Proceedings of the Edinburgh Mathematical Society (1999) 42, 43–63 ©

ON DUALITY BETWEEN K- AND J-SPACES

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(Received 14th October 1996)

We study the relationship between the dual of the K-space defined by means of a polygon and the J-space generated by the dual N-tuple. The results complete the research started in [4]. Special attention is paid to the case when the N-tuple is formed by Banach lattices

1991 *Mathematics subject classification:* 46B70, 46M35.

Introduction

This paper deals with K- and J-interpolation methods defined by means of polygons. These constructions were introduced in [7]. They are similar to the real interpolation method but they work for N-tuples of Banach spaces ($N \geq 3$) instead of couples. For special choices of the polygon, they recover (the first case of) multidimensional methods studied by Sparr [14] and by Fernandez [11]. The polygon and its elements (vertices and sides) play an important role when deriving some classical results of interpolation theory (see [7] and [6]). Some related constructions to the methods associated to polygons that works not only for N-tuples but also for infinite families of Banach spaces can be found in [3].

In this paper we conclude the study initiated in [4] on duality between K- and J-spaces. There, two of the present authors proved that duals may fail to be intermediate spaces with respect to the dual N-tuple, but however the dual of a J-space is always a closed subspace of a K-space.

We consider here the case of the K-space. In Section 2 we show that its dual is embedded in a J-space generated with respect to the dual N-tuple and that the two spaces coincide provided $\Sigma(\overline{A})' = \Delta(\overline{A'})$.

We also give a sufficient condition for equality $\Sigma(\overline{A})' = \Delta(\overline{A'})$, and we prove that under such condition the dual of the J-space does coincide with a K-space.

In Section 3 we study the special case when the N-tuple is formed by Banach lattices. We establish that for these N-tuples equality $\Sigma(\overline{A})' = \Delta(\overline{A'})$ always holds, that the dual

* Supported in part by DGICYT (PB94-0252).
† The participation of the first and fourth authors to this work benefited from the joint French-Spanish cooperation programme Picasso (contract no. 94029.)

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of a K-space coincides with a J-space and that, conversely, the dual of a J-space is a K-space.

1. Preliminaries

Let $\overline{A} = \{A_1, \ldots, A_N\}$ be a Banach N-tuple, i.e., N Banach spaces all of them continuously embedded in a common linear Hausdorff space. We denote by $\Sigma(\overline{A})$ their sum and by $\Delta(\overline{A})$ their intersection. These spaces become Banach spaces when normed by

$$\|a\|_{\Sigma(\overline{A})} = \inf\left\{\sum_{j=1}^{N} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, a_j \in A_j\right\}$$

and

$$\|a\|_{\Delta(\overline{A})} = \max_{1 \le j \le N}\left\{\|a\|_{A_j}\right\}.$$

Since we are interested in duality, in what follows we assume that $\overline{A}$ is regular, meaning that $\Delta(\overline{A})$ is dense in A_j for $j = 1, \ldots, N$. This ensures that we can identify the dual space A_j^* of A_j with a subspace A_j' of $\Delta(\overline{A})^*$. So the family $\overline{A'} = \{A_1', \ldots, A_N'\}$ is a Banach N-tuple too. As in the case of couples (see [2]) we have

$$\Delta(\overline{A})^* = \Sigma(\overline{A'}) \qquad \text{(equal norms)}$$

but now equality

$$\Sigma(\overline{A})' = \Delta(\overline{A'})$$

does not hold in general (see[9]). We only have that

$$\Sigma(\overline{A})' \text{ is a closed subspace of } \Delta(\overline{A'}) \text{ and norms coincide on } \Sigma(\overline{A})'. \tag{1}$$

Let now $\Pi = \overline{P_1, \ldots, P_N}$ be a convex polygon in the plane $\mathbf{R}^2$ with vertices $P_j = (x_j, y_j)$. By means of Π we introduce the following family of norms on $\Sigma(\overline{A})$ (resp. $\Delta(\overline{A})$)

$$K(t, s; a) = \inf\left\{\sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, a_j \in A_j\right\}, \qquad t, s > 0$$

$$\text{(resp.} \qquad J(t, s; a) = \max_{1 \le j \le N}\left\{t^{x_j} s^{y_j} \|a\|_{A_j}\right\}, \qquad t, s > 0).$$

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Given (α, β) in the interior of Π $[(\alpha, \beta) \in \operatorname{Int} \Pi]$ and $1 \leq q \leq \infty$, the space $\overline{A}_{(\alpha,\beta),q;K}$ consists of all $a \in \Sigma(\overline{A})$ which have a finite norm

$$\|a\|_{(\alpha,\beta),q;K} = \left(\sum_{(m,n)\in\mathbf{Z}^2} (2^{-\alpha m-\beta n} K(2^m, 2^n; a))^q \right)^{\frac{1}{q}} \qquad \text{if } 1 \leq q < \infty$$

$$\|a\|_{(\alpha,\beta),q;K} = \sup_{(m,n)\in\mathbf{Z}^2} \{2^{-\alpha m-\beta n} K(2^m, 2^n; a)\} \qquad \text{if } q = \infty.$$

The space $\overline{A}_{(\alpha,\beta),q;J}$ is formed by all elements $a \in \Sigma(\overline{A})$ which can be represented as

$$a = \sum_{(m,n)\in\mathbf{Z}^2} u_{m,n} \qquad (\text{convergence in } \Sigma(\overline{A}))$$

with $u_{m,n} \in \Delta(\overline{A})$ and

$$\left(\sum_{(m,n)\in\mathbf{Z}^2} (2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n}))^q \right)^{\frac{1}{q}} < \infty$$

(the sum should be replaced by the supremum if $q = \infty$). The norm in $\overline{A}_{(\alpha,\beta),q;J}$ is

$$\|a\|_{(\alpha,\beta),q;J} = \inf\left\{ \left(\sum_{(m,n)\in\mathbf{Z}^2} (2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n}))^q \right)^{\frac{1}{q}} \right\}$$

where the infimum is taken over all representations $(u_{m,n})$ of a as above.

In general

$$\overline{A}_{(\alpha,\beta),q;J} \neq \overline{A}_{(\alpha,\beta),q;K}$$

because the so-called fundamental lemma of interpolation theory (see [**2**, Lemma 3.3.2]) does not extend to our context of N-tuples. We only have now the continuous embedding

$$\overline{A}_{(\alpha,\beta),q;J} \hookrightarrow \overline{A}_{(\alpha,\beta),q;K}$$

see [**7**, Thm. 1.3]. The argument given there also shows that if $(u_{m,n}) \subset \Delta(\overline{A})$ with

$$\left(\sum_{(m,n)\in\mathbf{Z}^2} (2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n}))^q \right)^{\frac{1}{q}} < \infty$$

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then the series $\sum_{(m,n)\in\mathbb{Z}^2} u_{m,n}$ is absolutely convergent in $\Sigma(\overline{A})$. The following result will be useful for our reasonings in Section 3. In the proof we use some ideas introduced in [**7**, Thm. 1.3].

Lemma 1.1. *If* $(\alpha,\beta)\notin\Pi$ *and* $1\le q\le\infty$, *or* $(\alpha,\beta)\notin \mathrm{Int}\,\Pi$ *and* $1\le q<\infty$, *then*

$$\overline{A}_{(\alpha,\beta),q;K}=\{0\}$$

for any Banach N-tuple $\overline{A}$.

Proof. Since

$$\min_{1\le j\le N}\{t^{x_j}s^{y_j}\}K(1,1;a)\le K(t,s;a)$$

we have that

$$\left(\int_0^\infty\int_0^\infty\left(\min_{1\le j\le N}\{t^{x_j-\alpha}s^{y_j-\beta}\}\right)^q\frac{dt}{t}\frac{ds}{s}\right)^{\frac{1}{q}}K(1,1;a)\le\|a\|_{(\alpha,\beta),q;K}.$$

Hence, it suffices to show that the integral

$$\begin{aligned}I&=\int_0^\infty\int_0^\infty\left(\min_{1\le j\le N}\{t^{x_j-\alpha}s^{y_j-\beta}\}\right)^q\frac{dt}{t}\frac{ds}{s}\\&=\int_{-\infty}^\infty\int_{-\infty}^\infty\left(\min_{1\le j\le N}\{e^{u(x_j-\alpha)+v(y_j-\beta)}\}\right)^q du\,dv\end{aligned}$$

is not finite.

If $P=(\alpha,\beta)\notin\Pi$ [resp. $P\in\Pi$ but $P\notin\mathrm{Int}\,\Pi$], we can find $1\le j\le N$ such that

$$P-P_j=\lambda_j(P_{j+1}-P_j)+\mu_j(P_{j-1}-P_j)$$

with some of the scalars λ_j,μ_j, say λ_j, being <0 [resp. $\lambda_j=0$ and $\mu_j\ge0$] (see Fig. 1.1). Put

$$\Gamma_j=\{(u,v)\in\mathbb{R}^2: u(x_j-\alpha)+v(y_j-\beta)\le u(x_k-\alpha)+v(y_k-\beta),\ k=1,\dots,N\}.$$

We claim that

$$(u,v)\in\Gamma_j\qquad\text{if and only if}\qquad\begin{cases}u^*=\langle(u,v),P_{j+1}-P_j\rangle\ge0\\ \qquad\text{and}\\ v^*=\langle(u,v),P_{j-1}-P_j\rangle\ge0\end{cases}$$

where $\langle\cdot,\cdot\rangle$ stands for the scalar product of $\mathbb{R}^2$.

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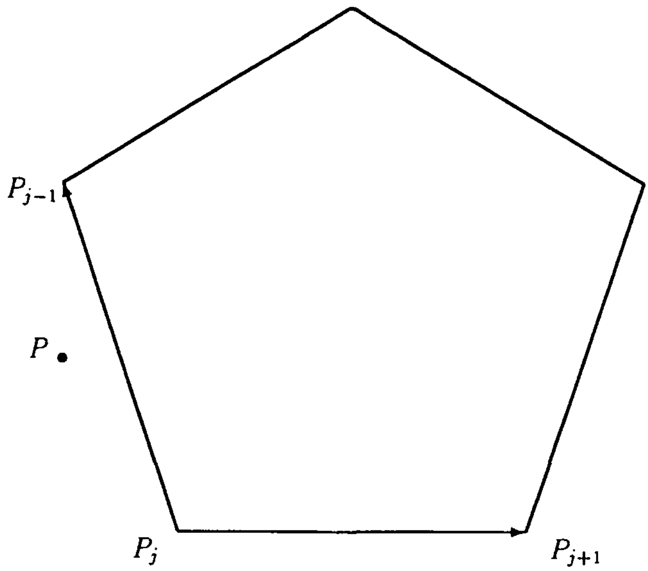

FIGURE 1.1

Indeed, if $(u, v) \in \Gamma_j$ then inequalities

$$u(x_j - \alpha) + v(y_j - \beta) \leq u(x_{j+1} - \alpha) + v(y_{j+1} - \beta)$$
$$u(x_j - \alpha) + v(y_j - \beta) \leq u(x_{j-1} - \alpha) + v(y_{j-1} - \beta)$$

imply that

$$u^* = \langle (u, v), P_{j+1} - P_j \rangle \geq 0 \qquad \text{and} \qquad v^* = \langle (u, v), P_{j-1} - P_j \rangle \geq 0.$$

Reciprocally, suppose that $u^* \geq 0$ and $v^* \geq 0$. Given any $1 \leq k \leq N$ with $k \neq j-1$, $j+1$, convexity of Π yields that

$$P_k - P_j = \rho(P_{j+1} - P_j) + \eta(P_{j-1} - P_j)$$

for some positive numbers ρ, η. Whence

$$\langle (u, v), P_k - P_j \rangle = \rho u^* + \eta v^* \geq 0.$$

So

$$ux_j + vy_j \leq ux_k + vy_k$$

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or equivalently

$$u(x_j-\alpha)+v(y_j-\beta)\le u(x_k-\alpha)+v(y_k-\beta).$$

Hence $(u,v)\in\Gamma_j$.

In order to show that I is not finite, we estimate it from below using Γ_j. We have

$$\begin{aligned} I &\ge \int\int_{\Gamma_j} e^{(u(x_j-\alpha)+v(y_j-\beta))q}\mathrm{d}u\mathrm{d}v \\ &= \frac{1}{|J_j|}\int_0^\infty\int_0^\infty e^{(-\lambda_j u^*-\mu_j v^*)q}\mathrm{d}u^*\mathrm{d}v^* \end{aligned}$$

where we have made the change of variables

$$\begin{aligned} u^* &= \langle (u,v), P_{j+1}-P_j\rangle \\ v^* &= \langle (u,v), P_{j-1}-P_j\rangle \end{aligned}$$

and we have written

$$J_j=\begin{vmatrix} x_{j+1}-x_j & y_{j+1}-y_j \\ x_{j-1}-x_j & y_{j-1}-y_j \end{vmatrix}.$$

Clearly the last integral is not finite because $\lambda_j<0$ [resp. $\lambda_j=0$]. □

Let $q<\infty$, then it is easy to check that $\Delta(\overline{A})$ is dense in $\overline{A}_{(\alpha,\beta),q;J}$. We designate by $(\overline{A}_{(\alpha,\beta),q;J})'$ the subspace of $\Delta(\overline{A})^*$ that can be identified with the dual space $(\overline{A}_{(\alpha,\beta),q;J})^*$ of $\overline{A}_{(\alpha,\beta),q;J}$. If $\Delta(\overline{A})$ is dense in $\overline{A}_{(\alpha,\beta),q;K}$, we define $(\overline{A}_{(\alpha,\beta),q;K})'$ in a similar way. Note that in general $\Delta(\overline{A})$ is not dense in the K-space (see [**5**]).

Let us see some examples:

Example 1.2. If Π is equal to the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$ and $0<\alpha,\beta<1$, K- and J-spaces coincide with those studied by Fernandez in [**11**] for 4-tuples.

Example 1.3. If we take Π equal to the simplex $\{(0,0),(1,0),(0,1)\}$ and $\alpha>0,\beta>0$ with $\alpha+\beta<1$, then the resulting spaces are the same as those considered by Sparr in [**14**] for 3-tuples.

As we said in the Introduction, Section 3 is devoted to N-tuples of Banach lattices. Recently Asekritova and Krugljak [**1**] have proved that Sparr K- and J-spaces coincide on 3-tuples of function lattices. In fact, their result works for any function lattice N-tuple interpolated by Sparr K- and J-methods constructed by means of $N-1$ parameters (see [**14**]). However, let us point out that J- and K-spaces associated to

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polygons might be different even on those particular N-tuples if $N > 3$ (see [**8**, Example 1.25] or [**6**, Example 3.4]). We refer to [**6**, §3] for other results on relationships between general Sparr constructions and K- and J-method associated to polygons.

2. Duality between K- and J-spaces

We start describing the dual of the K-space:

Theorem 2.1. *Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon. Let $(\alpha, \beta) \in Int\,\Pi$, $1 \leq q < \infty$, and $\frac{1}{q} + \frac{1}{q'} = 1$. Assume that $\overline{A} = \{A_1, \dots, A_N\}$ is a regular N-tuple, and let $\overline{A'} = \{A'_1, \dots, A'_N\}$ be the dual N-tuple. Then the restriction of any continuous linear functional on $\overline{A}_{(\alpha,\beta),q;K}$ to $\Delta(\overline{A})$ regarded as an element of $\Sigma(\overline{A'})$ belongs to $\overline{A'}_{(\alpha,\beta),q';J}$. If $\Delta(\overline{A})$ is dense in $\overline{A}_{(\alpha,\beta),q;K}$, then we have the continuous inclusion (with norm less than or equal to one)*

$$(\overline{A}_{(\alpha,\beta),q;K})' \hookrightarrow \overline{A'}_{(\alpha,\beta),q';J},$$

and the two spaces coincide with equality of norms provided that $\Sigma(\overline{A})' = \Delta(\overline{A'})$.

Proof. For $(m, n) \in \mathbb{Z}^2$, let $F_{m,n}$ be the Banach space $\Sigma(\overline{A})$ normed by $K(2^m, 2^n; \cdot)$. Set

$$\begin{aligned} Y &= \ell_q(2^{-\alpha m-\beta n}F_{m,n}) \\ &= \left\{ (z_{m,n}) : \|(z_{m,n})\|_Y = \left(\sum_{(m,n)\in\mathbb{Z}^2} (2^{-\alpha m-\beta n}K(2^m, 2^n; z_{m,n}))^q \right)^{\frac{1}{q}} < \infty \right\} \end{aligned}$$

and let $\mathcal{D}$ be the diagonal of Y, that is to say, the collection of all constant sequences belonging to Y.

According to the definition of $\overline{A}_{(\alpha,\beta),q;K}$, this space is isometric to D. Hence

$$(\overline{A}_{(\alpha,\beta),q;K})^* = \mathcal{D}^* = \frac{Y^*}{\mathcal{D}^\perp}.$$

In order to describe the quotient, we denote by $H_{m,n}$ the space $\Sigma(\overline{A})'$ endowed with the norm $J(2^m, 2^n; \cdot)$ and we put $G_{m,n}$ for $\Delta(\overline{A'})$ normed by $J(2^m, 2^n; \cdot)$. By (1), we know that

$$F'_{m,n} = H_{-m,-n} \qquad \text{(equal norms).}$$

Take now

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$$f \in (\overline{A}_{(\alpha,\beta),q;K})^* = \frac{\ell_{q'}(2^{\alpha m+\beta n}F^*_{m,n})}{\mathcal{D}^{\perp}}.$$

Given any $\varepsilon > 0$, we can find a sequence $(f_{m,n}) \subset \Sigma(\overline{A})'$ such that

$$f(a) = \sum_{(m,n)\in\mathbf{Z}^2} f_{m,n}(a), \qquad a \in \Delta(\overline{A})$$

and

$$\left(\sum_{(m,n)\in\mathbf{Z}^2} (2^{\alpha m+\beta n} J(2^{-m}, 2^{-n}; f_{m,n}))^{q'}\right)^{\frac{1}{q'}} \le (1+\varepsilon)\|f\|_{(\overline{A}_{(\alpha,\beta),q;K})^*}.$$

It follows from the definition of the J-space that $f \in \overline{A'}_{(\alpha,\beta),q';J}$ with

$$\|f\|_{(\alpha,\beta),q';J} \le \|f\|_{(\overline{A}_{(\alpha,\beta),q;K})^*}.$$

Suppose next that $\Delta(\overline{A})$ is dense in $\overline{A}_{(\alpha,\beta),q;K}$, and let $(\overline{A}_{(\alpha,\beta),q;K})'$ be the subspace of $\Delta(\overline{A})^*$ that can be identified with $(\overline{A}_{\alpha,\beta,q;K})^*$. One has

$$\begin{aligned}(\overline{A}_{(\alpha,\beta),q;K})' &= \left\{f = \sum_{(m,n)\in\mathbf{Z}^2} f_{m,n} : (f_{m,n}) \in \ell_{q'}(2^{\alpha m+\beta n}H_{-m,-n})\right\}\\ &\hookrightarrow \left\{f = \sum_{(m,n)\in\mathbf{Z}^2} f_{m,n} : (f_{m,n}) \in \ell_{q'}(2^{\alpha m+\beta n}G_{-m,-n})\right\}\\ &= \overline{A'}_{(\alpha,\beta),q';J}.\end{aligned}$$

Finally, if also $\Sigma(\overline{A})' = \Delta(\overline{A'})$, then

$$H_{m,n} = G_{m,n}, \qquad (m,n) \in \mathbb{Z}^2,$$

and consequently

$$(\overline{A}_{(\alpha,\beta),q;K})' = \overline{A'}_{(\alpha,\beta),q';J}$$

with equality of norms. □

Next we establish a sufficient condition on the N-tuple $\overline{A}$ for $\Delta(\overline{A})$ to be dense in the K-space and for equality $\Sigma(\overline{A})' = \Delta(\overline{A'})$.

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Definition 2.2. We say that the N-tuple $\overline{A} = \{A_1, \ldots, A_N\}$ satisfies condition $(\mathcal{H})$ if there is a positive constant $C > 0$ such that for any $\varepsilon > 0$ and any finite sets $F_j \subset A_j$ there is a linear operator $P : \Sigma(\overline{A}) \longrightarrow \Delta(\overline{A})$ such that the restriction of P to each A_j defines a bounded operator from A_j into A_j with

$$\|P\|_{\mathcal{L}(A_j, A_j)} \le C \qquad (j = 1, \ldots, N)$$

and moreover

$$\|a_j - Pa_j\|_{A_j} < \varepsilon \qquad \text{for all } a_j \in F_j \qquad (j = 1, \ldots, N).$$

Note that $P : \Sigma(\overline{A}) \longrightarrow \Delta(\overline{A})$ is bounded. In fact, using the closed graph theorem, it is not hard to check that $P : A_j \longrightarrow \Delta(\overline{A})$ is bounded for $j = 1, \ldots, N$. Hence

$$\|P\|_{\mathcal{L}(\Sigma(\overline{A}), \Delta(\overline{A}))} \le \max_{1 \le j \le N} \left\{ \|P\|_{\mathcal{L}(A_j, \Delta(\overline{A}))} \right\} < \infty.$$

Theorem 2.3. *If $\overline{A} = \{A_1, \ldots, A_N\}$ is an N-tuple satisfying the condition $(\mathcal{H})$, then A is regular and $\Sigma(\overline{A})' = \Delta(\overline{A'})$. Furthermore, for any $(\alpha, \beta) \in \mathit{Int}\, \Pi$ and any $1 \le q < \infty$, the intersection $\Delta(\overline{A})$ is dense in $\overline{A}_{(\alpha,\beta),q;K}$.*

Proof. Regularity of $\overline{A}$ is clear since $Pa \in \Delta(\overline{A})$ for any $a \in \Sigma(\overline{A})$. In order to show equality $\Sigma(\overline{A})' = \Delta(\overline{A'})$ take $f \in \Delta(\overline{A'})$ and let f_j be its continuous extension to A_j. Given any $a_j \in A_j$ with $\sum_{j=1}^{N} a_j = 0$, we claim that

$$\sum_{j=1}^{N} f_j(a_j) = 0.$$

Indeed, according to condition $(\mathcal{H})$, we can find a sequence of operators $(P_r) \subset \mathcal{L}(\Sigma(\overline{A}), \Delta(\overline{A}))$ such that

$$\|P_r\|_{\mathcal{L}(A_j, A_j)} \le C \qquad \text{and} \qquad \lim_{r \to \infty} \|a_j - P_r a_j\|_{A_j} = 0 \qquad (j = 1, \ldots, N).$$

Whence,

$$\begin{aligned} \sum_{j=1}^{N} f_j(a_j) &= \lim_{r \to \infty} \sum_{j=1}^{N} f_j(P_r a_j) = \lim_{r \to \infty} \sum_{j=1}^{N} f(P_r a_j) \\ &= \lim_{r \to \infty} f\left(P_r \sum_{j=1}^{N} a_j \right) = \lim_{r \to \infty} f(0) = 0. \end{aligned}$$

Hence we can extend continuously f to $\Sigma(\overline{A})$ by putting

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$$\hat{f}\left(\sum_{j=1}^{N} a_j\right) = \sum_{j=1}^{N} f_j(a_j) \qquad \text{for} \qquad \sum_{j=1}^{N} a_j \in \Sigma(\overline{A}) \qquad \text{with } a_j \in A_j.$$

Therefore

$$\Sigma(\overline{A})' = \Delta(\overline{A'}).$$

Let us check the density of $\Delta(\overline{A})$ in $\overline{A}_{(\alpha,\beta),q;K}$.
Take any $a \in \overline{A}_{(\alpha,\beta),q;K}$ and $\varepsilon > 0$. We can find $M \in \mathbb{N}$ so that

$$\left(\sum_{(m,n)\notin[-M,M]^2} (2^{-\alpha m-\beta n}K(2^m, 2^n; a))^q\right)^{\frac{1}{q}} \leq \frac{\varepsilon}{2(1+C)}.$$

Using condition $(\mathcal{H})$, choose $P \in \mathcal{L}(\Sigma(\overline{A}), \Delta(\overline{A}))$ satisfying

$$\left(\sum_{(m,n)\in[-M,M]^2} (2^{-\alpha m-\beta n}K(2^m, 2^n; a-Pa))^q\right)^{\frac{1}{q}} \leq \frac{\varepsilon}{2}.$$

Then $Pa \in \Delta(\overline{A})$ and

$$\begin{aligned} \|a-Pa\|_{(\alpha,\beta),q;K} &\leq \left(\sum_{(m,n)\in[-M,M]^2} (2^{-\alpha m-\beta n}K(2^m, 2^n; a-Pa))^q\right)^{\frac{1}{q}} \\ &\quad + \left(\sum_{(m,n)\notin[-M,M]^2} (2^{-\alpha m-\beta n}K(2^m, 2^n; a-Pa))^q\right)^{\frac{1}{q}} \\ &\leq \frac{\varepsilon}{2} + (1+C)\left(\sum_{(m,n)\notin[-M,M]^2} (2^{-\alpha m-\beta n}K(2^m, 2^n; a))^q\right)^{\frac{1}{q}} \leq \varepsilon. \end{aligned}$$

The proof is complete. □

So

$$(\overline{A}_{(\alpha,\beta),q;K})' = \overline{A'}_{(\alpha,\beta),q';J}$$

if $\overline{A}$ satisfies condition $(\mathcal{H})$. Next we show that under this assumption the dual of the J-space coincides with a K-space.

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Theorem 2.4. *Let $\overline{A} = \{A_1, \ldots, A_N\}$ be a Banach N-tuple satisfying the condition $(\mathcal{H})$. Then for $1 \leq q < \infty$*

$$(\overline{A}_{(\alpha,\beta),q;J})' = \overline{A'}_{(\alpha,\beta),q';K}$$

with equal norms.

Proof. For $m, n \in \mathbb{Z}^2$, let $G_{m,n}$ be the space $\Delta(\overline{A})$ normed by $J(2^m, 2^n; \cdot)$. We know from [**4**, Thm. 2.3], that $(\overline{A}_{(\alpha,\beta),q;J})'$ consists of all those $f \in \overline{A'}_{(\alpha,\beta),q';K}$ satisfying the following property:

$$\begin{aligned}&\text{Whenever } (u_{m,n}) \in \ell_q(2^{-\alpha m-\beta n}G_{m,n}) \text{ with } \sum_{(m,n)\in\mathbf{Z}^2} u_{m,n} = 0 \text{ (convergence in } \Sigma(\overline{A})),\\ &\text{then } \sum_{(m,n)\in\mathbf{Z}^2} f(u_{m,n}) = 0.\end{aligned} \tag{2}$$

Since $\overline{A'}_{(\alpha,\beta),q';K}$ may contain functionals that do not belong to $\Sigma(\overline{A})'$, in general

$$(\overline{A}_{(\alpha,\beta),q;J})' \subsetneqq \overline{A'}_{(\alpha,\beta),q';K}$$

but the norms coincide in $(\overline{A}_{(\alpha,\beta),q;J})'$. Our task is to establish that under the assumption on $\overline{A}$, all elements of $\overline{A'}_{(\alpha,\beta),q';K}$ satisfy (2).

Take any $(u_{m,n}) \in \ell_q(2^{-\alpha m-\beta n}G_{m,n})$ with $\sum_{(m,n)\in\mathbf{Z}^2} u_{m,n} = 0$ (convergence in $\Sigma(\overline{A})$), and let $f \in \overline{A'}_{(\alpha,\beta),q';K}$. Then

$$\begin{aligned}\sum_{(m,n)\in\mathbf{Z}^2} J(2^m, 2^n; u_{m,n})K(2^{-m}, 2^{-n}; f) &\leq \sum_{(m,n)\in\mathbf{Z}^2} 2^{-\alpha m-\beta n}J(2^m, 2^n; u_{m,n})2^{\alpha m+\beta n}K(2^{-m}, 2^{-n}; f)\\ &\leq \left(\sum_{(m,n)\in\mathbf{Z}^2} (2^{-\alpha m-\beta n}J(2^m, 2^n; u_{m,n}))^q\right)^{\frac{1}{q}} \|f\|_{\overline{A'}_{(\alpha,\beta),q';K}} < \infty.\end{aligned}$$

Hence, for any $\varepsilon > 0$, we can find $M \in \mathbb{N}$ such that

$$\sum_{(m,n)\notin[-M,M]^2} J(2^m, 2^n; u_{m,n})K(2^{-m}, 2^{-n}; f) \leq \frac{\varepsilon}{2(1+C)}.$$

Using condition $(\mathcal{H})$ we can now choose $P \in \mathcal{L}(\Sigma(\overline{A}), \Delta(A))$ so that

$$\|P\|_{\mathcal{L}(A_j,A_j)} \leq C \qquad (j = 1, \ldots, N)$$

and

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$$\sum_{(m,n)\in[-M,M]^2} \|u_{m,n} - P(u_{m,n})\|_{\Delta(\bar{A})} \le \frac{\varepsilon}{2\|f\|_{\Delta(\bar{A})^*}}$$

(if $\|f\|_{\Delta(\bar{A})^*} = 0$ then (2) is clearly satisfied). Therefore we have

$$\begin{aligned}
\left|\sum_{(m,n)\in\mathbf{Z}^2} f(u_{m,n})\right| &= \left|\sum_{(m,n)\in\mathbf{Z}^2} f(u_{m,n}) - f\left(P\left(\sum_{(m,n)\in\mathbf{Z}^2} u_{m,n}\right)\right)\right| \\
&= \left|\sum_{(m,n)\in\mathbf{Z}^2} f(u_{m,n}) - \sum_{(m,n)\in\mathbf{Z}^2} f(P(u_{m,n}))\right| \\
&\le \sum_{(m,n)\notin[-M,M]^2} |f(I-P)(u_{m,n})| \\
&\quad + \sum_{(m,n)\in[-M,M]^2} |f(u_{m,n} - P(u_{m,n}))| \\
&\le (1+C) \sum_{(m,n)\notin[-M,M]^2} J(2^m, 2^n; u_{m,n}) K(2^{-m}, 2^{-n}; f) \\
&\quad + \sum_{(m,n)\in[-M,M]^2} \|f\|_{\Delta(\bar{A})^*} \|u_{m,n} - P(u_{m,n})\|_{\Delta(\bar{A})} \\
&\le (1+C)\frac{\varepsilon}{2(1+C)} + \|f\|_{\Delta(\bar{A})^*} \frac{\varepsilon}{2\|f\|_{\Delta(\bar{A})^*}} = \varepsilon.
\end{aligned}$$

This establishes (2) and completes the proof. □

In the case of Banach couples, a similar condition to $(\mathcal{H})$ was used for other purposes by A. Persson [13] (see also the paper by Edmunds and Teixeira [15]).

Doing minor changes in the arguments given by A. Persson [13], it is not hard to check that if X is a locally compact space endowed with a positive measure μ and $1 \le p_1, p_2, \ldots, p_N < \infty$, then the N-tuple

$$\{L_{p_1}(X,\mu), \ldots, L_{p_N}(X,\mu)\}$$

satisfies condition $(\mathcal{H})$.

A stronger condition than $(\mathcal{H})$ has been used by Favini [10] to study duality for an interpolation method that extends the classical complex method to N-tuples.

3. The case of Banach lattices

We introduce first some notations referring to Banach lattices (see, for example [12]).

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Recall that given a Riesz space $\mathcal{Z}$, an order ideal of $\mathcal{Z}$ is simply a linear subspace Y which is hereditary for the order of $\mathcal{Z}$, i.e.,

$$y \in Y, z \in \mathcal{Z} \text{ and } |z| \leq |y| \text{ imply } z \in Y.$$

In particular, Y is a sublattice of $\mathcal{Z}$.

By a *Banach lattice N-tuple* we mean an N-tuple $\overline{X} = \{X_1, \ldots, X_N\}$ of Banach lattices which are order ideals of a common topological Riesz space $\mathcal{Z}$, with continuous inclusions. (Equivalently we can define order preserving, continuous injections j_k from X_k into a common topological Riesz space $\mathcal{Z}$ such that $j_k(X_k)$ are order ideals of $\mathcal{Z}$).

A Banach lattice N-tuple is in particular a Banach N-tuple. Note that in this case the intersection $\Delta(\overline{X})$ and the sum $\Sigma(\overline{X})$ are also order ideals of the ambient space $\mathcal{Z}$. Moreover if $\Delta(\overline{X})$ is dense in each X_k for $k = 1, \ldots, N$, then the conjugate i_k^* of the inclusion $i_k : \Delta(\overline{X}) \longrightarrow X_k$ defines an order preserving inclusion of X_k^* into the Banach lattice $\Delta(\overline{X})^*$, and the images $X_k' = i_k^*(X_k^*)$ are order ideals of $\Delta(\overline{X})^*$. Hence $\overline{X'} = \{X_1', \ldots, X_N'\}$ has a structure of Banach lattice N-tuple.

The most common situation is the case where $\mathcal{Z} = L_0(\Omega, \mathcal{A}, \mu)$ the space of all (classes of) measurable functions relative to a measure space $(\Omega, \mathcal{A}, \mu)$, equipped with topology of local convergence in measure. Banach lattices which are order dense order ideals of L_0 are called Köthe function spaces. By Vulikh-Lozanovskii theory ([**16**]) their duals are representable as generalized Köthe function spaces (i.e., Banach lattices which are L_0-order ideals without the order density assumption). In this case we talk about *Köthe N-tuples* instead of Banach lattice N-tuples.

Next we show that Banach lattices N-tuples enjoy certain properties that general Banach N-tuples fail.

Proposition 3.1. *For every regular Banach lattice N-tuple $\overline{X}$, the equality $\Sigma(\overline{X})' = \Delta(\overline{X'})$ holds.*

Proof. According to (1) in Section 1, it suffices to prove that if an element f of $\Delta(\overline{X})^*$ is continuous for each norm $\|\cdot\|_{X_j}$, $j = 1, \ldots, N$, then it is continuous for the norm of $\Sigma(\overline{X})$ (by density of $\Delta(\overline{X})$ in $\Sigma(\overline{X})$, f is then the restriction of a unique element f of $\Sigma(\overline{X})^*$, which proves the inclusion $\Delta(\overline{X'}) \subseteq \Sigma(\overline{X})'$).

To this end, recall the following well-known fact: For every $x \geq 0$ in $\Sigma(\overline{X})$, its norm in $\Sigma(\overline{X})$ is given by

$$\|x\|_{\Sigma(\overline{X})} = \inf\left\{\sum_{j=1}^{N} \|x_j\|_{X_j} : x = \sum_{j=1}^{N} x_j, x_j \in X_j, x_j \geq 0\right\}.$$

This formula can be easily proved by induction on N. (If $N = 2$ and $x = x_1 + x_2$ then $x = x_1' + x_2'$ with $x_j' = (x_j \vee 0) \wedge x$ and $\|x_j'\|_{X_j} \leq \|x_j\|_{X}$.)

Using this fact we derive for any $x \in \Sigma(\overline{X})$, by decomposing x into positive and negative parts, that

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$$\|x\|_{\Sigma(\overline{X})} = \inf\left\{\sum_{j=1}^{N} \|x_j\|_{X_j} : x = \sum_{j=1}^{N} x_j,\, x_j \in X_j,\, |x_j| \le |x|\right\}.$$

In particular, if $x \in \Delta(\overline{X})$ we obtain

$$\|x\|_{\Sigma(\overline{X})} = \inf\left\{\sum_{j=1}^{N} \|x_j\|_{X_j} : x = \sum_{j=1}^{N} x_j,\, x_j \in \Delta(\overline{X})\right\}.$$

Hence if $f \in \Delta(\overline{X})^*$ is bounded for the norm of X_j, $j = 1, \ldots, N$ and $x \in \Delta(\overline{X})$ we have

$$\begin{aligned} |f(x)| &\le \inf\left\{\sum_{j=1}^{N} |f(x_j)| : x = \sum_{j=1}^{N} x_j,\, x_j \in \Delta(\overline{X})\right\} \\ &\le \max_{1 \le j \le N}\left\{\|f_j\|_{X_j^*}\right\}\|x\|_{\Sigma(\overline{X})} \end{aligned}$$

i.e., f has norm less or equal than $\max_{1\le j\le N}\{\|f_j\|_{X_j^*}\}$ with respect to the norm $\|\cdot\|_{\Sigma(\overline{X})}$. □

Proposition 3.2. *Let* $\Pi = \overline{P_1 \ldots P_N}$ *be a convex polygon with vertices* $P_j = (x_j, y_j)$, *let* $(\alpha, \beta) \in \operatorname{Int} \Pi$ *and* $1 \le q < \infty$. *If* $\overline{X}$ *is a regular Banach lattice N-tuple, then* $\Delta(\overline{X})$ *is dense in* $\overline{X}_{(\alpha,\beta),q;K}$.

Proof. Let $f \in \overline{X}_{(\alpha,\beta),q;K}$ and $\varepsilon > 0$. Choose a natural number M such that

$$\sum_{(m,n)\notin[-M,M]^2} (2^{-\alpha m-\beta n} K(2^m, 2^n; f))^q < \varepsilon^q.$$

Besides, we claim that for every $\delta > \infty$ there exists $h \in \Delta(\overline{X})$ so that

$$\begin{aligned} &\text{(i)} && |h| \le |f|, \\ &\text{(ii)} && \|f - h\|_{\Sigma(\overline{X})} \le \delta. \end{aligned}$$

Indeed, by density of $\Delta(\overline{X})$ in $\Sigma(\overline{X})$ we can find $h_0 \in \Delta(\overline{X})$ satisfying condition (ii). Then $h = (h_0 \wedge |f|) \vee (-|f|)$ fulfils the desired conditions.

For such an h, we have

$$K(2^m, 2^n; f - h) \le \delta \sup_{1\le j\le N} \{2^{mx_j + ny_j}\}$$

hence

$$\sum_{(m,n)\in[-M,M]^2} (2^{-\alpha m-\beta n} K(2^m, 2^n; f - h))^q \le C(M)^q \delta^q$$

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with

$$C(M)=\left(\sum_{(m,n)\in[-M,M]^2}\left(2^{-\alpha m-\beta n}\sup_{1\le j\le N}\{2^{mx_j+ny_j}\}\right)^q\right)^{\frac{1}{q}},$$

while

$$\left(\sum_{(m,n)\notin[-M,M]^2}(2^{-\alpha m-\beta n}K(2^m,2^n;f-h))^q\right)^{\frac{1}{q}}\le\left(\sum_{(m,n)\notin[-M,M]^2}(2^{-\alpha m-\beta n}K(2^m,2^n;f))^q\right)^{\frac{1}{q}}$$
$$+\left(\sum_{(m,n)\notin[-M,M]^2}(2^{-\alpha m-\beta n}K(2^m,2^n;h))^q\right)^{\frac{1}{q}}\le 2\varepsilon.$$

The choice $\delta<\frac{\varepsilon}{C(M)}$ now shows that

$$\|f-h\|^q_{(\alpha,\beta),q;K}\le 2^q\varepsilon^q+C(M)^q\delta^q\le(2^q+1)\varepsilon^q.$$

The proof is complete. □

From Theorem 2.1, Propositions 3.1 and 3.2 we get

Corollary 3.3. *Let* Π *be a convex polygon, let* $(\alpha,\beta)\in Int\,\Pi$, $1\le q<\infty$ *and* $\frac{1}{q}+\frac{1}{q'}=1$. *If* $\overline{X}$ *is any regular Banach lattice N-tuple, then*

$$(\overline{X}_{(\alpha,\beta),q;K})'=\overline{X'}_{(\alpha,\beta),q';J}.$$

Next we focus our attention on the dual of the J-space.

Theorem 3.4. *Let* $\Pi=\overline{P_1\ldots P_N}$ *be a convex polygon with vertices* $P_j=(x_j,y_j)$, *let* $(\alpha,\beta)\in Int\,\Pi$, $1\le q<\infty$ *and* $\frac{1}{q}+\frac{1}{q'}=1$. *If* $\overline{X}=\{X_1,\ldots,X_N\}$ *is a regular Banach lattice N-tuple, then*

$$(\overline{X}_{(\alpha,\beta),q;J})'=\overline{X'}_{(\alpha,\beta),q';K}.$$

According to [**4**, Thm. 2.3] (see the beginning of the proof of Theorem 2.4 above) we know that

$$(\overline{X}_{(\alpha,\beta),q;J})'\subseteq\overline{X'}_{(\alpha,\beta),q';K}$$

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with equality of norms, so we have only to prove the reverse inclusion. We first establish two auxiliary results.

Lemma 3.5. *The space* $(\overline{X}_{(\alpha,\beta),q;J})'$ *contains the band generated by* $\Delta(\overline{X'})$ *in* $\overline{X'}_{(\alpha,\beta),q';K}$.

Proof. Clearly $\Delta(\overline{X'}) \subseteq \overline{X'}_{(\alpha,\beta),q';K}$ and whenever $(u_{m,n}) \subseteq \Delta(\overline{X})$ with

$$\sum_{(m,n)\in\mathbf{Z}^2} u_{m,n} = 0 \qquad (\text{convergence in } \Sigma(\overline{X}))$$

then for every $f \in \Delta(\overline{X'}) = \Sigma(\overline{X})'$ it follows that

$$\sum_{(m,n)\in\mathbf{Z}^2} f(u_{m,n}) = f\left(\sum_{(m,n)\in\mathbf{Z}^2} u_{m,n}\right) = 0.$$

Hence, by [**4**, Thm. 2.3], $\Delta(\overline{X'}) \subseteq (\overline{X}_{(\alpha,\beta),q;J})'$. Moreover, the spaces $X'_1, \dots, X'_N$, and $(\overline{X}_{(\alpha,\beta),q;J})'$ are order ideals in $\Delta(\overline{X})'$, and their identification with the dual spaces $X^*_1, \dots X^*_N$ and $(\overline{X}_{(\alpha,\beta),q;J})^*$ conserves the lattice operations, as well as arbitrary suprema or infima. In other words, this isomorphism is order continuous. Since dual Banach lattices are monotonically complete (i.e., norm bounded upward directed subsets have a supremum) so does in particular $(\overline{X}_{(\alpha,\beta),q;J})'$. But this last space is clearly a closed order ideal of $\overline{X'}_{(\alpha,\beta),q';K}$, therefore it must be a band of $\overline{X'}_{(\alpha,\beta),q';K}$. □

Lemma 3.6. *Let* $\tilde{\Pi}$ *be any triangle, let* $(\alpha,\beta) \in Int\,\tilde{\Pi}$ *and* $1 \le p \le \infty$. *If* $\overline{Y} = \{Y_1, Y_2, Y_3\}$ *is a Banach lattice triple, then* $\Delta(\overline{Y})$ *is order dense in* $\overline{Y}_{(\alpha,\beta),p;K}$. *(In other words, the band generated by* $\Delta(\overline{Y})$ *in* $\overline{Y}_{(\alpha,\beta),p;K}$ *is the whole space.)*

(Note that we do not assume that $\Delta(\overline{Y})$ is order dense in each $Y_j, j = 1, 2, 3$.)

Proof. Let $\mathcal{Z}$ be the ambient Riesz space containing the lattices Y_j, $j = 1, 2, 3$ as order ideals. We have to prove that the space $\Delta(\overline{Y})^{\perp}$ consisting of those elements in $\overline{Y}_{(\alpha,\beta),p;K}$ which are disjoint from $\Delta(\overline{Y})$ reduces to $\{0\}$. So suppose that $z \in \mathcal{Z}$ is some non-negative, non-zero element of $\Sigma(\overline{Y})$, which is disjoint from $\Delta(\overline{Y})$. Then there is some $z' \in \Sigma(\overline{Y})$, $0 \le z' \le z$, $z' \ne 0$ which is disjoint from one of the spaces Y_j, $j = 1, 2, 3$. For, if not, for every $j = 1, 2, 3$ and $0 \le z' \le z$, $z' \ne 0$ there exists $0 \le z''_j \le z'$, $z''_j \ne 0$ which belongs to Y_j. Reasoning inductively, we find $z''_1 \ge z''_2 \ge z''_3 \gneq 0$. Whence $z''_3 \in \Delta(\overline{Y})$, $0 \le z''_3 \le z$ and $z''_3 \ne 0$, a contradiction.

To complete the proof it suffices to show that $z' \notin \overline{Y}_{(\alpha,\beta),p;K}$. In order to fix the ideas, suppose that $z = z' \perp Y_1$. Then $z \in Y_2 + Y_3$ (recall that in the decomposition $z = \sum_{j=1}^3 z_j$, $z_j \in Y_j$ we may assume that $0 \le z_j \le z$; hence $z_1 \le z$ and thus $z_1 = 0$). Moreover

$$K(2^m, 2^n; z) = \inf\{2^m \|z_2\|_{Y_2} + 2^n \|z_3\|_{Y_3} : z = z_2 + z_3, z_j \in Y_j, z_j \ge 0\}.$$

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That is to say, the K-functional of z relative to the triple $\{Y_1, Y_2, Y_3\}$ coincide with that relative to the triple $\{\{0\}, Y_2, Y_3\}$. But a simple calculation shows that if one of the spaces A_1, A_2, A_3 is zero, then

$$\overline{A}_{(\alpha,\beta),p;K} = \{0\}$$

for every $p \in [1, \infty]$ and $(\alpha, \beta) \in \operatorname{Int} \tilde{\Pi}$ (see also Lemma 1.1). This gives the result. $\square$

Now we are in a position to give

Proof of Theorem 3.4. We start with the case $N = 3$ (Sparr's case). As we have seen in Lemma 3.6 the band generated by $\Delta(\overline{X'})$ in $\overline{X'}_{(\alpha,\beta),q';K}$ is equal to $\overline{X'}_{(\alpha,\beta),q';K}$. Hence, using Lemma 3.5, we conclude the desired equality

$$(\overline{X}_{(\alpha,\beta),q;J})' = \overline{X'}_{(\alpha,\beta),q';K}.$$

For the general case we will proceed by induction on $N \geq 4$.

In the space $\overline{X'}_{(\alpha,\beta),q';K}$ consider the band V_j consisting of all those elements which are disjoint (in the lattice $Z = \Delta(\overline{X})^*$) from the space X'_j, and let W be the band generated by $\Delta(\overline{X'})$. We claim that $W + \sum_{j=1}^{N} V_j$ generates $\overline{X'}_{(\alpha,\beta),q';K}$ as a band.

Indeed,

$$\left(W + \sum_{j=1}^{N} V_j\right)^{\perp} = W^{\perp} \bigcap \cap_{j=1}^{N} V_j^{\perp} = \Delta(\overline{X'})^{\perp} \bigcap \cap_{j=1}^{N} V_j^{\perp},$$

and the same reasoning as in the beginning of the proof of Lemma 3.6 shows that this intersection reduces to zero.

By Lemma 3.5 we know that $W \subseteq (\overline{X}_{(\alpha,\beta),q;J})'$, so to complete the proof it is enough to show that each band V_j is also included in $(\overline{X}_{(\alpha,\beta),q;J})'$.

The argument given at the end of the proof of Lemma 3.6 shows that

$$V_j \subseteq \overline{Y}_{(\alpha,\beta),q';K} \quad \text{where} \quad \overline{Y} = \{Y_1, \ldots, Y_N\} \quad \text{with} \quad Y_k = \begin{cases} X'_k & \text{if } k \neq j \\ \{0\} & \text{if } k = j. \end{cases}$$

This interpolation space may also be viewed as the K-interpolation space of the $(N-1)$-tuple $\overline{X^{(j)}} = \{X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_N\}$ relative to the polygon $\Pi^{(j)} = \overline{P_1, \ldots, P_{j-1}, P_{j+1}, \ldots, P_N}$, where the P_k's are the vertices of the original polygon Π. Whence, by Lemma 1.1, we may have $V_j \neq \{0\}$ only if $(\alpha, \beta) \in \operatorname{Int} \Pi^{(j)}$ in the case $1 \leq q' < \infty$ [resp. $(\alpha, \beta) \in \Pi^{(j)}$ in the case $q' = \infty$].

Let us first consider the case $(\alpha, \beta) \in \operatorname{Int} \Pi^{(j)}$. According to the induction hypothesis

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$$\left(\overline{X'^{(j)}}\right)^{\Pi^{(j)}}_{(\alpha,\beta),q';K} = \left[\left(\overline{X^{(j)}}\right)^{\Pi^{(j)}}_{(\alpha,\beta),q;J}\right]'.$$

Thus

$$V_j \subseteq \left[\left(\overline{X^{(j)}}\right)^{\Pi^{(j)}}_{(\alpha,\beta),q;J}\right]'. \tag{3}$$

On the other hand, the natural inclusion $\Delta(\overline{X}) \subseteq \Delta(\overline{X^{(j)}})$ extends to an injection of norm less than or equal to 1

$$\overline{X}^{\Pi}_{(\alpha,\beta),q;J} \hookrightarrow \left(\overline{X^{(j)}}\right)^{\Pi^{(j)}}_{(\alpha,\beta),q;J}. \tag{4}$$

The reason is that for any $u \in \Delta(\overline{X})$ we have

$$J(t, s; u; \overline{X^{(j)}}) \le J(t, s; u; \overline{X}) \qquad \text{for every } t, s > 0.$$

So, putting

$$G_{m,n} = (\Delta(\overline{X}), J(2^m, 2^n; \cdot; \overline{X})) \quad \text{and} \quad G^{(j)}_{m,n} = \left(\Delta(\overline{X^{(j)}}), J(2^m, 2^n; \cdot; \overline{X^{(j)}})\right), (m, n) \in \mathbb{Z}^2,$$

it follows that the vector valued sequence space $\ell_q(2^{-\alpha m-\beta n}G_{m,n})$ is naturally included with norm ≤ 1 into $\ell_q(2^{-\alpha m-\beta n}G^{(j)}_{m,n})$. Therefore if $x = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n}$ is any J-representation of $x \in \overline{X}^{\Pi}_{(\alpha,\beta),q;J}$ with $(u_{m,n}) \in \ell_q(2^{-\alpha m-\beta n}G_{m,n})$, then it is also a representation of x in $\left(\overline{X^{(j)}}\right)^{\Pi^{(j)}}_{(\alpha,\beta),q;J}$ because $(u_{m,n}) \in \ell_q(2^{-\alpha m-\beta n}G^{(j)}_{m,n})$. This establishes inclusion (4) and shows that its norm is ≤ 1.

Dualizing (4) we obtain that

$$\left[\left(\overline{X^{(j)}}\right)^{\Pi^{(j)}}_{(\alpha,\beta),q;J}\right]' \subseteq \left(\overline{X}_{\Pi_{(\alpha,\beta,),q;J}}\right)' \tag{5}$$

with the inclusion having norm ≤ 1.

Combining (3) and (5) we get

$$V_j \subseteq \left(\overline{X}^{\Pi}_{(\alpha,\beta,),q;J}\right)'$$

which completes the proof in this case.

Let us consider now the case where $q' = \infty$ and $(\alpha, \beta) \in \Pi^{(j)} \backslash \operatorname{Int} \Pi^{(j)} = \partial\Pi^{(j)}$. This means that (α, β) belongs to the diagonal $[P_{j-1}, P_{j+1}]$ of the polygon Π.

In the present situation, a non zero element of V_j cannot be disjoint from the space X'_{j-1} nor from X'_{j+1}, because, using Lemma 1.1 again, this would imply that (α, β) belongs to the polygon $\Pi^{(j-1,j)} = \overline{P_1, \ldots, P_{j-2}, P_{j+1}, \ldots, P_N}$ (resp. to $\Pi^{(j,j+1)}$), which

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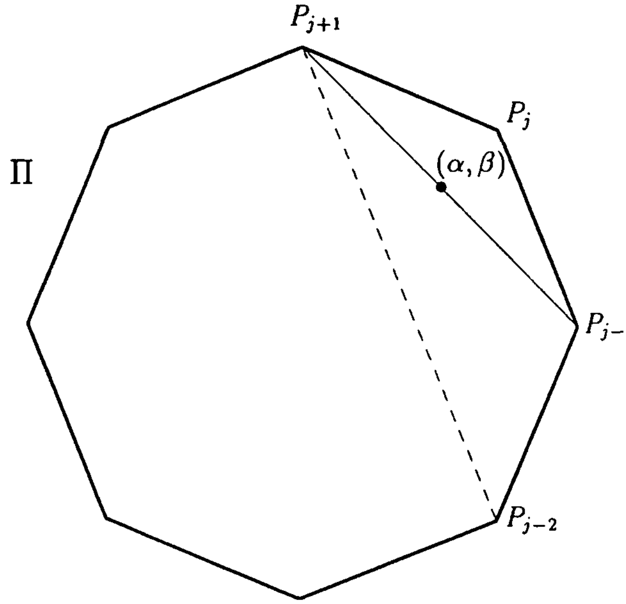

FIGURE 3.1

is impossible by convexity of the original polygon Π (see Fig. 3.1). Hence $X'_{j-1} \cap X'_{j+1} \cap V_j$ is order dense in V_j.

It remains to show that $X'_{j-1} \cap X'_{j+1} \cap V_j \subseteq \overline{X}'_{(\alpha,\beta),1;J}$. Note first that if $(u_{m,n}) \in \ell_1\{2^{-\alpha m-\beta n}G_{m,n}\}$ with $\sum_{(m,n)\in\mathbb{Z}^2} u_{m,n} = 0$ in $\Sigma(\overline{X})$, then the series also converges to zero in $X_{j-1} + X_{j+1}$. Indeed, put

$$\begin{aligned} M &= \sum_{(m,n)\in\mathbb{Z}^2} 2^{-\alpha m-\beta n} \max\{2^{mx_{j-1}+ny_{j-1}} \|u_{m,n}\|_{X_{j-1}}, 2^{mx_{j+1}+ny_{j+1}} \|u_{m,n}\|_{X_{j+1}}\} \\ &\le \sum_{(m,n)\in\mathbb{Z}^2} 2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n}; \overline{X}) < \infty. \end{aligned}$$

Let

$$\bar{v} = (x_{j-1} - \alpha, y_{j-1} - \beta) \quad \text{and} \quad \bar{w} = (x_{j+1} - \alpha, y_{j+1} - \beta).$$

Since these vectors have opposite directions, the half planes

$$\mathcal{P}_+ = \{\bar{\sigma} \in \mathbb{R}^2 : \bar{\sigma} \cdot \bar{v} \ge 0\} \quad \text{and} \quad \mathcal{P}_- = \{\bar{\sigma} \in \mathbb{R}^2 : \bar{\sigma} \cdot \bar{w} > 0\}$$

form a partition of $\mathbb{R}^2$. Hence

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$$\sum_{(m,n)\in\mathbf{Z}^2} \|u_{m,n}\|_{X_{j-1}+X_{j+1}} \le \sum_{(m,n)\in\mathbf{Z}^2\cap\mathcal{P}_+} \|u_{m,n}\|_{X_{j-1}} + \sum_{(m,n)\in\mathbf{Z}^2\cap\mathcal{P}_-} \|u_{m,n}\|_{X_{j+1}} \le M$$

and so $\sum_{(m,n)\in\mathbf{Z}^2} u_{m,n}$ converges to 0 in $X_{j-1} + X_{j+1}$.

Take now any $f \in X'_{j-1} \cap X'_{j+1} \cap V_j \subseteq X'_{j-1} \cap X'_{j+1} \cap \overline{X'}_{(\alpha,\beta),\infty;K}$ and any $(u_{m,n}) \in \ell_1(2^{-\alpha m-\beta n}G_{m,n})$ such that $\sum_{(m,n)\in\mathbf{Z}^2} u_{m,n} = 0$ in $\Sigma(\overline{X})$. By our previous observation we have

$$\sum_{(m,n)\in\mathbf{Z}^2} f(u_{m,n}) = f\left(\sum_{(m,n)\in\mathbf{Z}^2} u_{m,n}\right) = 0.$$

Hence according to [**4**, Thm. 2.3], we conclude that $f \in (\overline{X}_{(\alpha,\beta),1;J})'$.

This finishes the proof of Theorem 3.4. □

Remark 3.7. As we pointed out in Section 1, if $\overline{X}$ is a Köthe triple then (see [**1**])

$$\overline{X}_{(\alpha,\beta),q;J} = \overline{X}_{(\alpha,\beta),q;K}$$

(with equivalent norms). For such a triple, $\Delta(\overline{X})$ turns out to be not only order dense in $\overline{X}_{(\alpha,\beta),p;K}$ (see Lemma 3.6), but norm dense provided $p < \infty$.

Remark 3.8. It turns out to be possible to describe the dual $(\overline{X}_{(\alpha,\beta),q;K})^*$ even in the case of a non regular Banach lattice N-tuple (generalizing Corollary 3.3). This will be done elsewhere.

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