A General Theorem on the Nine-points Circle.

By V. RAMASWAMI AIYAR, M.A.

THEOREM: *If any conic be inscribed in a given triangle and a confocal to it pass through the circumcentre, then the circle through the intersection of these two confocals touches the nine-points circle of the triangle.*

DEMONSTRATION: Let X (Fig. 10) be any conic inscribed in the triangle ABC; O, H, N its circumcentre, orthocentre and nine-points centre; let R be the circumradius.

Let X be any conic inscribed in the triangle ABC; P, Q its foci; M its centre; and α, β its semi-axes.

Let Y be a confocal to X passing through the circumcentre O; and let ρ be the radius of the circle through the intersections of X and Y. We have to show that this circle touches the nine-points circle of ABC.

This will be proved if we show that $\rho = \frac{1}{2}R \pm MN$. This can be shown with the aid of the following propositions:

Lemma I. The circle passing through the intersections of the confocals

$x^2/a^2 + y^2/b^2 = 1$ and $x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1$ is $x^2 + y^2 = a^2 + b^2 + \lambda$;

this circle is the *mutual orthoptic circle* of the two confocals.

Lemma II. If P and Q be the foci of any conic X inscribed in a triangle ABC we have

$$(R^2 - OP^2)(R^2 - OQ^2) = 4\beta^2 R^2.$$

[Professor Genese, *Educatianal Times*, Q. 10879; for a solution see p. 37, Vol. 57 of the *Mathematical Reprints.*]

Lemma III. Any conic X being inscribed in a triangle ABC its director circle cuts the *polar circle* of the triangle orthogonally. The centre of the polar circle is the orthocentre H and the square of its radius $= -\frac{1}{2}(R^2 - OH^2)$.

Now by lemma I. applied to the confocals X and Y we have

$$\rho^2 = \beta^2 + \left(\frac{OP \pm OQ}{2}\right)^2$$
$$= \tfrac{1}{4}(OP^2 + OA^2 + 4\beta^2) \pm \tfrac{1}{2}OP \cdot OQ \qquad . \qquad . \quad (1)$$

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Lemma II. gives

$$R^4 - R^2(OP^2 + OQ^2 + 4\beta^2) + OP^2 . OQ^2 = 0 \quad . \quad . \quad (2)$$

In (1) and (2) the expression $OP^2 + OQ^2 + 4\beta^2$ occurs; this is readily seen to be equal to $2(\alpha^2 + \beta^2 + OM^2)$ . . . (3)

Again by lemma III. we have

$$(\alpha^2 + \beta^2) - \tfrac{1}{2}(R^2 - OH^2) = MH^2;$$

$$\therefore \quad \alpha^2 + \beta^2 + OM^2 = OM^2 + MH^2 + \tfrac{1}{2}(R^2 - OH^2)$$

$$= \tfrac{1}{2}R^2 + 2MN^2 \quad . \quad . \quad . \quad . \quad (4)$$

By (3) and (4) we have

$$OP^2 + OQ^2 + 4\beta^2 = R^2 + 4MN^2 \quad . \quad . \quad . \quad (5)$$

Using this in (2) we get a pretty simple result

$$OP . OQ = 2R . MN \quad . \quad . \quad . \quad . \quad (6)$$

Now making use of (5) and (6) in equation (1) we get

$$\rho^2 = \tfrac{1}{4}R^2 + MN^2 \pm R . MN$$

$$= (\tfrac{1}{2}R \pm MN)^2$$

$\therefore \ \rho = \tfrac{1}{2}R \pm MN$; and the theorem is proved.

Corollary.—A beautiful theorem, due to Mr M'Cay, of which Feuerbach's theorem is a particular case, is itself a particular case of the theorem now given; Mr M'Cay's theorem may be thus stated: "If either axis of a conic inscribed in a given triangle pass through the circumcentre, then the corresponding auxiliary circle of the conic touches the nine-points circle of the triangle." [See *Casey's Conics*, 2nd Edition, p. 329.]

On the Geometical Represetation of Elliptic Integrals of the First Kind.

By Alex. Morgan, M.A., B.Sc.

[*See page* 2 *of present volume.*]

Dr T. B. Sprague, M.A., F.R.S.E., was elected President in room of the Rev. John Wilson, deceased.

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