TOPOLOGIES ON BOOLEAN ALGEBRAS
DEFINED BY IDEALS AND DUAL IDEALS

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Introduction. In the paper [5], Rema used the well-known fact that in a Boolean algebra \( B = \langle B; \lor, \land, '; 0, 1 \rangle \) the binary operation \( d: B \times B \rightarrow B \) defined by \( d(a, b) = (a \land b') \lor (b \land a') \) is a "metric" operation to show that, if \( D \) is any dual ideal of \( B \), then the sets \( U_p = \{(x, y): d(x, y) \leq p\} \), where \( p \in D \), form a base for a uniformity of \( B \), the resulting topological space \( \langle B; T[D] \rangle \) being called an auto-topologized Boolean algebra. Recently, Kent and Atherton [1, 4] exhibited a family of topologies on an arbitrary lattice \( L \) defined in terms of ideals and dual ideals. More specifically, if \( I \) and \( D \) are respectively an ideal and a dual ideal of \( L \), then the \( T[I: D] \) topology on \( L \) is the topology defined by taking the sets of the form \( a^* \cap b^+ \), where \( a \in I, b \in D, a^* = \{x \in L : x \geq a\} \) and \( b^+ = \{x \in L : x \leq b\} \), as sub-base for the open sets. It is these topologies that are studied in this paper.

It is first shown that a \( T[I: D] \) topology on a Boolean algebra \( B \) is an auto-topology if and only if \( I \) is the "Boolean-complement" of \( D \). The property of a topology on \( B \) being a \( T[I: D] \) (auto-)topology is shown to be "productive" as well as being "c-hereditary" in that, if \( S \) is a complete subalgebra of a Boolean algebra endowed with a \( T[I: D] \) (auto-)topology, then the subspace topology on \( S \) is a \( T[I: D] \) (auto-)topology. Necessary and sufficient conditions are then established for a \( T[I: D] \) topology to be Hausdorff and employed to show that a Hausdorff \( T[I: D] \) topology is totally disconnected whereas an auto-topology is Hausdorff if and only if it is totally disconnected. Various connectedness properties of \( T[I: D] \) topologies are studied in some detail and it is shown, in particular, that such a topology is connected if and only if \( I \) is contained in the "lower section" of \( D \) and that an auto-topology \( T[D] \) is locally connected if and only if \( D \) is a principal dual ideal. Finally, we show that a Boolean algebra admits a compact, Hausdorff \( T[I: D] \) topology if and only if it is complete and atomic.

Notation and terminology. The topological concepts and results referred to throughout the paper can be found in [3], while the lattice-theoretic results are to be found in [2]. If \( S \) is a nonempty subset of a Boolean algebra \( B \), then we denote the set \( \{a'; a \in S\} \) by \( S' \) and refer to it as the Boolean complement of \( S \). The usual partial ordering of \( B \) will be denoted by \( \leq \) and \([a, b]\) will denote the interval \( \{x \in B : a \leq x \leq b\} \). For the sake of brevity we frequently write \( a \land b \) instead of \( a \land b \) for the lattice meet of \( a \) and \( b \) and \( a \lor b \) for the lattice join. The symbols \( \subseteq, \cup, \cap \) will be reserved for set inclusion, union and intersection respectively.

1. Theorem 1.1. \( T[I: D] \) is an auto-topology if and only if \( D = I' \) and, when this condition is satisfied, \( T[I: D] = T[D] \).

Proof. Suppose that \( T[I: D] \) coincides with the auto-topology \( T[F] \) defined by the dual ideal \( F \) of \( B \); then the set \( \{U_f[a]: f \in F\} \), where \( U_f[a] = \{x: d(x, a) \leq f\} \), forms a base for...
the $T[F]$ neighbourhood system of the point $a \in \mathcal{B}$, and the set \{\{a \land q, a \lor p\} : p \in D, q \in I\} forms a base for the $T[I : D]$ neighbourhood system of $a \in \mathcal{B}$. Now $d(x, a) \leq f \land^* a \land f' \leq x \leq a \lor f$, so that $U_f[a] = [a \land f', a \lor f]$, and it follows that $\forall a \in \mathcal{B}, \forall p \in D, \forall q \in I, \exists f \in F$ such that $[a \land f', a \lor f] \subseteq [a \land q, a \lor p]$. On taking $a = 0$, we deduce that every element in $D$ contains some element in $F$ and this implies that $D \subseteq F$. Furthermore, on taking $a = 1$, it follows that $\forall q \in I, \exists f \in F$ such that $q \leq f'$, or equivalently $f \leq q'$, and so $q' \in F$, which implies that $I \subseteq F'$. Similarly $\forall a \in \mathcal{B}, \forall f \in F, \exists p \in D$ and $q \in I$ such that $[a \land q, a \lor p] \subseteq [a \land f', a \lor f]$. Taking $a = 0$, we have that $\forall f \in F, \exists p \in D$ such that $p \leq f$ and so $f \in D$, which shows that $F \subseteq D$. Again, taking $a = 1$, we have that $\forall f \in F, \exists q \in I$ such that $f' \leq q$ and this implies that $F' \subseteq I$. In summary then, $D = F$ and $I = F'$; whence $D = I'$.

The converse has been established by Atherton [1] who showed that, if this condition is satisfied, then $T[I : D] = T[D]$.

A property $\mathcal{P}$ of a topology on a Boolean algebra is said to be productive if and only if the product of any family of Boolean algebras, each being endowed with a topology possessing the property $\mathcal{P}$, also possesses $\mathcal{P}$.

**Theorem 1.2.** The property of being a $T[I : D]$ topology is productive.

**Proof.** Suppose that $\{<\mathcal{B}_a; T[I_a : D_a]> \}_{a \in \Lambda}$ is an arbitrary family of Boolean algebras each endowed with a $T[I : D]$ topology. Let $D$ be the subset of the direct product $\mathcal{B}$ of the $\mathcal{B}_a$'s consisting of all functions $f \in \mathcal{B}$ with the property that $f(a) = 1_a, \forall a \in \Lambda$, except when $a$ is in some finite subset $\{a_1, a_2, \ldots, a_n\}$ of $\Lambda$, in which case $f(a_i) \in D$.

Similarly, let $I$ be the subset of $\mathcal{B}$ consisting of all functions $f \in \mathcal{B}$ with the property that $f(a) = 0_a, \forall a \in \Lambda$, except when $a$ is in some finite subset $\{a_1, a_2, \ldots, a_n\}$ of $\Lambda$, in which case $f(a_i) \in I$. Then it is easily shown that $D$ is a dual ideal and $I$ an ideal of $\mathcal{B}$, and we prove that the product topology $\prod_{a \in \Lambda} T[I_a : D_a]$ on $\mathcal{B}$ coincides with the topology $T[I : D]$.

To this end, let $f \in U \in \prod_{a \in \Lambda} T[I_a : D_a]$; then, by definition of the product topology, there exist open sets $U_j \in T[I_j : D_j] (\alpha_j \in \Lambda, 1 \leq j \leq m)$ such that the corresponding sub-basic open sets $U^*_j = \{f \in \mathcal{B} : (f(a_j) \in U_j)\}_{a \in \Lambda}$ satisfy $f \in \bigcap_{j=1}^m U^*_j \subseteq U$. Now, since $f(a_i) \in U_{a_i}$ and $U_{a_i} \in T[I_{a_i} : D_{a_i}]$, it follows that $\exists p_{a_i} \in D_{a_i}$ and $q_{a_i} \in I_{a_i}$ such that $[f(a_i) \land p_{a_i}, f(a_i) \lor q_{a_i}] \subseteq U_{a_i}(1 \leq i \leq n)$. Let $q \in \mathcal{B}$ be defined by $q(a) = 0_a, \forall a \in \Lambda$ except where $a = a_j$, when $q(a_j) = q_{a_j}(1 \leq j \leq m)$, and let $p \in \mathcal{B}$ be defined by $p(a) = 1_a, \forall a \in \Lambda$ except where $a = a_j$, when $p(a_j) = p_{a_j}(1 \leq j \leq m)$. Then $p \in D, q \in I$ and $[f \land q, f \lor p]$ is a $T[I : D]$-open neighbourhood of $f$ which is contained in $\bigcap_{j=1}^m U^*_j$; for if $g \in [f \land q, f \lor p]$, then, in particular, $f(a_j) \land q(a_j) \leq g(a_j) \leq f(a_j) \lor p(a_j)$, so that $g(a_j) \in [f(a_j) \land p_{a_j}, f(a_j) \lor q_{a_j}], (1 \leq j \leq m)$, whence $g \in \bigcap_{j=1}^m U^*_j$. Thus $\prod_{a \in \Lambda} T[I_a : D_a] \subseteq T[I : D]$.

Conversely, suppose that $f \in U \in T[I : D]$; then $\exists p \in D, q \in I$ such that $f \in [f \land q, f \lor p] \subseteq U$. Now suppose that $p(a) = 1_a, \forall a \in \Lambda - J, p(a_j) \in D_{a_j} \forall a_j \in J$, where $J$ is a finite subset of $\Lambda$, and $q(\beta) = 0_\beta, \forall \beta \in \Lambda - K, q(\beta_k) \in I_{\beta_k} \forall \beta_k \in K$, where $K$ is a finite subset of $\Lambda$. Let $L = J \cup K$ and, for
each $\gamma \in L$, consider the sub-basic $\prod_{\alpha \in \Lambda} T[I_\alpha : D_\alpha]$-open set $U^*_\gamma = \{ b \in \mathcal{B} : b(\gamma) \in U_\gamma \}$, where $U_\gamma$ is the basic $T[I_\gamma : D_\gamma]$-open set $[f(\gamma) \land q(\gamma), f(\gamma) \lor p(\gamma)]$. Now $f \in \bigcap_{\gamma \in L} U^*_\gamma \subseteq \{ q \land q, q \lor p \} \subseteq U$; for, if $g \in \bigcap_{\gamma \in L} U^*_\gamma$, then $g(\gamma) \in U_\gamma, \forall \gamma \in L$, or, equivalently, $f(\gamma) \land q(\gamma) \leq g(\gamma) \leq f(\gamma) \lor p(\gamma), \forall \gamma \in L$, and, if $\alpha \in \Lambda - J$, so that $\alpha \in \Lambda - J$ and $\alpha \in \Lambda - K$, then $q(\alpha) = 0_\alpha$ and $p(\alpha) = 1_\alpha$, which implies that $f(\alpha) \land q(\alpha) \leq g(\alpha) \leq f(\alpha) \lor p(\alpha), \forall \alpha \in \Lambda$, i.e., $g \in \{ q \land q, q \lor p \}$. Hence $T[I : D] \subseteq \prod_{\alpha \in \Lambda} T[I_\alpha : D_\alpha]$ and therefore equality holds.

**Corollary 1.3.** The property of being an auto-topology is productive.

**Proof.** If each of the topologies $T[I_\alpha : D_\alpha]$ in the theorem is an auto-topology, then, by Theorem 1.1, $D_\alpha = I'_\alpha \forall \alpha \in \Lambda$, and it is easily shown that the associated ideal $I$ and dual idea $D$ of $\mathcal{B}$ satisfy $D = I'$. Hence the product topology on $\mathcal{B}$ is an auto-topology.

A property $\mathcal{P}$ of a topology on a Boolean algebra is said to be c-hereditary if and only if the subspace topology on any complete subalgebra of a Boolean algebra endowed with a topology possessing the property $\mathcal{P}$ also possesses $\mathcal{P}$.

**Theorem 1.4.** The property of being a $T[I : D]$ topology is c-hereditary.

**Proof.** Let $S$ be a complete subalgebra of the Boolean algebra $\mathcal{B}$; for each $p \in D$, let $t_p = \bigvee (p^+ \land S)$ and form the dual ideal $D_S$ in $S$ generated by the set $T_D = \{ t_p : p \in D \}$. Observe that, since $T_D$ is closed under finite meets, $D_S = \{ s \in S : s \geq t_q \text{ for some } q \in I \}$. For each $q \in I$, let $t_q = \bigwedge (q^* \land S)$, form the ideal $I_q$ in $S$ generated by the set $T_I = \{ t_q : q \in I \}$ and, once again, observe that $I_q = \{ s \in S : s \leq t_q \text{ for some } q \in I \}$. We show that the subspace topology $T[I : D]S$ on $S$ is identical with $T[I_q : D_S]S$. Let $a \in U \in T[I : D]S$; then $T[D]S = \{ t_p : p \in D \}$ and, once again, observe that $I_q = \{ s \in S : s \leq t_q \text{ for some } q \in I \}$. We show that the subspace topology $T[I : D]S$ on $S$ is identical with $T[I_q : D_S]S$. Let $a \in U \in T[I : D]S$; then $T[D]S = \{ t_p : p \in D \}$ and, once again, observe that $I_q = \{ s \in S : s \leq t_q \text{ for some } q \in I \}$. We show that the subspace topology $T[I : D]S$ on $S$ is identical with $T[I_q : D_S]S$. Let $a \in U \in T[I_q : D_S]S$; then $T[D]S = \{ t_p : p \in D \}$ and, once again, observe that $I_q = \{ s \in S : s \leq t_q \}$. Hence $\mathcal{P}$ is c-hereditary.

**Corollary 1.5.** The property of being an auto-topology is c-hereditary.

**Proof.** If the topology $T[I : D]$ of the theorem is an auto-topology, then $D = I'$ and it suffices to show that $D_S = I'_S$. To this end let $s \in D_S$; then $\exists p \in D$ such that $s \geq t_p = \bigvee (p^+ \land S)$. Now, since $p = q^*$ for some $q \in I$, $t_p = \bigwedge (p^+ \land S) = \bigwedge (q^* \land S) = t_q$ and so $s' \leq t_q$, which implies that $s' \in I_q$, or, equivalently, $s \in I'_q$. Similarly, if $s \in I'_q$, so that $s = r'$ for some $r \in I_q$, then $\exists q \in I$ such that $t_q \leq s$. Now $q = p'$ for some $p \in D$, and so $t_q = [\bigwedge (q^* \land S)]' = \bigvee (q^* \land S) = \bigvee (p^+ \land S) = t_p$, which implies that $t_p \leq s$ and therefore $s \in D_q$. Hence $I'_q \subseteq D_q$, completing the proof.
2. Connectedness properties. Prior to establishing necessary and sufficient conditions for $T[I : D]$ to be Hausdorff, we recall that an ideal (dual ideal) in the pseudo-complemented lattice $L$ of all ideals (dual ideals) of a Boolean algebra $B$ is said to be (algebraically) dense if and only if its pseudo-complement is the zero element of $L$. We remark that an ideal $I$ of $B$ is dense if and only if its upper section $I^* = \{x \in B : x \geq q, \forall q \in I\}$ contains only the element 1, while a dual ideal $D$ is dense if and only if its lower section $D^+ = \{x \in B : x \leq p, \forall p \in D\}$ contains only the element 0.

**Theorem 2.1.** The topology $T[I : D]$ is Hausdorff if and only if both $I$ and $D$ are dense.

**Proof.** Suppose that $T[I : D]$ is Hausdorff and $x \in I^*$ but $x \neq 1$; then $\exists p_1 \in D$ and $\exists q_1, q_2 \in I$ such that $[q_2, 1] \cap x \wedge q_1, x \vee p_1] = \emptyset$, which gives a contradiction on observing that the element $q = q_1 \vee q_2 \in I$ satisfies $q_2 \leq q$ and $x \wedge q_1 = q_1 \leq q \leq x \vee p_1$ and therefore lies in the intersection. Hence $I^* = \{1\}$. Similarly, suppose that $x \in D^+$ but $x \neq 0$; then $\exists p_1, p_2 \in D$ and $\exists q_1 \in I$ such that $[0, p_2] \cap [x \wedge q_1, x \vee p_1] = \emptyset$, which, on observing that $p = p_1 \wedge p_2 \in D$ satisfies $p \leq p_2$ and $x \wedge q_1 \leq x \leq p \leq x \vee p_1$, gives a contradiction. Hence $D^+ = \{0\}$.

Conversely, suppose that both $I$ and $D$ are dense, but $T[I : D]$ is not Hausdorff; then there exist distinct points $a, b \in B$ such that every open neighbourhood of $a$ meets every open neighbourhood of $b$. Hence $[a \wedge q, a \vee p] \cap [b \wedge q, b \vee p] \neq \emptyset, \forall p \in D, \forall q \in I$. But $\exists x \in B$ satisfying

$$x \in [a \wedge q, a \vee p] \cap [b \wedge q, b \vee p] \iff aq \vee bq \leq x \leq (a \vee p)(b \vee p)$$

$$\iff (a' \wedge b')p' x \vee (a \wedge b)q' x' = 0$$

$$\iff (a' \wedge b')(a \wedge b)p' q = 0$$

$$\iff d(a, b)p' q = 0$$

and so it follows that $d(a, b)q \leq p, \forall p \in D, \forall q \in I$. Whence

$$d(a, b)q \in D^+ = \{0\}, \forall q \in I \iff q \leq d'(a, b), \forall q \in I$$

$$\iff d'(a, b) \in I^* = \{1\}$$

$$\iff d(a, b) = 0$$

$$\iff a = b,$$

giving a contradiction and therefore proving that $T[I : D]$ is Hausdorff.

**Corollary 2.2.** An auto-topology $T[D]$ is Hausdorff if and only if $D$ is a dense dual ideal.

**Theorem 2.3.** If $T[I : D]$ is Hausdorff, then it is totally disconnected.

**Proof.** It is, of course, well known that a Hausdorff, zero-dimensional space is totally disconnected and so, in proving the theorem, it suffices to show that each basic open set $[a \wedge q, a \vee p] \ (p \in D, q \in I)$ is clopen. Now $x \in \text{Cl} [a \wedge q_1, a \vee p_1]$, the closure of $[a \wedge q_1, a \vee p_1]$,
if and only if every neighbourhood of \( x \) meets \( [a \land q, a \lor p] \) or, equivalently, \( \mathcal{F} = [x \land q, x \lor p] \cap [a \land q, a \lor p] \neq \emptyset, \forall p \in D, \forall q \in I \). But
\[
\exists y \in \mathcal{F} \iff xq \lor aq_1 \leq y \leq (x \lor p)(a \lor p_1)
\]
\[
\iff (x'p' \lor a'p_1')y' \lor (xq \lor aq_1)y' = 0
\]
\[
\iff (x'p' \lor a'p_1')(xq \lor aq_1) = 0
\]
\[
\iff aq_1 \land x'p' = 0 \quad \text{and} \quad a'p_1' \land xq = 0
\]
\[
\iff ax'q_1 \leq p \quad \text{and} \quad q \leq a \land x' \lor p_1.
\]
Hence \( \mathcal{F} \neq \emptyset, \forall p \in D, \forall q \in I \iff ax'q_1 \leq p, \forall p \in D \) and \( q \leq a \land x' \lor p_1, \forall q \in I \iff ax'q_1 \in D^+ = \{0\} \) and \( a \land x' \lor p_1 \in I^* = \{1\} \iff ax'q_1 = 0 \) and \( a'p_1' \land xq = 0 \). It follows now that \( [a \land q_1, a \lor p_1] \) is clopen and the theorem is proved.

**Corollary 2.4.** An auto-topology is Hausdorff if and only if it is totally disconnected.

**Proof.** It is well known that \( \text{cmp}(a) \), the component of \( a \), is contained in the intersection of all clopen sets containing the point \( a \) and so, since the \( T[D] \)-open sets \( [0, p] \) (\( p \in D \)) are clopen, it follows that \( \text{cmp}(0) \subseteq \bigcap_{p \in D} [0, p] = D^+ \). We show that the subspace \( D^+ \) is indiscrete and therefore connected. To this end, let \( V \) be an open set containing the element \( l \) in the subspace \( D^+ \), so that \( V = U \cap D^+ \) for some \( T[D] \)-open set \( U \) containing \( l \). Then \( \exists p \in D \) such that \( [l \land p, l \lor p] \land D^+ \subseteq V \). Furthermore, \( D^+ \subseteq [l \land p, l \lor p] \); for, if \( x \in D^+ \), so that \( x \leq p, \forall p \in D \), then \( d(x, l) \leq x \lor l \leq p \) and so \( x \in [l \land p, l \lor p] \). It follows now that \( D^+ = V \) and so the only open sets in the subspace \( D^+ \) are itself and the empty set. Hence \( D^+ \) is an indiscrete subspace. Now \( \text{cmp}(0) \) is the largest connected set containing the element \( 0 \) and so, by the connectedness of \( D^+ \), \( \text{cmp}(0) = D^+ \). Hence, if \( \langle B; T[D] \rangle \) is totally disconnected, \( D^+ = \text{cmp}(0) = \{0\} \) and it follows, by Corollary 2.2, that \( \langle B; T[D] \rangle \) is Hausdorff.

**Theorem 2.5.** The topology \( T[I : D] \) is connected if and only if \( I \subseteq D^+ \).

**Proof.** Suppose that \( T[I : D] \) is connected. Let \( I_m \) be an arbitrary maximal ideal in \( \mathcal{B} \) and let \( p \in D, q \in I \) be given; then the set \( \{[a \land q, a \lor p] : a \in I_m\} \) forms an open cover of \( I_m \) and, by a well-known property of maximal ideals, the set \( \{[b \land q, b \lor p] : b \in I_m^*\} \) forms an open cover of \( \mathcal{B} \) - \( I_m \). Hence the open sets of \( U = \bigcup_{a \in I_m} [a \land q, a \lor p] \), \( V = \bigcup_{b \in I_m^*} [b \land q, b \lor p] \) cover \( \mathcal{B} \) and therefore cannot be disjoint. This implies that \( \exists a, c \in I_m \) such that \( [a \land q, a \lor p] \cap [c' \land q, c' \lor p] \neq \emptyset \iff \exists x \in \mathcal{B} \) such that
\[
(a \lor c')q \leq x \leq ac' \lor p \iff (a' \lor c')p'x \lor (a \lor c')q'x = 0
\]
\[
\iff p'q(a' \lor c')(a \lor c') = 0
\]
\[
\iff qp'd(a, c') = 0 \iff q'p' \leq d(a, c) \in I_m.
\]
Hence \( q'p' \in I_m \), so that, since \( I_m \) is an arbitrary maximal ideal and the intersection of all maximal ideals of \( \mathcal{B} \) contains only the element \( 0 \), it follows that \( q \leq p, \forall p \in D, \forall q \in I \). Therefore \( I \subseteq D^+ \).

Conversely, suppose that \( I \subseteq D^+ \) and let \( C \) be any clopen subset of \( \langle B; T[I : D] \rangle \). Then either \( C = \emptyset \) or \( \exists a \in C \). In the latter case suppose that \( \exists b \in \mathcal{B} - C \). Then \( \exists p_1 \in D, q_1 \in I \) such
\[
\text{that } q_1 \leq p, \forall p \in D, \forall q \in I.
\]
that $[b \land q_1, b \lor p_1] \subseteq \mathfrak{B} - C$. Also, since $C$ is open, $\exists p_2 \in D, q_2 \in I$ such that $[a \land q_2, a \lor p_2] \subseteq C$ and so these intervals are disjoint. But $I \subseteq D^+ \leftrightarrow q \leq p, \forall p \in D, \forall q \in I$ and we observe that the element $s = bq_1 \lor aq_2$ lies in their intersection, giving a contradiction. Therefore, if $C$ is clopen, then either $C = \emptyset$ or $C = \mathfrak{B}$; whence the space is connected.

**Theorem 2.6.** An auto-topology $T[D]$ is locally connected if and only if $D$ is a principal dual ideal.

**Proof.** Suppose that $T[D]$ is a locally connected auto-topology on the Boolean algebra $\mathfrak{B}$. Then there exists a base $\sigma$ for $T[D]$ consisting of connected open sets. Let $U$ be any member of $\sigma$ containing the least element of $\mathfrak{B}$. Then $\exists p \in D$ such that $p^+ \subseteq U$, which, since $p^+$ is a nonempty clopen set and therefore clopen in the subspace $U$ of $T[D]$, implies that $p^+ = U$. Now suppose that $D$ is non-principal. Then $\exists p_1 \in D$ such that $p_1 < p$ and so the clopen set $p_1^+$ is properly contained in the connected set $p^+$, giving a contradiction. Hence $D$ is a principal dual ideal of $\mathfrak{B}$.

Conversely, suppose that $T[D]$ is induced by the principal dual ideal $D = p^*$ generated by $p$. Then it is obvious that the set $\{[a \land p', a \lor p] : a \in \mathfrak{B}\}$ forms a base for $T[D]$. Furthermore these intervals are connected sets; for otherwise there exists a nonempty clopen set $U_a$, containing the element $a$, in the subspace $[a \land p', a \lor p]$ and distinct from it, which, since $U_a$ must be $T[D]$-open, implies that $[a \land p', a \lor p] \subseteq U_a$. It follows that $T[D]$ is locally connected.

The preceding theorem characterizes principal dual ideals of a Boolean algebra $\mathfrak{B}$ in terms of a property of the associated auto-topologies, while Corollary 2.2. may be regarded as a characterization of dense dual ideals of $\mathfrak{B}$. The following theorem characterizes, in the same way, the maximal dual ideals of $\mathfrak{B}$.

**Theorem 2.7.** If $T[D]$ is an auto-topology on $\mathfrak{B}$, induced by the dual ideal $D$, then $D$ is maximal if and only if $\langle B; T[D]\rangle$ is non-discrete and, for all $a \in \mathfrak{B}$, either $a^*$ or $a^+$ is an open set.

**Proof.** If $D$ is a maximal dual ideal of $\mathfrak{B}$, then it is proper, so that $T[D]$ is non-discrete; and, furthermore, if $a \in \mathfrak{B}$, then either $a \in D$ or $a' \in D$. In the first case the set $U_a[a] = a^+$ is open, while in the second the set $U_a[a] = a^*$ is open.

Conversely, suppose that $T[D]$ is a non-discrete auto-topology on $\mathfrak{B}$ with the property that, for all $a \in \mathfrak{B}$, either $a^+$ or $a^*$ is open. Then $D$ is proper and, furthermore, if $a^+ \in D$, $\exists p \in D$ such that $U_p[a] = [a \land p', a \lor p] \subseteq a^+$, which implies that $a \lor p \leq a$, or, equivalently, $p \leq a$, and so $a \in D$. In the event that $a^* \in D$, $\exists p \in D$ such that $U_p[a] = [a \land p', a \lor p] \subseteq a^*$, which implies that $a \land p' \geq a$, or, equivalently, $p \leq a'$, and so $a' \in D$. Hence $D$ is a proper dual ideal possessing the property that, for all $a \in \mathfrak{B}$, either $a \in D$ or $a' \in D$ and $D$ is therefore maximal.


**Theorem 3.1.** If a Boolean algebra admits a compact, Hausdorff $T[I : D]$ topology, then it is complete.

**Proof.** Let $X$ be any nonempty subset of a Boolean algebra $\mathfrak{B}$ admitting a compact,
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Hausdorff $T[I : D]$ topology and let $\mathcal{L}_X$ be the set of all lower bounds of $X$; then $\mathcal{L}_X$ is an ideal of $\mathcal{B}$ and consequently the identity map $n : \mathcal{L}_X \to \mathcal{L}_X$ is a net in $\mathcal{L}_X$ which, since $\langle B ; T[I : D] \rangle$ is compact, has a cluster point $c$. Let $p \in D$, $q \in I$ be given; then, since $n$ is frequently in the open neighbourhood $[c \land q, c \lor p]$, it follows that $\forall a \in \mathcal{L}_X, \exists b \in \mathcal{L}_X$ such that $b \geq a$ and $b \in [c \land q, c \lor p]$. Whence $c \land q < b \leq c \lor p$, from which it follows that $c \land q \in \mathcal{L}_X$ and $a \leq c \lor p$, $\forall a \in \mathcal{L}_X$. Hence, since $p$ and $q$ were arbitrarily chosen, it follows that $c \land q \leq x, \forall x \in X, \forall q \in I \iff q \leq c \lor x, \forall x \in X, \forall q \in I \iff c \lor x \in \mathcal{I}^*, \forall x \in \mathcal{L}_X$.

Also

$$a \leq c \lor p, \forall a \in \mathcal{L}_X, \forall p \in D \iff a \land c \leq p, \forall a \in \mathcal{L}_X, \forall p \in D \iff a \land c \in D^+.$$  

But $T[I : D]$ is Hausdorff, or, equivalently, $I^+ = \{1\} = D^* = \{0\}$, and so $c \lor x = 1, \forall x \in X$, and $a \land c = 0, \forall a \in \mathcal{L}_X$, i.e., $c \in \mathcal{L}_X$ and $a \leq c, \forall a \in \mathcal{L}_X$, so that $c$ is the greatest lower bound on the set $X$. It follows that $\mathcal{B}$ is complete.

**Theorem 3.2.** A Boolean algebra admits a compact, Hausdorff $T[I : D]$ topology if and only if it is complete and atomic.

**Proof.** Let $\mathcal{B}$ be a Boolean algebra and suppose that $\langle B ; T[I : D] \rangle$ is compact and Hausdorff. Then, by the preceding theorem, $\mathcal{B}$ is complete and it remains only to show that $\mathcal{B}$ is atomic. To this end, let $p$ be an arbitrary element in the dual ideal $D$ distinct from the element 1. Then $\exists q \in I$ such that $q \leq p$; otherwise $p \not\in q, \forall q \in I$, so that $p \in I^* = \{1\}$, whence $p = 1$. Let $I_p$ be any prime ideal of $\mathcal{B}$ such that $p \in I_p$ but $q \not\in I_p$, the existence of such an ideal being well known. Now $\mathcal{C} = \{[a \land q, a \lor p], [b \land q, b \lor p] : a \in I_p, b \in \mathcal{B} - I_p\}$ is an open cover of $\mathcal{B}$ and so, since $T[I : D]$ is compact, $\exists$ a finite sub-cover $\mathcal{C}^* = \{[a_i \land q, a_i \lor p], [b_j \land q, b_j \lor p] : 1 \leq i \leq m, 1 \leq j \leq n\}$ of $\mathcal{B}$. We assert that $\mathcal{C}^{**} = \{[a_i \land q, a_i \lor p] : 1 \leq i \leq m\}$ is an open cover of $I_p$; for, if not, $\exists a \in I_p$ such that $a \not\in [b_j \land q, b_j \lor p]$ for some $j$. But $a \in I_p$ and $b_j \land q \leq a$ implies that $b_j \land q \in I_p$, which, since $I_p$ is prime, implies that either $b_j \in I_p$ or $q \in I_p$, both of which give a contradiction. Hence $I_p \subseteq \bigcup_{i=1}^{m} [a_i \land q, a_i \lor p]$ so that $x \in I_p \to x \leq \bigvee_{i=1}^{m} (a_i \lor p) = p \lor \bigvee_{i=1}^{m} a_i \in I_p$. Therefore $I_p$ is a principal ideal of $\mathcal{B}$ generated by $m$, say. But an ideal in $\mathcal{B}$ is prime if and only if it is maximal and so it follows that $m$ is a maximal element in $\mathcal{B}$. Furthermore, since the complement of a maximal element in $\mathcal{B}$ is an atom, we have shown that $\forall p \in D (p \not= 1), \exists$ an atom $a \leq p'$.

Let $a_p$ be the join of all atoms contained in $p'$, which exists since $\mathcal{B}$ is complete; we show that $p' = a_p$. For, if $p' > a_p > 0$, let $x$ be the relative complement of $a_p$ in the Boolean interval $[0, p']$, so that $0 < x < p'$ and $a_p \land x = 0$. Then $p < x'$, which implies that $x' \in D$ and $x' \not= 1$. Therefore $\exists$ an atom $b \leq x' = x$, whence $b < p'$, so that $b$ is an atom contained in $p'$, which implies that $b \leq a_p$. Then $0 < b \leq a_p \land x = 0$, so that $b = 0$, giving a contradiction. Hence $\forall p \in D, p'$ is the join of all atoms it contains. Now we show that every element of $\mathcal{B}$ contains...
an atom. Since \( T[I : D] \) is Hausdorff, \( D^+ = \{0\} \) and therefore, since \( \bigwedge_{p \in D} p \in D^+ \), it follows that \( \bigwedge_{p \in D} p = 0 \), which implies that \( \bigvee_{p \in D} p' = 1 \). Each \( p' \) is, as we have shown, a join of atoms of \( B \) and therefore the element 1 is the join of all atoms of \( B \). Let \( A \) be the set of all atoms of \( B \) and suppose that some element \( x \in B \) contains no member of \( A \). Then \( a \land x = 0 \), \( \forall a \in A \) and so \( 0 = \bigvee_{a \in A} (a \land x) = x \land \bigvee_{a \in A} a = x \land 1 = x \). Therefore every nonzero element of \( B \) contains an atom and so \( B \) is atomic.

Conversely, if \( B \) is complete and atomic, or, equivalently, \( B = 2^N \) for some cardinal \( N \), then, since each two-element Boolean algebra endowed with the discrete topology is a \( T[I : D] \) topologized Boolean algebra and the property of being such a topology is productive, it follows that \( B \) admits a compact, Hausdorff \( T[I : D] \) topology.

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