Spherical Simplexes in \(n\)-dimensions.

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1. All points in an \(n\)-space equidistant from a fixed point (the centre) constitute what may be called a spherical continuum of the \(n^{th}\) order,—the continuum being of \(n-1\) dimensions ((\(n-1\))-dimensional spread) and of the 2nd degree. Any region of this spherical continuum bounded by \(n\) \((n-1)\)-dimensional linear continua or primes (spaces of \(n-1\) dimensions), passing through the centre shall be called a spherical simplex of the \(n^{th}\) order. This spherical simplex is bounded by \(n\) faces, spherical simplexes of the \((n-1)^{th}\) order, each of which in turn is bounded by \(n-1\) spherical simplexes of the \((n-2)^{th}\) order, and so on till we reach spherical triangles, arcs and lastly points, the vertices. The total number of spherical simplexes of different orders connected with one of the \(n^{th}\) order is \(2^n - 2\). The \(n\) spherical continua of the \((n-1)^{th}\) order which contain the faces of the spherical simplex of the \(n^{th}\) order determine a set of \(2^n\) spherical simplexes of the same order, \(2^{n-1}\) pairs, the two spherical simplexes of a pair being symmetrically situated with respect to the centre and therefore congruent.\(^1\)

Let every \(n-1\) of the \(n\) primes intersect one another in lines which meet the spherical continuum in the vertices denoted by the numerals 1, 2, ..., \(n\); and let \((12...m)\) denote a spherical simplex of the \(m^{th}\) order lying on the spherical continuum \(K(12...m)\) of the same order in which the \(m\)-dimensional linear continuum \(P(12...m)\) passing through the centre intersects \(K(12...m)\); also let \((12...m, u...v, y...w)\) denote the angle between the spaces \(P(12...mu...v)\) and \(P(12...my...w)\) in the space \(P(1...mu...vy...w)\) in which they lie.

Without loss of generality we shall regard the radius as of unit length. Then let

\[
\begin{pmatrix}
1 & \cos(12) & \cos(13) & \ldots & \cos(1m) \\
\cos(12) & 1 & \cos(23) & \ldots & \cos(2m) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\cos(1m) & \cos(2m) & \ldots & \ldots & 1
\end{pmatrix} = \Delta(12...m) = \sin^2(12...m),
\]

\(^1\)The subject has been studied by Schläfli in the second chapter of his book *Theorie der vielfachen Kontinuität*. The field of his investigation is different from that of the present paper and the method adopted is totally different.

Also vide Coolidge, *Non Euclidean Geometry*. 
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denoting the square of $m!$ times the content of the join of the points 1, 2, ..., $m$ and the centre.\(^1\)

Also, as in the ordinary geometry, with each face are associated two \textit{poles}; and hence $2^n$ spherical simplexes can be formed, having for their vertices poles of these faces; and among them there is one $(1'2'\ldots n')$ whose vertices lie on the same side of the corresponding faces of the original spherical simplex as the latter's opposite vertices.

Now $(1\ldots n-1)$ and $(1\ldots n-2\ldots n)$ intersect one another in $(1\ldots n-2)$. In $P(1\ldots n-1)$ and $P(1\ldots n-2\ldots n)$ let $O_P$ and $O_Q$ be drawn respectively, through the centre $O$, perpendiculars to $P(1\ldots n-2)$ to meet $K(1\ldots n)$ in $p$ and $q$; then $(pq)$ measures the angle $(1\ldots n-2, n-1, n)$. And the faces of $(1'\ldots n')$ of which 1, 2, ..., $n-2$ are the poles intersect one another in $((n-1)'n')$; let this arc intersect $K(1\ldots n-1)$ and $K(1\ldots n-2\ldots n)$ in $p$ and $q$ respectively. Then it may be seen that $(pq) + ((n-1)'n') = \pi$, or $((n-1)'n') = \pi - (1\ldots n-2, n-1, n)$; similarly for others. So the relation between the spherical simplexes is of a dual character; and this gives rise to a duality between theorems relating to a spherical simplex.

In what follows we shall suppose that all di-spherical simplexes, \textit{i.e.} circular arcs, $(mn)$ are less than $\pi$. Also we shall use $N$ to denote the set of numbers from 1 to $n$, $N_p$ to denote all of these but the number $p$, $N_{pq}$ all but $p$ and $q$ and so on.

2. Consider the perpendicular $h_p$ let fall from $p$ on $P(N_p)$. Equating the values of $h_p$ we have the $n-1$ quantities

\[(1) \sin (pq) \sin (N_p, p, N_{pq}) \text{ all equal to one another for the same value of } p.\]

Again, since $h_p = \frac{\sin (N_p)}{\sin (N_{pq})}$, equating the values of $\sin (N)$ we have the $n-1$ quantities

\[(2) \sin (pq) \sin (p, q, N_{pq}) \text{ equal to one another for the same value of } p.\]

Also, from above,

\[(3) \frac{\sin (N_p)}{\sin (N_q)} = \frac{\sin (p, q, N_{pq})}{\sin (q, p, N_{pq})}.\]

In general, let $M$ denote any set of $m$ numbers ($m < n$) containing a particular number $p$, and consider the perpendiculars let fall from

\(^1\)This is in accordance with the nomenclature used by Prof. von Staudt, who has called the function $\sqrt{A(123)}$ the sine of the solid angle that the spherical triangle subtends at the centre of the sphere. \textit{Crelle} 24 (1842), 252.
Then for all values of $M$ the $\frac{(n - 1)!}{(m - 1)! (n - m)!}$ quantities

\[(4) \frac{\sin (M)}{\sin (M_p)} \sin (\bar{M}_p, p, N_M) \text{ are equal to one another for the same value of } p. \] It follows that, for a particular value of $M$ denoting a set of $m$ numbers $p q r \ldots$, the $m$ quantities

\[(5) \frac{\sin (N_i)}{\sin (M_i)} \sin (\bar{M}_i, i, N_M) \text{ are equal to one another for } i = p, q, r, \ldots, \] obtained by equating the values of $\sin (N)$.

In particular, when $m = n - 1$, the $n - 1$ quantities

\[(6) \frac{\sin (N_i)}{\sin (N_{pq})} \sin (\bar{N}, p, q) \text{ are equal for the same value of } p. \] Consequently we have the $n (n - 1)$, $2$ quantities

\[(7) \frac{\sin (N_i)}{\sin (N_{ij})} \sin (\bar{N}, i, j) \text{ equal to one another and to } \frac{\sin (N)}{\Pi \sin (N_j)}, \text{ where } \] II denotes the continued product for

\[j = 1, 2, \ldots, p - 1, p + 1, \ldots, q - 1, q + 1, \ldots, n \text{ and } i = 1, 2, \ldots, n. \] It immediately follows that

\[(8) \sin (N) = \frac{\sin (N_{pq}, p, q)}{\sin (N_{ij})} \sin (N_i) \sin (N_{ij}), \text{ for all values of } p \text{ and } q. \]

Again substituting in (6) the value of $\sin (N_i)$ from (8) which is

\[\sin (N_i) = \frac{\sin (N_{pq}, p, r)}{\sin (N_{pqr})} \sin (N_p) \sin (N_{pr}), \] we have

\[(9) \frac{\sin (N_{pq}, p, q)}{\sin (N_{pqr}, p, q)} = \frac{\sin (N_{pq}, p, r)}{\sin (N_{pqr}, p, r)} = \frac{\sin (N_{pq}, q, r)}{\sin (N_{pqr}, q, r)}. \]

Similarly substituting the values of $\frac{\sin (N_i)}{\sin (N_{ij})}$ from (1) in (6), we

\[1 \text{ Schl"{a}fli, loc. cit } \S 20 (4), \text{ who has given only one of these forms. The formula is also proved by the consideration of a parallelochesm of the } n \text{th order whose content is equal to the product of the contents of two of its adjacent faces (parallelochesms of } (n - 1)\text{th order) multiplied by the sine of the angle between the faces and divided by the content of the parallelochesm of } (n - 2)\text{th order in which the faces intersect.} \]
have, for fixed values of \( q \) and \( r \), the \((n - 2)\) quantities

(10) \( \sin(q, r, N_{pq}) \sin(N_{pq}, p, q) \) equal for all values of \( p \).

Proceeding in this way we may have a number of formulae connecting the elements of a spherical simplex, e.g., equating corresponding terms from (1), (4), (6) we shall have a number of other formulae. But we have given above only the more fundamental which we shall have occasion to use hereafter. It may be seen that from (6), when \( n = 4 \),

\[
\frac{\sin(12, 3, 4)}{\sin(12)} \cdot \frac{\sin(34, 1, 2)}{\sin(34)} = \frac{\sin(13, 2, 4)}{\sin(13)} \cdot \frac{\sin(24, 1, 3)}{\sin(24)}
\]

\[
= \frac{\sin(14, 2, 3)}{\sin(14)} \cdot \frac{\sin(23, 1, 4)}{\sin(23)}
\]

\[
= \frac{\sin^2(1234)}{\sin(123) \sin(124) \sin(134) \sin(234)}.
\]

\[\Delta = \begin{vmatrix}
\cos(n - 1) & \cos(n - 1) & \cos(2n - 1) & \ldots & \cos(n - 2n - 1) \\
\cos(1n) & 1 & \cos(12) & \ldots & \cos(1n - 2) \\
\cos(2n) & \cos(12) & 1 & \ldots & \cos(2n - 2) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\cos(n - 2n) & \cos(1n - 2) & \cos(2n - 2) & \ldots & 1
\end{vmatrix}
\]

It may therefore be seen that

\[
\Delta \left( q \frac{P}{q} N_{pq} \right) = \frac{1}{\Delta(N_{pq})} \left[ \Delta(N_{pq}) \Delta(p, q, N_{pq}) - \Delta(q, N_{pq}) \Delta(q, N_{pq}) \right]
\]

\[
= \frac{\sin^2(N_{pq}) \sin(N_{pr}) \sin(N_{qr})}{\sin^2(N_{pq})}
\]

\[
\times [\cos(N_{pq}, p, q) - \cos(N_{pq}, p, r) \cos(N_{pq}, q, r)].
\]

Accordingly, for all \( n - 2 \) values of \( r \)
\[ (12), (13) \]
\[
\cos(\overline{N}_{pq}, p, q) = \frac{\sin^2(\overline{N}_{pq}) \cdot \sin(N_r) \cot(\overline{N}_{pqr}, p, q) - \sin(N_p) \sin(\overline{N}_{pqr}, p, r) \cot(\overline{N}_{pqr}, q, r)}{\sin(N_p) \sin(N_r)}
\]
\[
= \frac{\cos(\overline{N}_{pq}, p, q) - \cos(\overline{N}_{pqr}, p, r) \cos(\overline{N}_{pqr}, q, r)}{\sin(\overline{N}_{pqr}, p, r) \sin(\overline{N}_{pqr}, q, r)}
\]

Also, since \( \Delta \left( \frac{p}{q} N_{pq} \right) = \sin(N) \sin(\overline{N}_{pq}) \cot(\overline{N}_{pq}, p, q) \), we have

\[ (14) \]
\[
\sin(N) = \Delta \left( \frac{p}{q} N_{pq} \right) \tan(\overline{N}_{pq}, p q) \cosec(\overline{N}_{pq}), \text{ for all values of } p \text{ and } q.
\]

Moreover, as in the ordinary geometry, if the three angles in the numerator of \( (13) \) be denoted by \( A, B, C \) and their sum by \( 2S \), we shall express the usual \( 2 \sqrt{\sin S \sin(S - A) \sin(S - B) \sin(S - C)} \) by \( \sin(\overline{N}_{pqr}, pqr) \) and regard the angle as the spherical trihedral angle at \( (N_{pqr}) \) formed by the spherical continua \( K(N_{pq}), K(N_{pr}) \) and \( K(N_{qr}) \). Then,

\[ (15) \]
\[
\sin(N) = \frac{\sin(N_p) \sin(N_q) \sin(\overline{N}_{pqr}, pqr)}{\sin(N_p) \sin(N_{pqr}, p, r) \sin(N_{pqr}, q, r)}, \text{ from (8)}
\]
\[
= \frac{\sin(N_{pqr}) \sin(N_{pqr}) \sin(N_{pqr}) \sin(N_{pqr}) \sin(\overline{N}_{pqr}, r, s)}{\sin^2(\overline{N}_{pqr}) \cdot \sin(\overline{N}_{pqr}, pqr)}, \text{ from (8a)}
\]
\[
= \sin(N_{pqr}) \sin(N_{pqr}) \sin(N_{pqr}) \sin(N_{pqr}) \sin(\overline{N}_{pqr}, r, s) \sin(\overline{N}_{pqr}, q, s) \sin(\overline{N}_{pqr}, p, s) \sin(\overline{N}_{pqr}, pqr) \sin(N_{pqr})
\]

Proceeding thus it may be seen ultimately, taking \( p = n, q = n - 1, \ldots, \) that

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\(1\) Owing to the importance of formula \( (13) \) we append another proof by the method of projection: Let \( a \) be the projection of \( n \) on \( P(N_n) \), \( b \) and \( c \) the projections of \( a \) on \( P(N_{n-2}) \) and \( P(N_{n-1}) \) respectively, \( d \) the projection of \( b \) or \( c \) on \( P(N_{n-1} n-2) \). \( e \) the projection of \( c \) on \( bd \) and \( f \) that of \( a \) on \( ce \). Then \( db = de + eb \). But \( db = nd \cos(N_{n-1} n-2, n-1, n) \)

\[
dc = cd \cos(N_{n-1} n-2, n-2, n-1) = nd \cos(\overline{N}_{n-1} n-2, n-2, n-1) \cos(\overline{N}_{n-1} n-2, n-2, n-1)
\]

\[
db = eb = cos(\overline{N}_{n-1} n-2, n-2, n-1)
\]

\[
dc = nd \cos(\overline{N}_{n-1} n-1, n-2, n) \sin(\overline{N}_{n-1} n-2, n-2, n-1) \cos(\overline{N}_{n-1} n-1, n-1, n)
\]
(16) \( \sin(N) = \sin(12) \sin(13) \ldots \sin(1n) \sin(\overline{1}, 2, 3) \sin(\overline{1}, 2, 4) \)
\[ \ldots \sin(\overline{1}, 2, n) \sin(\overline{12}, 3, 4) \ldots \sin(\overline{12}, 3, n) \sin(\overline{123}, 4, 5) \]
\[ \ldots \sin(\overline{1 \ldots n-4}, n-3, n) \sin(\overline{1 \ldots n-3}, n-2, n-1, n) \].

We shall express the coefficient of \( \sin(12) \ldots \sin(1n) \) by \( \sin(\overline{123} \ldots \overline{n}) \)
and regard the angle as a spherical polyhedroidal angle at 1 formed
by the arcs \( \overline{12}, \ldots, \overline{1n} \).

§ 4. From (13),
\[
\cos(\overline{NP}, p, q) + \cos(\overline{NP}, p, r) \cos(\overline{Nq}, q, r) = \frac{\sin^2(\overline{NP}, pqr) \cos(\overline{Npqr}, p, q)}{\sin^2(\overline{Npqr}, p, q) \sin(\overline{Npqr}, p, r) \sin(\overline{Npqr}, q, r) \sin^2(\overline{NP}), p, q) \sin(\overline{Npqr}, p, r) \sin(\overline{Npqr}, q, r)}
\]
\[
\sin^2(N) \sin^4(\overline{NP}), p, q) \sin(\overline{Npqr}, p, r) \sin(\overline{Npqr}, q, r)
\]
\[
\cos(\overline{NP}, p, q) \sin(\overline{NP}, p, r) \sin(\overline{Npqr}, q, r).
\]
Therefore
\[
(17) \cos(\overline{NP}, p, q) = \frac{\cos(\overline{NP}, p, q) + \cos(\overline{NP}, p, r) \cos(\overline{Np}, q, r)}{\sin(\overline{NP}, p, r) \sin(\overline{Npqr}, q, r)}.
\]

Moreover, as in the ordinary geometry, if the three angles in the
numerator of (17) be denoted by \( A, B, C \) and their sum by \( 2S \), we
shall express the usual \( 2\sqrt{-\cos S \cos(S-A) \cos(S-B) \cos(S-C)} \)
by \( \sin(p' q' r') \), in accordance with what has been said at the end of §1.
It may then be seen that
\[
(18) \sin^2(N) = \frac{\sin(\overline{NP}), \sin(\overline{Nq}), \sin(\overline{Nr}) \sin(p' q' r')}{\sin(\overline{NPqr})}
\]
\[
= \frac{\sin(p' q' r')}{\sin^4(\overline{NPqr})} \sin^2(\overline{NP}), \sin^2(\overline{Np}), \sin(\overline{Npqr}, p, q) \sin(\overline{Npqr}, p, r) \sin(\overline{Npqr}, q, r)
\]
\[
\cos(\overline{NP}, p, q) \sin(\overline{NP}, p, r) \sin(\overline{Npqr}, q, r).
\]
Proceeding thus it may be seen ultimately, taking \( p = n, \)
\( q = n - 1, \ldots \) that
\[
(19) \sin(N) = \sin(12) \sin(13) \ldots \sin(1n) \sin(\overline{1}, 2, 3) \ldots \sin(\overline{1}, 2, n)
\]
\[
\sin(12, 3, 4) \ldots \sin(12, 3, n) \ldots \sin(1 \ldots n - 4, n - 3, n) \sin^2((n-2)'
\]
\[
(n-1)' n') / \sin(1 \ldots n-2, n-1, n) \sin(1 \ldots n - 3 n - 1, n - 2, n)
\]
\[
\sin(1 \ldots n - 3 n, n - 2, n - 1).
\]
Comparing (16) with (19),
\[
(20) \quad \sin^2(p' q' r') = \sin (\overline{N}_{pq}, p, q) \sin (\overline{N}_{pr}, p, r) \sin (\overline{N}_{qr}, q, r) \sin (\overline{N}_{pqr}, pqr)
\]

Also from (18), for all sets of values of \( p, q, r \), the quantities
\[
(21) \quad \frac{\sin (p' q' r')}{\sin (\overline{N}_{pqr}) \Pi \sin (\overline{N}_i)}
\]
are equal, \( \Pi \) denoting the continued product for \( i \neq p, q, r \).

§ 5. (i) The arc \((pu)\) drawn from a vertex \( p \) to meet the opposite face \((N\_p)\) orthogonally in \( u \) is an altitude of \((N)\).

Since in this case
\[
\frac{\sin (\overline{N}_u)}{\sin (\overline{N}_{pq})} = \frac{\sin (\overline{N}_{qr})}{\sin (\overline{N}_{pr})} = \ldots = \sin (pu),
\]
we have
\[
\sin (pu) = \frac{\sin (N_u)}{\sin (N_p)} \sin (\overline{N}_{pq}, p, q), \text{ from (6) and } = \sin (pq) \sin (q, p, N_p),
\]
from (1). Accordingly
\[
(22) \quad \sin (pu) = \frac{\sin (N)}{\sin (N_p)} = \sin (pq) \sin (qr, p, s) \ldots \sin (\overline{N}_{pqr}, p, t)
\]

(ii) Let \( u \) be any point on \((N_p)\). Then from (13),
\[
\sin (\overline{N}_{pqr}, q, r) \cos (\overline{N}_{pqr}, p, u) = \sin (\overline{N}_{pqr}, q, u) \cos (\overline{N}_{pqr}, p, r) + \sin (\overline{N}_{pqr}, r, u) \cos (\overline{N}_{pqr}, p, q).
\]

Or
\[
(23) \quad \sin (N_p) \sin (N_q) \sin (N_r) \cos (\overline{N}_{pqr}, p, u) = \sin (N_q) \sin (N_p) \cot (N_{pqr}, p, u) + \sin (N_r) \sin (N_p) \cot (N_{pqr}, p, q).
\]

(iii) Let \( u \) be any point in \((qr)\). If we multiply the known formula in the ordinary geometry
\[
\sin (23) \cos (1u) = \sin (2u) \cos (13) + \sin (3u) \cos (12)
\]
successively by sines of the altitudes from (4) to (23), from (5) to (234) and so on, we shall have
\[
\cos (pu) \sin (N_p) = \cos (pq) \sin (N_{pq} u) + \cos (pr) \sin (N_{pr} u).
\]

Now divide both sides by \( \sin (pu) \sin (N_p) \) and multiply and divide the right side by \( \sin (u, p, N_{pq}) \) and apply (3) to the numerator.
We shall have
\[ \cot(\theta) = \frac{1}{\sin(N) \{\cos(pq) \sin(N_q) \sin(p,u,N_{pq}) + \cos(pr) \sin(N_r) \sin(p,u,N_{pr})\}} \]
\[ = \frac{\sin(N_{qr})}{\sin(N_{qr}, q, r)} \left\{ \frac{\cos(pq) \sin(p, u, N_{pq})}{\sin(N_r)} + \frac{\cos(pr) \sin(p, u, N_{pr})}{\sin(N_q)} \right\} \]
\[ = \frac{\sin(N_{qrs})}{\sin(N_{qrs}, q,r)} \left\{ \frac{\cos(pq) \sin(p, u, N_{pq})}{\sin(N_{qrs}, s, q)} + \frac{\cos(pr) \sin(p, u, N_{pr})}{\sin(N_{qrs}, s, r)} \right\} \]

Proceeding thus we have ultimately, taking
\[ p = 1, q = n, r = n - 1, s = n - 2, \ldots \]

(24) \[\sin(1 \ldots n - 2, n - 1, n) \cot(1u)\]
\[ = \frac{\cot(1n - 1) \sin(1, u, 2 \ldots n - 2 n)}{\sin(1, 2, n - 1) \sin(12, 3, n - 1) \ldots \sin(1 \ldots n - 3, n - 2, n - 1)} \]
\[ + \frac{\cot(1n) \sin(1, u, 2 \ldots n - 1)}{\sin(1, 2, n) \sin(12, 3, n) \ldots \sin(1 \ldots n - 3, n - 2, n)}. \]

§ 6. The content of a simplex (linear) with \( n + 2 \) vertices in a space of \( n \) dimensions vanishes. Therefore the relation between the arcs joining a point \( n + 1 \) with \( n \) other points \( N \) on \( K(N) \) is

(25) \[\sin(1 \ldots n + 1) = 0. \]
Or, squaring,
\[ \Sigma \cos^2(i, n + 1) \Delta(N_i) - \Delta(N) - 2 \Sigma \cos(i, n + 1) \cos(j, n + 1) \Delta(i, N_{ij}) = 0, \]
\[ i, j = 1, 2, \ldots, n; \ i \neq j \]
or
\[ \Sigma \cos^2(i, n + 1) \sin^2(N_i) - \sin^2(N) \]
\[ = 2 \Sigma \cos(i, n + 1) \cos(j, n + 1) \sin(N_i) \sin(N_i) \cos(N_{ij}, i, j). \]

Now if the \( n \) points \( N \) lie on a spherical continuum of \((n - 1)\)th order, \( \sin(N) = 0 \) and it is seen that the above relation reduces to

(26) \[\Sigma \cos(i n + 1) \sin(N_i) = 0, \]
which is the relation between the arcs joining any point \( n + 1 \) with \( n \) other points \( N \) on a spherical continuum of \((n - 1)\)th order.

Again if the arcs \((n + 1)\) be produced to meet the spherical continuum of \((n - 1)\)th order of which \( n + 1 \) is the pole in points \( 1', 2', \ldots, n' \), then
(27) \( \sum \sin(i'i') \sin(N_j) = 0, \)
which is the relation between the arcs drawn perpendicular to one spherical continuum of \((n - 1)\)th order from \(n\) points on another of the same order.

Also, if in (25) we substitute \(\frac{p^2n + 2 + q^2n + 2}{2qn + 2} - \frac{pq^2}{2p^2n + 2} \) for \(\cos (pq)\),

where \(\overline{pq}\) is the distance joining \(p\) and \(q\), we obtain

\[
\begin{vmatrix}
0 & 1 & 1 & \ldots & 1 \\
1 & 0 & \frac{12}{12} & \ldots & \frac{1n + 2}{1n + 2} \\
1 & \frac{12}{12} & 0 & \ldots & \frac{2n + 2}{2n + 2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \frac{1n + 2}{1n + 2} & \frac{2n + 2}{2n + 2} & \ldots & 0 \\
\end{vmatrix} = 0
\]

which is the identical relation connecting the distances between \(n + 2\) points in an \(n\)-space.

And if in (25) we substitute \(1 - \frac{\overline{pq}^2}{2R^2}\) for \(\cos (pq)\) and \(V\) denotes the content of the simplex with the \(n + 1\) points as vertices, we obtain

\[
(-1)^n 2^{n+1} (Rn! V)^2 = \begin{vmatrix}
0 & \frac{12}{12} & \frac{13}{13} & \ldots & \frac{1n + 1}{1n + 1} \\
\frac{12}{12} & 0 & \frac{23}{23} & \ldots & \frac{2n + 2}{2n + 2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1n + 1}{1n + 1} & \frac{2n + 2}{2n + 2} & \frac{0}{0} & \ldots & 0 \\
\end{vmatrix}
\]

giving the radius of a spherical continuum of \(n\)th order circumscribing a simplex whose vertices are the \(n + 1\) points.

Moreover, if the \(n + 1\) points lie on a spherical continuum of \((n - 1)\)th order, \(V = 0\) and substituting \(2R \sin \frac{1}{2} (pq)\) for \(\overline{pq}\) in (29) we have

\[
\begin{vmatrix}
0 & \sin^2 \frac{1}{2} (12) & \sin^2 \frac{1}{2} (13) & \ldots & \sin^2 \frac{1}{2} (1n + 1) \\
\sin^2 \frac{1}{2} (12) & 0 & \sin^2 \frac{1}{2} (23) & \ldots & \sin^2 \frac{1}{2} (2n + 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sin^2 \frac{1}{2} (1n + 1) & \sin^2 \frac{1}{2} (2n + 1) & \ldots & 0 \\
\end{vmatrix} = 0,
\]

which is the relation between the arcs joining \(n + 1\) points on a spherical continuum of \((n - 1)\)th order.

Finally, put \((1n + 1) = (2n + 1) = \ldots = (nn + 1) = r\) in (25); \(r\) is
then the spherical radius of the small (in the ordinary sense) spherical continuum of \((n - 1)\)th order circumscribing \((N)\). Accordingly,

\[
(31) \ \sec^2 r \sin^2(N) = \sum \sin^2(N_i) - 2 \sum \sin(N_i) \sin(N_j) \cos(N_{ij}, i, j).
\]

Also from (29) we have

\[
\begin{vmatrix}
0 & 1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & \frac{12^2}{12} & \frac{13^2}{13} & \ldots & 1n+1^2 & R^2 \\
1 & \frac{12^2}{12} & 0 & \frac{23^2}{23} & \ldots & 2n+1^2 & R^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1n+1^2 & 2n+1^2 & \ldots & \ldots & 0 & R^2 \\
1 & R^2 & R^2 & \ldots & \ldots & R^2 & 0
\end{vmatrix} = 0
\]

This, on substitution of \(2R \sin \frac{1}{2}(pq)\) for \(\overline{pq}\) and \(R\) for \((p n + 1)\), gives us

\[
(32) \ \sin^2 r = -2
\begin{vmatrix}
0 & 1 & 1 & 1 & \ldots & 1 \\
1 & 0 & \sin^2 \frac{1}{2}(12) & \sin^2 \frac{1}{2}(13) & \ldots & \sin^2 \frac{1}{2}(1n) \\
1 & \sin^2 \frac{1}{2}(12) & 0 & \sin^2 \frac{1}{2}(23) & \ldots & \sin^2 \frac{1}{2}(2n) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \sin^2 \frac{1}{2}(1n) & \sin^2 \frac{1}{2}(2n) & \ldots & \ldots & 0 \\
\sin^2 \frac{1}{2}(12) & \sin^2 \frac{1}{2}(13) & \ldots & \sin^2 \frac{1}{2}(1n) \\
\sin^2 \frac{1}{2}(12) & 0 & \sin^2 \frac{1}{2}(23) & \ldots & \sin^2 \frac{1}{2}(2n) \\
\sin^2 \frac{1}{2}(1n) & \sin^2 \frac{1}{2}(2n) & \ldots & \ldots & 0
\end{vmatrix}.
\]