), it suffices to show that \( P_i \) restricts to a nonzero function on \( M(m, i) \) via the morphism \( \hat{\theta} \). Since

\[
P_i \in S^i(S^m(E_i)) \simeq S^i(S^{m}(E_i^* )) \simeq \mathcal{P}^i(\mathcal{P}^m(E_i)),
\]

\( P_i \) is the pullback of a function \( \tilde{P}_i \in \mathcal{P}^i(\mathcal{P}^m(E_i)) \) via the restriction map \( r : \mathcal{P}^m(E) \to \mathcal{P}^m(E_i) \). Thus, it suffices to prove that \( \tilde{P}_i \) restricts to a nonzero function on \( M(m, i) \) via \( \theta : M(m, i) \to \mathcal{P}^m(E_i) \simeq S^m(E_i)^* \) defined as the composite \( \theta = r \circ \hat{\theta} \). (Observe that this \( \theta \) coincides with the map \( \theta \) defined just before Lemma 3.2.) Now, by Lemma 3.3, the induced map

\[
\theta^* : S^* (S^m(E_i)) \to \mathcal{P}^{m*}(M(m, i))
\]

coincides with the map \( \varphi \) (up to a nonzero scalar multiple in any degree). Since \( \tilde{P}_i \) is the unique (up to a nonzero multiple) nonzero element of \( [S^i(S^m(E_i))]_{\text{SL}(E_i)} \) (by Proposition 2.3), it suffices to show that \( \varphi|_{[S^i(S^m(E_i))]_{\text{SL}(E_i)}} \neq 0 \). This follows from Proposition 3.4 and Theorem 5.6. The ‘in particular’ part follows from Corollary 2.4, and hence the theorem is proved.

Corollary 6.2. With the notation and assumptions as in the last theorem (in particular, assuming the validity of Conjecture 4.3 for \( m \)), for any dominant integral weight \( \lambda \) for \( GL(E) \) of the form \( \lambda = \sum_{i=1}^{m} n_i \delta_i, \quad n_i \in \mathbb{Z}_+ \), the irreducible \( GL(E) \)-module \( V_E(m \lambda) \) occurs in \( \mathbb{C}[X] \) with nonzero multiplicity.

Proof. First of all, \( X \) being an irreducible variety, \( \mathbb{C}[X] \) is an integral domain. Take a \( B_E \)-eigenvecot \( \tilde{P}_i \in \mathbb{C}[X] \) of weight \( m \delta_i \) for any \( 1 \leq i \leq m \); which exists by the last theorem (assuming the validity of Conjecture 4.3). Now consider the function

\[
\tilde{P}_\lambda = \prod_{i=1}^{m} \tilde{P}_i^{n_i} \in \mathbb{C}[X].
\]

Clearly, \( \tilde{P}_\lambda \) is a nonzero \( B_E \)-eigenvecot of weight \( m \lambda \). This proves the corollary.

310
Representations supported by the orbit closure of the determinant

Let \( \mathcal{O} \) be the \( G \)-orbit \( G \cdot \mathcal{O} \subset Q \). Then, by a classical result due to Frobenius (cf. [Kum13, Proposition 2.1 and Corollary 2.3]), the isotropy subgroup \( G_{\mathcal{O}} \) of \( \mathcal{O} \) is a reductive subgroup. In particular, by a result of Matsushima, \( \mathcal{O} \) is an affine variety. Moreover, by Frobenius reciprocity, we get the following proposition.

**Proposition 6.3.** \( \mathbb{C}[\mathcal{O}] \simeq \bigoplus_{\lambda} V_E(\lambda) \otimes [V_E(\lambda)^*]^{G_{\mathcal{O}}} \) as \( G \)-modules, where the summation runs over all the dominant integral weights \( \lambda \) of \( G \) (i.e., \( \lambda \) runs over \( \sum_{i=1}^{m^2} n_i \delta_i, n_i \in \mathbb{Z}_+ \) for all \( 1 \leq i < m^2 \) and \( n_{m^2} \in \mathbb{Z} \)) and \( [V_E(\lambda)^*]^{G_{\mathcal{O}}} \) denotes the subspace of \( G_{\mathcal{O}} \)-invariants in the dual space \( V_E(\lambda)^* \). The action of \( G \) on the right is via its standard action on the first factor and it acts trivially on the second factor.

In particular, the multiplicity of \( V_E(\lambda) \) in \( \mathbb{C}[\mathcal{O}] \) is the dimension of the invariant space \([V_E(\lambda)^*]^{G_{\mathcal{O}}}\).

It is easy to see that if \([V_E(\lambda)^*]^{G_{\mathcal{O}}} \neq 0\), then \( |\lambda| := \sum_{i=1}^{m^2} i n_i \in m\mathbb{Z} \), where (as earlier) \( \lambda = \sum_{i=1}^{m^2} n_i \delta_i \).

Applying [BLMW11, Proposition 5.2.1], we get that for any polynomial representation \( V_E(\lambda) \) (i.e., \( \lambda = \sum_{i=1}^{m^2} n_i \delta_i \) with all \( n_i \in \mathbb{Z}_+ \)) with \( |\lambda| = md, d \in \mathbb{Z}_+ \),

\[
\dim[V_E(\lambda)^*]^{G_{\mathcal{O}}} = sk_{\lambda,d,\delta,m},
\]

where \( \delta_m \) (as earlier) is the partition \( \delta_m : (1 \geq 1 \geq \cdots \geq 1) \) \((m \text{ factors})\), \( \lambda \) is the partition \((n_1 + \cdots + n_{m^2}) \geq n_2 + \cdots + n_{m^2} \geq n_3 + \cdots + n_{m^2} \geq \cdots \geq n_{m^2} \geq 0 \) and \( sk_{\lambda,d,\delta,m} \) is the symmetric Kronecker coefficient (i.e., the multiplicity of the irreducible \( S_{dm} \)-module \( W_{\lambda} \) in the symmetric product \( S^2(W_{dm}) \)), where, as earlier, \( W_{\lambda} \) denotes the irreducible \( S_{dm} \)-module corresponding to the partition \( \lambda \).

As a corollary of (30), and Proposition 6.3, we get the following result (since \( \mathbb{C}[\mathcal{O}] \hookrightarrow \mathbb{C}[\mathcal{O}] \) as a \( G \)-module).

**Corollary 6.4.** For any irreducible polynomial representation \( V_E(\lambda) \) of \( G \), such that \( |\lambda| = dm \), for \( d \in \mathbb{Z}_+ \), the multiplicity \( \mu(\lambda) \) of \( V_E(\lambda) \) in \( \mathbb{C}[\mathcal{O}] \) is bounded by:

\[
\mu(\lambda) \leq sk_{\lambda,d,\delta,m}.
\]

Observe that unless \( V_E(\lambda) \) is a polynomial representation of \( G \) and \( |\lambda| \in m\mathbb{Z}_+ \), we have \( \mu(\lambda) = 0 \).

As an immediate consequence of Corollaries 6.2 and 6.4, we get the following result.

**Corollary 6.5.** Let \( m \) be any positive even integer. Assume that Conjecture 4.3 is true for \( m \). Then, for any partition \( \lambda : (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0) \) (with at most \( m \) parts) of \( d \) (i.e., \( |\lambda| = d \)), the symmetric Kronecker coefficient

\[
sk_{\lambda,d,\delta,m} > 0.
\]

**Remark 6.6.** (a) Compare the above corollary with [BCI11, Theorem 1, §3].

(b) The following generalization of Theorem 6.1 holds by exactly the same proof. Let \( \mathcal{F} \in Q = S^m(E)^* \) be any (homogeneous) polynomial such that, writing \( \mathcal{F} \) as a sum of monomials (in a basis of \( E^* \)), some monomial with no repeated factors occurs with nonzero coefficient. Assume further that Conjecture 4.3 is true for \( m \), which is assumed to be even. Then, for any \( 1 \leq i \leq m \), the polynomial \( P_i \) does not vanish identically on the orbit \( \text{GL}(E) \cdot \mathcal{F} \).

In particular, this remark applies to \( \mathcal{F} = \mathfrak{P} \), where \( \mathfrak{P} \) is the function \( E \to \mathbb{C} \) taking any matrix \( A \in E : = \text{End} \mathfrak{v} \) to its permanent.

311
Thus, the irreducible $\text{GL}(E)$-module $V_{E}(m\delta_{i})$ occurs with multiplicity one in $\mathbb{C}[\text{GL}(E) \cdot \mathfrak{P}]$ for any $1 \leq i \leq m$ (assuming the validity of Conjecture 4.3 for $m$). Moreover, $V_{E}(d\delta_{i})$, for any $d < m$ and $1 \leq i \leq m^{2}$, does not occur in $\mathbb{C}[\text{GL}(E) \cdot \mathfrak{P}]$ (cf. Corollary 2.4).

Acknowledgements
I am grateful to P. Bürgisser and C. Ikenmeyer for pointing out a gap in an earlier version of the paper which resulted in this greatly modified version. The gap did not alter the main Theorem 6.1, however unfortunately now the validity of the theorem is proved only under the hypothesis that the Latin square conjecture is valid.

I thank J. Landsberg for helpful conversations and correspondence; in particular, §5 is greatly influenced by discussions with him and Proposition 5.5(b) is due to him. I also thank the referee for some helpful suggestions and Jeroen Zuiddam for pointing out some typos.

Partial support from the NSF grants is gratefully acknowledged.

References


Shrawan Kumar shrawan@email.unc.edu
Department of Mathematics, University of North Carolina,
Chapel Hill, NC 27599-3250, USA