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Mr CHARLES TWEEDIE, President, in the Chair.

On the Fractional Infinite Series for cosecx, secx, cotx, and tanx.

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The infinite products for sina; and cosa; are most conveniently obtained in a rigorous way from the well-known factorial expressions for sin^nθ and cos^nθ which, when n is an even integer, take the forms

(i) \( \sin n\theta = 2^{n-1} \sin \theta \cos \left( \frac{\sin^2 \frac{\pi}{n} - \sin^2 \theta}{n} \right) \left( \frac{\sin^2 \frac{2\pi}{n} - \sin^2 \theta}{n} \right) \ldots \left( \frac{\sin^2 \frac{n-2\pi}{2n} - \sin^2 \theta}{2n} \right) \)

(ii) \( \cos n\theta = 2^{n-1} \left( \frac{\sin^2 \frac{\pi}{2n} - \sin^2 \theta}{2n} \right) \left( \frac{\sin^2 \frac{3\pi}{2n} - \sin^2 \theta}{2n} \right) \ldots \left( \frac{\sin^2 \frac{n-1\pi}{2n} - \sin^2 \theta}{2n} \right) \)

\( \theta \) being put equal to \( \frac{\pi}{n} \) and \( n \) made infinitely great.*

It is then usual to obtain the fractional infinite series for cotx and tanx by logarithmic differentiation—a process in which the treatment of the remainder is somewhat involved—and to deduce those for cosecx and secx by the use of certain elementary trigonometrical identities.

It seems, however, a more fundamental process to obtain from (i) and (ii) expressions for \( \cos \theta \cosec n\theta \), \( \sec n\theta \), \( \sec \theta \cot n\theta \), and \( \sec \theta \tan n\theta \) in partial fractions; the degree in sinθ of the denominator in each of these functions being higher than that of the numerator; and then to proceed to the limit as in the case of the products.

* Cf. Hobson’s Trigonometry, Chap. XVII.
I. When \( n \) is even

\[
\frac{\cos^ \vartheta}{\sin^ \vartheta \sin n^ \vartheta} \text{ can be written in the form } \sum_{r=0}^{n-1} \frac{A_r}{\sin^ 2 \vartheta n - \sin^ 2 \vartheta},
\]

where \( A_0 = \left[ -\frac{\sin^ \vartheta \cos^ \vartheta}{\sin n^ \vartheta} \right]_{\vartheta=0} = \frac{1}{n} \)

and \( A_r = \left[ \frac{\left( \sin^ r \pi \vartheta - \sin^ 2 \vartheta \right) \cos^ \vartheta}{\sin^ \vartheta \cdot \sin n^ \vartheta} \right]_{\vartheta=\pi r/n} \)

\[
= \left[ \frac{\left( \sin^ r \pi \vartheta + \sin \vartheta \right) \cos^ \vartheta \sin^ r \pi \vartheta - \sin^ \vartheta}{\sin^ \vartheta \cdot \sin n^ \vartheta} \right]_{\vartheta=\pi r/n} \]

\[
= -\frac{2 \cos^ r \pi \vartheta}{n \cos r \pi} = \frac{(-)^{r-1}}{n} \cdot 2 \cos^ r \pi \vartheta, \quad (r = 1, 2, \ldots, \frac{n}{2} - 1);
\]

\[\therefore \, \cos^ \vartheta \cosec n^ \vartheta = \frac{1}{n \sin \vartheta} + \frac{2 \sin^ \vartheta}{n} \sum_{r=1}^{n-1} (-)^{r-1} \frac{\cos^ r \pi \vartheta}{\sin^ 2 \vartheta n - \sin^ 2 \vartheta} \]

[When \( n \) is odd,]

\[\cosec n^ \vartheta = \frac{1}{n \sin \vartheta} + \frac{2 \sin^ \vartheta}{n} \sum_{r=1}^{n-1} (-)^{r-1} \frac{\cos^ r \pi \vartheta}{\sin^ 2 \vartheta n - \sin^ 2 \vartheta} \]

Putting \( \vartheta = \frac{2x}{n} \), we get

\[
\cos^x n \cdot \cosec x = \frac{1}{\sin^x \frac{x}{n}} \sum_{r=1}^{n-1} (-)^{r-1} \frac{\cos^ r \pi \vartheta}{\sin^ 2 \vartheta n - \sin^ 2 \vartheta} + (-)^{k}. R
\]

where \( k \) is any integer less than \( \left( \frac{n}{2} - 1 \right) \).
It is obvious that $R$ is positive and less than
\[
\frac{2\sin \frac{x}{n} \cos^2 \left(\frac{(k+1)\pi}{n}\right)}{n \left\{ \sin^2 \left(\frac{(k+1)\pi}{n}\right) - \sin^2 \frac{x}{n} \right\}}
\]
provided $n$ is so great that $k$ can be chosen greater than $\frac{x}{\pi}$; for the angles $\frac{r\pi}{n}$ are increasing acute angles and therefore the terms of $R$ are in descending order of numerical magnitude.

\[
\therefore \cos \frac{x}{n} \cosec x = \frac{1}{n \sin \frac{x}{n}} + 2\sin \frac{x}{n} \cdot \sum_{1}^{k} (-)^{r-1} \cdot \frac{\cos^2 \left(\frac{(r+1)\pi}{n}\right)}{n^2 \left\{ \sin^2 \left(\frac{(r+1)\pi}{n}\right) - \sin^2 \frac{x}{n} \right\}}
\]
\[
+ (-)^{k} \cdot \epsilon \left(\frac{2\sin \frac{x}{n} \cos \left(\frac{(k+1)\pi}{n}\right)}{n \left\{ \sin^2 \left(\frac{(k+1)\pi}{n}\right) - \sin^2 \frac{x}{n} \right\}}\right),
\]
where $\epsilon$ is a positive proper fraction;

\[
\therefore \text{proceeding to the limit when } n \text{ becomes infinitely great,}
\]
\[
\cosec x = \frac{1}{x} + \sum_{1}^{\infty} (-)^{r-1} \cdot \frac{2x}{r\pi^2 - x^2} + (-)^{k} \cdot \epsilon \cdot \frac{2x}{(k+1)\pi^2 - x^2},
\]
$\epsilon$ being the limiting positive fractional value of $\epsilon$.

Hence the greater we make the finite number $k$ the more nearly is cosecx equal to $\frac{1}{x} + \sum_{1}^{\infty} (-)^{r-1} \cdot \frac{2x}{r\pi^2 - x^2}$, and the difference vanishes when $k$ becomes infinitely great.

\[i.e., \cosec x = \frac{1}{x} + \sum_{1}^{\infty} (-)^{r-1} \cdot \frac{2x}{r\pi^2 - x^2}, \text{ an absolutely convergent series;}
\]
\[= \frac{1}{x} + \frac{1}{\pi - x} - \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} + \ldots, \text{ a semi-convergent series; for all real values of } x, \text{ except } x = \pm r\pi.\]
II. When \( n \) is even,

\[
\sec n\theta = \frac{\sum_{r=1}^{n} A_r}{\sin^2(2r-1)\pi + \sin^2\theta},
\]

where \( A_r = \left[ \frac{\sin^2(2r-1)\pi + \sin^2\theta}{\cos\theta} \right] \theta = \frac{(2r-1)\pi}{2n} \)

\[
= \frac{2\sin(2r-1)\pi \cos(2r-1)\pi}{n\sin(2r-1)\pi} = \frac{(2r-1)\pi}{n} \cdot \frac{\sin(2r-1)\pi}{\sin^2(2r-1)\pi + \sin^2\theta};
\]

\[
\therefore \sec n\theta = \frac{1}{n} \sum_{r=1}^{\frac{n}{2}} (-1)^{r-1} \cdot \frac{\sin(2r-1)\pi}{\sin^2(2r-1)\pi + \sin^2\theta} ;
\]

[and when \( n \) is odd,

\[
\cos\theta \cdot \sec n\theta = \frac{1}{n} \sum_{r=1}^{\frac{n-1}{2}} (-1)^{r-1} \cdot \frac{\sin(2r-1)\pi \cos(2r-1)\pi}{\sin^2(2r-1)\pi + \sin^2\theta} .
\]

Putting \( \theta = \frac{x}{n} \), we get

\[
\sec x = \frac{1}{n} \sum_{r=1}^{\frac{k}{2}} (-1)^{r-1} \cdot \frac{\sin(2r-1)\pi}{\sin^2(2r-1)\pi + \sin^2\frac{x}{n}} + (-1)^x . \quad R
\]

and, as before, if \( n \) is so great that \( (2k-1) \) can be taken greater than \( \frac{2x}{\pi} \)

\[
R = \frac{1}{n} \cdot \frac{\sin(2k+1)\pi}{\sin^2(2k+1)\pi + \sin^2\frac{x}{n}} , \quad \text{for}
\]

\[
\frac{\sin(2r-1)\pi}{n} \left\{ \frac{\sin^2(2r+1)\pi}{2n} - \sin^2\frac{x}{n} \right\} \\
- \sin(2r+1)\pi \left\{ \frac{\sin^2(2r-1)\pi}{2n} - \sin^2\frac{x}{n} \right\}
\]
$$= 2 \sin \frac{\pi}{n} \left\{ \sin \frac{(2r - 1)\pi}{2n} \cdot \sin \frac{(2r + 1)\pi}{2n} + \sin^2 \frac{x}{n} \cdot \cos \frac{2r\pi}{n} \right\}$$

$$> 2 \sin \frac{\pi}{n} \left\{ \sin \frac{2r - 1}{2n} \cdot \sin \frac{2r + 1}{2n} \right\},$$

and \ldots the terms of R are in descending order of magnitude.

Hence, \(\sec x\)

$$= \frac{1}{n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin \frac{(2r - 1)\pi}{2n} \sin \frac{(2r + 1)\pi}{2n} + (-1)^k \sin \frac{2r\pi}{2n} \sin \frac{2r + 1}{2n}}{\sin^2 \frac{x}{n}},$$

and proceeding to the limit, we get, exactly as in I,

$$\sec x = 4 \pi \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2r - 1) \sin^2 \frac{2r\pi}{2n} + 4x^2}{2n},$$

a semi-convergent series;

$$= 2 \left\{ \frac{1}{\pi - 2x} + \frac{1}{\pi + 2x} - \frac{1}{3\pi - 2x} - \frac{1}{3\pi + 2x} + \ldots \right\};$$

for all real values of \(x\) except \(x = \pm \frac{(2r - 1)\pi}{2}\).

### III. When \(n\) is even

$$\sec \theta \cdot \cot n\theta = \prod_{k=1}^{n} \sin \frac{(3r - 1)\pi}{2n} \cdot \sin^2 \frac{\pi}{n} - \sin^2 \theta,$$

$$2^{n-1}, \sin \theta (1 - \sin^2 \theta) \prod_{k=1}^{n} \sin \frac{\pi}{n} - \sin^2 \theta,$$

\ldots

$$\cos \theta \sin \theta \cos \theta \sin \theta = \prod_{k=1}^{n} \frac{A_r}{\sin \frac{r\pi}{n} - \sin^2 \theta},$$

where \(A_0 = \left[ \begin{array}{c} -\sin \theta \cos \theta \\ \cos \theta \sin \theta \end{array} \right]_{\theta = 0} = -\frac{1}{n}, \)

\(A_r = \left[ \begin{array}{c} \cos \theta (\sin \frac{r\pi}{n} - \sin^2 \theta) \\ \sin \theta \cos \theta \cdot \sin \theta \end{array} \right]_{\theta = \frac{r\pi}{n}} = -\frac{2}{n}, \)

if \(r = 1, 2, \ldots, \frac{n}{2} - 1, \)
and \( A_n = \left[ \frac{\cos \theta \cos \theta}{\sin \theta \sin n\theta} \right]_{\theta = \frac{\pi}{2}} = \frac{1}{n} \); 

\[
\cdot \cdot \cdot \sec \theta \cot n\theta = \frac{1}{\sin \theta} - \frac{2\sin \theta}{n} \sum_{k=0}^{n-1} \frac{1}{\sin^2 \frac{\pi k}{n} - \sin^2 \theta} - \frac{\sin \theta}{n \cos^2 \theta}.
\]

[When \( n \) is odd, 

\[
\sec \theta \cot n\theta = \frac{1}{\sin \theta} - \frac{2\sin \theta}{n} \sum_{k=0}^{\frac{n-1}{2}} \frac{1}{\sin^2 \frac{\pi k}{n} - \sin^2 \theta}.
\]

Putting \( \theta = \frac{x}{n} \), we get

\[
\frac{\sec x}{n} = \cot x
\]

\[
= \frac{1}{\sin \frac{x}{n}} - \frac{\sin \frac{x}{n}}{n \cos^2 \frac{x}{n}} - 2\sin \frac{x}{n} \sum_{k=0}^{\frac{n-1}{2}} \frac{1}{\sin^2 \frac{\pi k}{n} - \sin^2 \frac{x}{n}} - \frac{2\sin \frac{x}{n}}{n} R,
\]

where 

\[
R = \sum_{k=1}^{n-1} \frac{1}{n^2 \left( \sin^2 \frac{\pi k}{n} - \sin^2 \frac{x}{n} \right)}
\]

\[
< \frac{n-1}{2} \sum_{k=1}^{n-1} \frac{1}{n^2 \left( \frac{4r^2}{n^2} - \frac{x^2}{n^2} \right)}, \text{ since } \phi > \sin \phi > \frac{2\phi}{\pi} \text{ if } 0 < \phi < \frac{\pi}{2};
\]

provided that \( n \) is so great that \( 2k \) can be taken greater than \( x \).

\[
\therefore R < \sum_{k+1}^{\frac{n-1}{2}} \frac{1}{4r^2 - x^2} < \sum_{k+1}^{\infty} \frac{1}{4r^2 - x^2}, \text{ the remainder after } k \text{ terms of a convergent infinite series.}
\]

Proceeding to the limit

\[
\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2x}{r^2 \pi^2 - x^2} - 2x \cdot R_1;
\]

and \( R_1 \), the limiting value of \( R \), can be made as small as we please by choosing \( k \) great enough.
\[
\cot x = \frac{1}{x} - \sum_{r=1}^{\infty} \frac{2x}{x^2 - r^2\pi^2 - 4x^2} \text{ an absolutely convergent series;}
\]
\[
= \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} - \ldots, \text{ a semi-convergent series; for all real values of } x, \text{ except } x = \pm r\pi.
\]

\[\text{IV. When } n \text{ is even,}
\]
\[
\sec \theta \csc \theta \cdot \tan \theta = \frac{n}{\cos \theta \cos \theta + \sin \theta \sin \theta} + \sum_{r=1}^{\infty} \frac{A_r}{\sin^2 \frac{(2r-1)n\pi}{2n} - \sin^2 \theta},
\]
where
\[
A_r = \sum_{r=1}^{\infty} \frac{\sin \theta \left( \frac{\sin \frac{(2r-1)n\pi}{2n} - \sin^2 \theta}{\sin \frac{(2r-1)n\pi}{2n}} \right)}{\sin \frac{(2r-1)n\pi}{2n} - \sin^2 \theta}.
\]
\[
\therefore \tan \theta = \frac{2\sin \theta \cos \theta}{n} \sum_{r=1}^{\infty} \frac{1}{\sin \frac{(2r-1)n\pi}{2n} - \sin^2 \theta}.
\]

\[\text{[When } n \text{ is odd,}
\]
\[
\tan \theta = \frac{1}{n} \tan \theta + \frac{2\sin \theta \cos \theta}{n} \sum_{r=1}^{\infty} \frac{1}{\sin \frac{(2r-1)n\pi}{2n} - \sin^2 \theta}.
\]

\[\text{Hence, exactly as in III.,}
\]
\[
\tan x = \frac{n}{2} \sum_{r=1}^{\infty} \frac{8x}{(2r-1)^2\pi^2 - 4x^2}, \text{ an absolutely convergent series;}
\]
\[
= 2 \left( \frac{1}{\pi - 2x} - \frac{1}{\pi + 2x} + \frac{1}{3\pi - 2x} - \frac{1}{3\pi + 2x} \ldots \right), \text{ a semi-convergent series; for all real values of } x, \text{ except } x = \pm \frac{(2r-1)\pi}{2}.
\]

The continuity of the algebraic expressions ensures that these results still hold good when \( x \) has complex values.