An Asymptotic Formula for a Class of Distribution Functions

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If \( x_1, x_2, \ldots, x_k, \ldots \) are independent random variables each of which is subjected to a distribution law \( \sigma = \sigma(x) \) independent of \( k \) and having a finite positive dispersion, then \( x_1 + x_2 + \ldots + x_n \) is known to obey the Gauss law as \( n \to +\infty \), no matter how \( \sigma(x) \) be chosen\(^1\). There arises, however, the question whether it is nevertheless possible to determine the elementary law \( \sigma(x) \) from the asymptotic behaviour of the distribution law of \( x_1 + x_2 + \ldots + x_n \) for very large but finite values of \( n \). It will be shown that the answer is affirmative under very general conditions.

Let the distribution function \( \sigma(x) \) be a solution of the moment problem

\[
(1) \quad \int_{-\infty}^{+\infty} x^m \, d\sigma(x) = M_m \quad (m = 0, 1, 2, \ldots; \sigma(-\infty) = 0),
\]

so that \( M_0 \) is the total probability, hence equal to 1. It is not required that (1) be a determined moment problem, \( i.e. \) that \( \sigma \) be uniquely determined by the conditions (1) if one normalises it by the requirement that \( 2\sigma(x) = \sigma(x+0) + \sigma(x-0) \). On excluding the case \( M_2 = 0 \) of the trivial distribution function \( \sigma(x) = \frac{1}{2} (1 + \text{sign } x) \) and replacing, if necessary, \( \sigma(x) \) by \( \sigma(ax) \), where \( a = M_2 > 0 \), we may suppose that \( M_2 = 1 \). Also, although the symmetry condition thereby imposed upon the law \( \sigma \) is not essential for our method, we assume for convenience that both ranges \((0, x)\) and \((-x, 0)\) of the random variable subjected to \( \sigma \) are equally probable, \( i.e. \) that

\[
(2) \quad \sigma(x) + \sigma(-x) = 1; \quad \text{hence } \sigma(0) = \frac{1}{2}, \quad M_{2n+1} = 0 \quad (n=0, 1, 2, \ldots).
\]

Accordingly, the characteristic function of \( \sigma \),

\[
(3') \quad L(t; \sigma) = \int_{-\infty}^{+\infty} e^{itx} \, d\sigma(x) \quad (-\infty < t < +\infty),
\]

\(^1\) Cf. P. Lévy, Calcul des Probabilités, Paris, 1925, pp. 233-235.
and its Fourier inversion\(^1\),
\[
\sigma(x) = \sigma(0) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} L(t; \sigma) (e^{-i\pi t} - 1) t^{-1} dt \quad (-\infty < x < +\infty),
\]
become respectively
\[
L(t; \sigma) = 2 \int_0^{+\infty} \cos(tx) d\sigma(x) \quad (-\infty < t < +\infty)
\]
and
\[
\sigma(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} L(t; \sigma) t^{-1} \sin(tx) dt.
\]

We finally suppose that for some sufficiently small \(\delta > 0\) and for some function \(\phi(t)\)
\[
\int_{-\infty}^{+\infty} |t \phi(t)|^{1/2} dt < +\infty, \quad \text{and} \quad L(t; \sigma) = O(1/\phi(t)) \quad \text{as} \quad t \to +\infty,
\]
consideration of \(t \to -\infty\) being unnecessary since \(L(t; \sigma)\) is an even function, and that
\[
L(t; \sigma) \to 0 \quad \text{as} \quad t \to \infty.
\]

A few remarks concerning the nature of the restriction imposed by conditions (5) and (5a) upon the behaviour of \(\sigma(x)\) are not out of place. According to Lévy\(^2\) the average of \(|L(t; \sigma)|^2\) in the whole range \(-\infty < t < +\infty\) always exists and is equal to the sum of the squares of all jumps of \(\sigma(x)\). Hence \(\sigma(x)\) is everywhere continuous if and only if the average of \(|L(t; \sigma)|^2\) is zero, a condition clearly satisfied whenever (5a) is satisfied, so that \(\sigma\) has no discontinuity points. However (5), (5a) are sufficiently general not to require the absolute continuity of \(\sigma\), i.e. the existence of a density of probability\(^3\). In fact (5) and (5a) are implied by
\[
L(t; \sigma) = O(|\log|t||^{-\delta}),
\]

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\(^1\) P. Lévy, op. cit., p. 167.

\(^2\) P. Lévy, op. cit., p. 171.

\(^3\) There exists a derivative \(\sigma'(x)\) up to a set of measure zero even if \(\sigma(x)\) is not absolutely continuous, but
\[
\sigma(x) = \int_{-\infty}^{x} \sigma'(y) dy
\]
holds if and only if \(\sigma(x)\) is absolutely continuous. It is meaningless to regard \(\sigma(x)\) as a density of probability if (i) is not valid.
where $\alpha > 0$ may be arbitrarily small, and there exist symmetric distribution functions which satisfy (5b) but are not absolutely continuous. Conversely, the absolute continuity of $\sigma$ does not imply (5) since the Riemann-Lebesgue lemma cannot be formulated by using a universal majorant which tends to zero. A sufficient condition for (5b), hence for (5) and (5a), is that there exist a density of probability satisfying a uniform Lipschitz condition of arbitrarily low index, or only the corresponding logarithmical estimate, and tending not too slowly to zero as $x \to \infty$. Another sufficient condition for (5) and (5a) is that $\sigma$ satisfy the Gauss postulate for error distributions, i.e., that there exist for every $x$ a probability density which does not increase when $x$ increases. In fact, in this case it is clear from (3), in virtue of the second mean-value theorem, that $L(t; \sigma) = O(t^{-1})$, so that (5b) is amply satisfied.

Let the random variables $x_1, x_2, \ldots, x_k, \ldots$ be such that $\sigma(x)$ represents the probability of the inequality $x_k < x$ for every $k$. Then if $\sigma_n(x)$ denotes the probability of the inequality $x_1 + x_2 + \cdots + x_n < x$, we have

\[(6) \quad L(t; \sigma_n) = L(t; \sigma)^n\]

in virtue of the supposed independence of the random variables. The fundamental limit theorem of the calculus of probability implies that the distribution function $\sigma(a_n x)$, where

\[a_n = M_2(\sigma_n)^{1/n} = n^{1/2} M_{1/2} = n^{1/4},\]

tends, as $n \to +\infty$, to the reduced Gaussian distribution function. Our purpose is to show that $\sigma_n(x)$ is capable of an infinite asymptotic development in the Poincaré sense, proceeding according to powers of $n^{-1}$. The rôle of assumption (5) is that of assuring the existence of such a development, formal treatments of which date back to Laplace. The coefficient of $(n^{-1})^m$ in the asymptotic series in question is a polynomial in $x$ having as coefficients polynomials in the moments (1) of the elementary law $\sigma$, and the coefficients of the

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2 Cf., e.g., P. Lévy, op. cit., pp. 184-185.


4 Cf., e.g., E. T. Whittaker and G. Robinson, The Calculus of Observations, London, 1924, p. 173; cf. also F. Zernike, Handbuch der Physik, 3 (1928) 450-51, where further references are also given.
latter polynomials are universal constants. The elementary laws occurring in the majority of applications satisfy (5) and are such that the Carleman condition
\[ \sum_{m=0}^{+\infty} M_{2m}^{-1/(2m)} = +\infty \]
of determinateness is fulfilled. Hence we obtain a method, at least in theory, for determining the elementary law \( \sigma(x) \) from the behaviour of the approximation of the iterated law to the Gauss distribution.

The function \( L(t; \sigma) \) has for every \( t \) derivatives of arbitrarily high order\(^1\) which may be obtained by formal differentiation of (3), so that
\[ L^{(m)}(t; \sigma) = i^m \int_{-\infty}^{+\infty} x^m e^{itx} d\sigma(x). \]
In fact, each of the integrals (7) is uniformly convergent with respect to \( t \), its integrand having as a majorant\(^2\) that of \( M_m \).

It is clear from (3) that \(|L(t; \sigma)| \leq 1\) for every \( t \) and \(|L(t; \sigma)| = 1\) for \( t = 0\). Suppose that \(|L(t; \sigma)| = 1\) for a fixed \( t \). Then
\[ \int_{-\infty}^{+\infty} (1 \pm \cos(tx)) d\sigma(x) = 0, \]
where \( 1 \pm \cos(tx) \geq 0 \) for every \( x \) and either \( t = 0 \) or else \( 1 \pm \cos(tx) > 0 \) for some \( x \). Hence either \( t = 0 \) or else \( \sigma(x) \) is a step-function having all its jumps at points \( x \) which form an arithmetical progression. The second case is excluded, \( \sigma \) being continuous in virtue of (5). Consequently\(^3\)
\[ |L(t; \sigma)| < 1 \quad \text{for every } t \neq 0. \]
Moreover, since the second derivative of (3) is negative at \( t = 0 \) in virtue of (7), and the first derivative \( L'(t; \sigma) \) vanishes at \( t = 0 \) because of (7) and (2), we have \( L'(t; \sigma) < 0 \) for sufficiently small values of \( t > 0 \). It follows therefore from \( L(0; \sigma) = 1 \) that \( L(t; \sigma) \) is positive and decreasing in the interval \( 0 < t \leq c \) if \( c \) is sufficiently small. Let \( c \) be so chosen and put
\[ K_n = \int_{c}^{+\infty} |t^{-1} L(t; \sigma)^n| \, dt. \]

\(^1\) It is not true, however, that \( L(t; \sigma) \) is necessarily regular-analytic along the \( t \)-axis.

\(^2\) In virtue of the Schwarz inequality it is sufficient to consider even values of \( m \).

\(^3\) It may be mentioned that (8) is actually false in the second case. In fact, \( L(t; \sigma) \) is then a periodic function so that \( L(t; \sigma) = 1 \) holds for some \( t \neq 0 \) since it holds for \( t = 0 \).
Now $L(t; \sigma)$ has in the interval $c \leq t < \infty$ a positive maximum $\theta < 1$ according to (8) and (5a). On the other hand, it is clear from (5) and (9) that $K_j < +\infty$ if $j$ is sufficiently large, so that $K_n < \theta^{n-j} K_j < +\infty$ for every $n > j$. Consequently

$$\left| \int_0^{+\infty} L(t; \sigma)^n t^{-1} \sin(tx) \, dt \right| < C\theta^n, \quad \text{where } 0 < \theta < 1,$$

for every $x$ and for every $n > j$, where $\theta$ and $C = K_j/\theta^j$ depend only upon $c$.

Since $L(0; \sigma) = 1$ and $L(t; \sigma)$ is positive and decreasing in the range $0 < t \leq c$, the function

$$s = s(t) = \left\{ -\log L(t; \sigma) \right\}^{\frac{1}{t}}$$

is positive and increasing in this range so that there exists an inverse function $t = t(s)$, where $0 \leq s \leq d$ and $d = \left\{ -\log L(c; \sigma) \right\}^{\frac{1}{t}}$. Now the derivative $L'(t; \sigma)$ is negative at every point of the range $0 < t \leq c$ and vanishes at $t = 0$ only in the first order in virtue of $L''(0; \sigma) = -1$; hence the function $s = s(t)$ vanishes at $t = 0$ exactly in the first order, and consequently $r(t) = t/s(t)$ is positive at $t = 0$. Upon placing, for a fixed value of $x$,

$$\pi f(x; s) = \sin(xt(s)) \frac{d}{ds}(s) = \left(0 \leq s \leq d\right),$$

where the dot denotes differentiation with respect to $s$, it follows from the Bürmann-Lagrange rule\(^1\) that all derivatives of $f(x; s)$ with respect to $s$ exist not only in the range $0 < s \leq d$, but at $s = 0$ as well, and, moreover, that the derivatives are given by the explicit formula

$$\left[ \frac{\partial^n f(x; s)}{\partial s^n} \right]_{s=0} = \frac{1}{\pi} \int_0^x \frac{\sin(tx) r(t)^{n+1}}{t} \, dt.$$

Setting

$$\chi_n(x) = \frac{1}{\pi} + \frac{1}{\pi} \int_0^x L(t; \sigma)^n t^{-1} \sin(tx) \, dt,$$

we have from (11) and (12)

$$\chi_n(x) = \frac{1}{\pi} + \int_0^d \exp(-ns^2)f(x; s) \, ds.$$

This function $\chi_n(x)$ admits for every fixed $x$ an asymptotic development\(^2\)

$$\frac{1}{\pi} + \sum_{k=1}^{+\infty} P_k(x) n^{-2k},$$

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1 Cf. P. L. Tchebychef, Oeuvres, vol. 1, St. Pétersburg, 1899, pp. 251-270, where analyticity of the functions is not required.

Thus
$$P_k = P_k(x) = \frac{\Gamma\left(\frac{1}{2} k\right)}{2 \Gamma(k)} \left(\frac{\partial^{k-1} f(x; \sigma)}{\partial x^{k-1}}\right)_{x=0},$$

or
$$P_k = P_k(x) = \frac{\Gamma\left(\frac{1}{2} k\right)}{2\pi \Gamma(k)} \left(\frac{\partial^{k-1} \sin(tx)}{\partial t^{k-1}} \frac{\tau(t)^k}{t}\right)_{t=0}.$$

Thus
$$P_{2k+1} = \frac{\Gamma\left(k + \frac{1}{2}\right)}{2\pi \Gamma(2k+1)} \sum_{\nu=0}^{k} \binom{2k}{2\nu} (-1)^{k-\nu} \frac{x^{2(k-\nu)+1}}{2(k-\nu)+1} \left\{\frac{d^{2\nu} \tau(t)^{2k+1}}{dt^{2\nu}}\right\}_{t=0}.$$

by the Leibniz rule for differentiation, while $P_{2k} = 0$ for every $x$ because of (2). In particular

$$P_1(x) = (2\pi)^{-\frac{1}{2}} x,$$
$$P_3(x) = (2\pi)^{-\frac{1}{2}} \left\{-\frac{x^3}{6} + (M_4 - 3) \frac{x}{8}\right\},$$
$$P_5(x) = (2\pi)^{-\frac{1}{2}} \left\{\frac{x^5}{40} - 5(M_4 - 3) \frac{x^3}{48} + (35M_6^2 - 8M_6 - 90M_4 + 75) \frac{x}{384}\right\}.$$

It is clear from (10) that the above asymptotic expansion of (14) is also an asymptotic development of

$$\frac{1}{\sigma} + \frac{1}{\pi} \int_0^{+\infty} L(t; \sigma) t^{-1} \sin(tx) \, dt.$$

Moreover, upon applying (4) to $\sigma_n(x)$ instead of $\sigma(x)$, we see from (6) that (19) is exactly $\sigma_n(x)$. Therefore (15) is an asymptotic development of $\sigma_n(x)$.

It may be mentioned that (15) can in certain cases be a convergent series. For example, if $\sigma(x)$ obey the Gauss law the asymptotic development (15) for $\sigma_n(x)$ is found from (18) to be the convergent power-series representation of $\sigma_n(x)$.