A General Theorem on the Nine-points Circle.

By V. RAMASWAMI AIYAR, M.A.

Theorem: If any conic be inscribed in a given triangle and a confocal to it pass through the circumcentre, then the circle through the intersection of these two confocals touches the nine-points circle of the triangle.

Demonstration: Let $X$ (Fig. 10) be any conic inscribed in the triangle $ABC$; $O$, $H$, $N$ its circumcentre, orthocentre and nine-points centre; let $R$ be the circumradius.

Let $X$ be any conic inscribed in the triangle $ABC$; $P$, $Q$ its foci; $M$ its centre; and $a$, $b$ its semi-axes.

Let $Y$ be a confocal to $X$ passing through the circumcentre $O$; and let $\rho$ be the radius of the circle through the intersections of $X$ and $Y$. We have to show that this circle touches the nine-points circle of $ABC$.

This will be proved if we show that $\rho = \frac{1}{2}R \pm MN$. This can be shown with the aid of the following propositions:

Lemma I. The circle passing through the intersections of the confocals

$$x^2 + y^2 = 1$$

and

$$x^2/(a^2 + \lambda) + y^2/(b^2 + \lambda) = 1$$

is

$$x^2 + y^2 = a^2 + b^2 + \lambda;$$

this circle is the mutual orthoptic circle of the two confocals.

Lemma II. If $P$ and $Q$ be the foci of any conic $X$ inscribed in a triangle $ABC$ we have

$$(R^2 - OP^2)(R^2 - OQ^2) = 4\beta^2 R^2.$$ [Professor Genese, Educational Times, Q. 10879; for a solution see p. 37, Vol. 57 of the Mathematical Reprints.]

Lemma III. Any conic $X$ being inscribed in a triangle $ABC$ its director circle cuts the polar circle of the triangle orthogonally. The centre of the polar circle is the orthocentre $H$ and the square of its radius

$$= -\frac{1}{4}(R^2 - OH^2).$$

Now by lemma I. applied to the confocals $X$ and $Y$ we have

$$\rho^2 = \beta^2 + \left(\frac{OP \pm OQ}{2}\right)^2$$

$$= \frac{1}{2}(OP^2 + OA^2 + 4\beta^2) \pm \frac{1}{2}OP \cdot OQ\ldots (1)$$
Lemma II. gives

\[ R^4 - R^2(\text{OP}^2 + \text{OQ}^2 + 4\beta^2) + \text{OP}^2 \cdot \text{OQ}^2 = 0 \]  

\[ \text{(2)} \]

In (1) and (2) the expression \( \text{OP}^2 + \text{OQ}^2 + 4\beta^2 \) occurs; this is readily seen to be equal to \( 2(a^2 + \beta^2 + \text{OM}^2) \)  

\[ \text{(3)} \]

Again by lemma III. we have

\[ (a^2 + \beta^2) - \frac{1}{2}(R^2 - \text{OH}^2) = \text{MH}^2; \]

\[ \therefore \quad a^2 + \beta^2 + \text{OM}^2 = \text{OM}^2 + \text{MH}^2 + \frac{1}{2}(R^2 - \text{OH}^2) = \frac{1}{2}R^2 + 2\text{MN}^2 \]  

\[ \text{(4)} \]

By (3) and (4) we have

\[ \text{OP}^2 + \text{OQ}^2 + 4\beta^2 = R^2 + 4\text{MN}^2 \]  

\[ \text{(5)} \]

Using this in (2) we get a pretty simple result

\[ \text{OP} \cdot \text{OQ} = 2R \cdot \text{MN} \]  

\[ \text{(6)} \]

Now making use of (5) and (6) in equation (1) we get

\[ \rho^2 = \frac{1}{4}R^2 + \text{MN}^2 \pm R \cdot \text{MN} \]

\[ = (\frac{1}{2}R \pm \text{MN})^2 \]

\[ \therefore \quad \rho = \frac{1}{2}R \pm \text{MN}; \text{ and the theorem is proved.} \]

Corollary.—A beautiful theorem, due to Mr M'Cay, of which Feuerbach's theorem is a particular case, is itself a particular case of the theorem now given; Mr M'Cay's theorem may be thus stated: "If either axis of a conic inscribed in a given triangle pass through the circumcentre, then the corresponding auxiliary circle of the conic touches the nine-points circle of the triangle." [See Casey's Conics, 2nd Edition, p. 329.]

On the Geometrical Representation of Elliptic Integrals of the First Kind.

By Alex. Morgan, M.A., B.Sc.

[See page 2 of present volume.]

Dr T. B. Sprague, M.A., F.R.S.E., was elected President in room of the Rev. John Wilson, deceased.