ON SCHOENEBERG’S THEOREM

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Let $S$ be a compact Riemann surface of genus $g \geq 2$ and $\sigma$ an automorphism (conformal self-homeomorphism) of $S$ of order $n$. Let $S^* = S/\langle \sigma \rangle$ have genus $g^*$. In [5], Schoeneberg gave a sufficient condition that a fixed point $P \in S$ of $\sigma$ should be a Weierstrass point of $S$, i.e., that $S$ should support a function that has a pole of order less than or equal to $g$ at $P$ and is elsewhere regular.

**Theorem (Schoeneberg).** $P$ is a Weierstrass point of $S$ provided that $g^* \not\in \lfloor g/n \rfloor$. ([x] denotes the integral part of $x$.)

By the uniformization theorem, $S$ can be represented as a quotient surface $U/K$, where $U$ denotes the upper half-plane and $K$ a Fuchsian group isomorphic to the fundamental group of $S$. Furthermore, $G$ will be a (finite) group of automorphisms of $U/K$ if and only if $G \cong \Gamma/K$, where $\Gamma$ is a Fuchsian group with compact quotient space $U/\Gamma$. Such groups are known to have a presentation of the following form:

Generators: $x_1, x_2, \ldots, x_r, a_1, b_1, \ldots, a_g^*, b_g^*$.

Relations: $x_i^{m_i} = 1 (i = 1, 2, \ldots, r), \prod_{j=1}^{r} x_j \prod_{k=1}^{g^*} [a_k, b_k] = 1$. \hspace{1cm} (1)

If the presentation is (1), the group is said to have signature $(g^*; m_1, \ldots, m_r)$. Such a group has a fundamental polygon $F_\Gamma$ in $U$ with hyperbolic area

$$\mu(F_\Gamma) = 2(g^*-1) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right).$$ \hspace{1cm} (2)

If $K$ is of index $n$ in $\Gamma$, then

$$n\mu(F_\Gamma) = \mu(F_K),$$ \hspace{1cm} (3)

and combining (2) and (3) gives a form of the Riemann-Hurwitz relation.

Here we sharpen Schoeneberg’s condition to criteria on the signature of the corresponding Fuchsian group. Our method uses results of Lewittes [3] which we have employed before [4]. (For other applications of similar methods, see [2].) In the proof of the theorem below we shall use the notation and results of [4].

**Theorem.** Let $\sigma$ be an automorphism of order $n$ of a compact Riemann surface $S = U/K$ of genus $g \geq 2$. Let $\Gamma$ be a Fuchsian group such that $\langle \sigma \rangle \cong \Gamma/K$. Let $\sigma$ have a fixed point $P \in S$. If $P$ is not a Weierstrass point, then $\Gamma$ has signature of one of the following forms:

(i) \( \left( \frac{g}{n}; n, n \right) \),

(ii) \( \left( \frac{g-(n-1)}{n}; n, n, n, n \right) \),
(iii) \((g^*, n, m_1, m_2)\), where \(2ng^* = 2g-1-n+\frac{m_1+m_2}{(m_1, m_2)}\) and the least common multiple of \(m_1, m_2\) is \(n\).

**Proof.** Let \(P\) have gap sequence \(\{\gamma_1, \gamma_2, \ldots, \gamma_g\}\) and choose a local parameter \(z\) at \(P\) such that locally \(\sigma^{-1}\) is \(z \rightarrow \varepsilon z\), where \(\varepsilon\) is a primitive \(n\)th root of unity. Letting \(\sigma\) act on the \(g\)-dimensional space of abelian differentials on \(S\) of the first kind, one obtains, with respect to a suitable basis, a diagonal representation of \(\sigma\) with entries \(\{\varepsilon^{\gamma_1}, \varepsilon^{\gamma_2}, \ldots, \varepsilon^{\gamma_g}\}\) [3].

Assume that \(\Gamma\) has the presentation (1). Let \(\pi : \Gamma \rightarrow \mathbb{Z}_n\) be the natural projection combined with the isomorphism \(\sigma \leftrightarrow 1\), where we write elements of \(\mathbb{Z}_n\) as residues modulo \(n\). Since the kernel of \(\pi\) contains no elements of finite order, each \(m_t\) divides \(n\). Assume that \(m_1 = n\) and adjust \(\pi\) so that, locally at \(P\), \(\sigma^{-1}\) is \(z \rightarrow \varepsilon z\) where \(\varepsilon = \exp[(2\pi/n)i]\). Now suppose that \(\pi(x_\mu) = \xi_\mu (\mu = 1, 2, \ldots, r)\). Then, if \(N_\nu\) denotes the multiplicity of \(\exp [2\pi\nu/n]i\) as an eigenvalue of \(\sigma\), we have

\[
N_0 = g^*,
\]

\[
N_\nu = g^* - 1 + \sum_{\nu \cdot \xi_\mu \equiv 0 \pmod{n}} \left(1 - \left\langle \frac{\nu \cdot \xi_\mu}{n} \right\rangle\right),
\]

(4)

where \(\langle x\rangle\) denotes the fractional part of \(x\). (See [4].)

As already noted above, \(\xi_1 = 1\). Also, from the relations (1), \(\sum_{i=1}^r \xi_i \equiv 0 \pmod{n}\). Let \(\sum_{i=1}^r \xi_i = an\). Then, from (4),

\[
N_1 = g^* - 1 + r - a, \quad N_{n-1} = g^* - 1 + a.
\]

Now suppose that \(P\) is not a Weierstrass point. Then \(\sigma\) has eigenvalues \(\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^g\) where \(\varepsilon = \exp[(2\pi/n)i]\). Thus, writing \(g = nk+l\), where \(0 \leq l < n\), we have \(N_0 = k, N_1 = k+1, N_2 = k+1, \ldots, N_i = k+1, N_{i+1} = k, \ldots, N_{n-1} = k\). Hence \(g^* = k\) and we consider three cases.

(i) \(l = 0\). Then \(g^* = \frac{g}{n}\). \(N_1 = N_{n-1} = g^*\). Thus \(a = 1\) and \(r = 2\) and, from the Riemann-Hurwitz relation, \(m_2 = n\).

(ii) \(l = n-1\). Then \(g^* = \frac{g-(n-1)}{n}\). \(N_1 = N_{n-1} = g^* + 1\). Thus \(a = 2, r = 4\) and, from the Riemann-Hurwitz relation, \(m_2 = m_4 = m_3 = m_4 = n\).

(iii) \(l \neq 0, n-1\). \(N_1 = g^* + 1, N_{n-1} = g^*\). Thus \(a = 1, r = 3\). By the Riemann-Hurwitz relation, we have

\[
\frac{2(g-1)}{n} = 2(g^* - 1) + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{m_2}\right) + \left(1 - \frac{1}{m_3}\right).
\]

But the least common multiple of \(m_2\) and \(m_3\) must be \(n [1]\). So \(\frac{n}{m_2} = \frac{m_2}{(m_2, m_3)}\) and (iii) follows.
Finally, we note that the conditions given in the theorem are, with a small number of exceptions for low values of $g$ and $n$, not generally necessary for $P$ to be a Weierstrass point.

REFERENCES


2. H. Larcher, Weierstrass points at the cusps of $\Gamma_0(16p)$ and the hyperellipticity of $\Gamma_0(n)$, *Canad. J. Math.* 22 (1971), 960–968.

