A General Theorem on the Nine-points Circle.

By V. Ramaswami Aiyar, M.A.

Theorem: If any conic be inscribed in a given triangle and a confocal to it pass through the circumcentre, then the circle through the intersection of these two confocals touches the nine-points circle of the triangle.

Demonstration: Let X (Fig. 10) be any conic inscribed in the triangle ABC; O, H, N its circumcentre, orthocentre and nine-points centre; let R be the circumradius.

Let X be any conic inscribed in the triangle ABC; P, Q its foci; M its centre; and \( a, b \) its semi-axes.

Let Y be a confocal to X passing through the circumcentre O; and let \( \rho \) be the radius of the circle through the intersections of X and Y. We have to show that this circle touches the nine-points circle of ABC.

This will be proved if we show that \( \rho = \frac{1}{2} R + MN \). This can be shown with the aid of the following propositions:

Lemma I. The circle passing through the intersections of the confocals

\[ x^2/a^2 + y^2/b^2 = 1 \text{ and } x^2/(a^2 + \lambda) + y^2/(b^2 + \lambda) = 1 \text{ is } x^2 + y^2 = a^2 + b^2 + \lambda; \]

this circle is the mutual orthoptic circle of the two confocals.

Lemma II. If P and Q be the foci of any conic X inscribed in a triangle ABO we have

\[ (R^2 - OP^2)(R^2 - OQ^2) = 4\rho^2R^2. \]

[Professor Genese, Educational Times, Q. 10879; for a solution see p. 37, Vol. 57 of the Mathematical Reprints.]

Lemma III. Any conic X being inscribed in a triangle ABC its director circle cuts the polar circle of the triangle orthogonally. The centre of the polar circle is the orthocentre H and the square of its radius

\[ = -\frac{1}{2}(R^2 - OH^2). \]

Now by lemma I. applied to the confocals X and Y we have

\[ \rho^2 = \beta^2 + \left(\frac{OP \pm OQ}{2}\right)^2 \]

\[ = \frac{1}{2}(OP^2 + OA^2 + 4\beta^2) \pm \frac{1}{2} OP \cdot OQ \quad \ldots (1) \]
Lemma II. gives

\[ R^4 - R^3 Q_1 + Q_1^2 = 0 \tag{2} \]

In (1) and (2) the expression \( Q_1^2 + 4Q_2 \) occurs; this is readily seen to be equal to \( 2(a^2 + \beta^2 + \alpha^2) \tag{3} \)

Again by lemma III. we have

\[ (a^2 + \beta^2) - \frac{1}{2}(R^2 - OH^2) = MH^2; \]

\[ \therefore \quad a^2 + \beta^2 + \alpha^2 = OM^2 + MH^2 + \frac{1}{2}(R^2 - OH^2) \]

\[ = \frac{1}{2}R^2 + 2MN^2 \tag{4} \]

By (3) and (4) we have

\[ Q_1^2 + 4Q_2 = R^2 + 4MN^2 \tag{5} \]

Using this in (2) we get a pretty simple result

\[ Q_1 Q_2 = 2R MN \tag{6} \]

Now making use of (5) and (6) in equation (1) we get

\[ \rho = \frac{1}{2}R + MN; \quad \text{and the theorem is proved.} \]

**Corollary.**—A beautiful theorem, due to Mr M'Cay, of which Feuerbach's theorem is a particular case, is itself a particular case of the theorem now given; Mr M'Cay's theorem may be thus stated: "If either axis of a conic inscribed in a given triangle pass through the circumcentre, then the corresponding auxiliary circle of the conic touches the nine-points circle of the triangle." [See *Casey's Conics*, 2nd Edition, p. 329.]

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**On the Geometrical Representation of Elliptic Integrals of the First Kind.**

By **Alex. Morgan**, M.A., B.Sc.

[See page 2 of present volume.]

Dr T. B. Sprague, M.A., F.R.S.E., was elected President in room of the Rev. John Wilson, deceased.