A General Theorem on the Nine-points Circle.

By V. RAMASWAMI AIYAR, M.A.

Theorem: If any conic be inscribed in a given triangle and a confocal to it pass through the circumcentre, then the circle through the intersection of these two confocals touches the nine-points circle of the triangle.

Demonstration: Let X (Fig. 10) be any conic inscribed in the triangle ABC; O, H, N its circumcentre, orthocentre and nine-points centre; let R be the circumradius.

Let X be any conic inscribed in the triangle ABC; P, Q its foci; M its centre; and α, β its semi-axes.

Let Y be a confocal to X passing through the circumcentre O; and let ρ be the radius of the circle through the intersections of X and Y. We have to show that this circle touches the nine-points circle of ABC.

This will be proved if we show that \( ρ = \frac{1}{2}R + MN \). This can be shown with the aid of the following propositions:

Lemma I. The circle passing through the intersections of the confocals

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] and \( \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \) is \( x^2 + y^2 = a^2 + b^2 + \lambda \); this circle is the mutual orthoptic circle of the two confocals.

Lemma II. If P and Q be the foci of any conic X inscribed in a triangle ABC we have

\[ (R^2 - OP^2)(R^2 - OQ^2) = 4\beta^2R^2. \]

[Professor Genese, Educational Times, Q. 10879; for a solution see p. 37, Vol. 57 of the Mathematical Reprints.]

Lemma III. Any conic X being inscribed in a triangle ABC its director circle cuts the polar circle of the triangle orthogonally. The centre of the polar circle is the orthocentre H and the square of its radius is \( -\frac{1}{2}(R^2 - OH^2) \).

Now by lemma I. applied to the confocals X and Y we have

\[ ρ^2 = \beta^2 + \left(\frac{OP \pm OQ}{2}\right)^2 \]

\[ = \frac{1}{4}(OP^2 + OA^2 + 4\beta^2) \pm \frac{1}{2}OP \cdot OQ \quad . \quad (1) \]
Lemma II. gives
\[ R^4 - R^2(\text{OP}^2 + \text{OQ}^2 + 4\beta^2) + \text{OP}^2 \cdot \text{OQ}^2 = 0 \] . . . (2)

In (1) and (2) the expression \( \text{OP}^2 + \text{OQ}^2 + 4\beta^2 \) occurs; this is readily seen to be equal to \( 2(a^2 + \beta^2 + \text{OM}^2) \) . . . . (3)

Again by lemma III. we have
\[ (a^2 + \beta^2) - \frac{1}{2}(R^2 - \text{OH}^2) = \text{MH}^2; \]
\[ \therefore \ a^2 + \beta^2 + \text{OM}^2 = \text{OM}^2 + \text{MH}^2 + \frac{1}{2}(R^2 - \text{OH}^2) \]
\[ = \frac{1}{2}R^2 + 2\text{MN}^2 \] . . . . . (4)

By (3) and (4) we have
\[ \text{OP}^2 + \text{OQ}^2 + 4\beta^2 = R^2 + 4\text{MN}^2 \] . . . . (5)

Using this in (2) we get a pretty simple result
\[ \text{OP} \cdot \text{OQ} = 2R \cdot \text{MN} \] . . . . . (6)

Now making use of (5) and (6) in equation (1) we get
\[ \rho^2 = \frac{1}{2}R^2 + \text{MN}^2 + R \cdot \text{MN} \]
\[ = \left(\frac{1}{2}R \pm \text{MN}\right)^2 \]
\[ \therefore \ \rho = \frac{1}{2}R \pm \text{MN}; \text{ and the theorem is proved.} \]

**Corollary.**—A beautiful theorem, due to Mr M'Cay, of which Feuerbach's theorem is a particular case, is itself a particular case of the theorem now given; Mr M'Cay's theorem may be thus stated: "If either axis of a conic inscribed in a given triangle pass through the circumcentre, then the corresponding auxiliary circle of the conic touches the nine-points circle of the triangle." [See *Casey's Conics*, 2nd Edition, p. 329.]

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**On the Geometrical Representation of Elliptic Integrals of the First Kind.**

By **Alex. Morgan, M.A., B.Sc.**

[See page 2 of present volume.]

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Dr T. B. Sprague, M.A., F.R.S.E., was elected President in room of the Rev. John Wilson, deceased.