ON VARIOUS TYPES OF BARRELLEDNESS AND THE HEREDITARY PROPERTY OF (DF)-SPACES

by T. HUSAIN† and YAU-CHUEN WONG

(Received 1 April, 1975; revised 20 October, 1975)

1. Introduction. Recently, Levin and Saxon [5], De Wilde and Houet [2] defined the $\sigma$-barrelledness while Husain [3] defined the countable barrelledness and countable quasi-barrelledness. It is well-known that barrelled spaces are countably barrelled, and countably barrelled spaces are $\sigma$-barrelled. It is natural to ask whether there is some condition for $\sigma$-barrelled (resp. countably barrelled) spaces to be countably barrelled (resp. barrelled). Using the concept of $S$-absorbent sequences of sets, we are able to give such conditions in Theorem 2.5 and Corollaries 2.6 and 2.7.

Valdivia [9], Saxon and Levin [8] have shown that every vector subspace with countable codimension of a barrelled space is barrelled. Also Levin and Saxon showed in [5] that this hereditary property is true for $\sigma$-barrelled spaces. In §3, we show that this hereditary property is also true for countably barrelled spaces as well as for $\sigma$-barrelled (DF)-spaces, which is a generalization of Valdivia [10, Theorem 3].

The final section is devoted to some properties of $S$-absorbent sequences of sets which extend some results of Valdivia [9], De Wilde and Houet [2].

2. The relationship between various types of barrelledness. Let $(E, T)$ be a Hausdorff locally convex space whose topological dual is denoted by $E'$. If $B$ is a subset of $E$ (resp. $E'$), then the polar of $B$, taken in $E'$ (resp. $E$), is denoted by $B^0$. By a topologizing family (t. family, for short) for $E'$ (resp. $E$) we mean a family $S$ consisting of (convex circled) $\sigma(E, E')$-bounded subsets of $E$ (resp. $\sigma(E', E)$-bounded subsets of $E'$) such that $\cup\{B: B \in S\} = E$ (resp. $E'$). For a t. family $S$ for $E'$ (resp. $E$), the topology on $E'$ (resp. $E$) of uniform convergence on $S$ is denoted by $T_S$.

Let $S$ be a t. family for $E'$. We denote by $S^b$ the family of all $T_S$-bounded subsets of $E'$. Clearly $S^b$ is again a t. family for $E$. The topology on $E$ of uniform convergence on $S$ is denoted by $T_S^b$, therefore we have $T_S^b = T_S$. Similarly we can define $S^{bb}$ and $T_S^{bb}$, where $S^{nb} = S^{bb\ldots b}$ and $T_S^{nb} = T_S^{bb\ldots b}$, the superscript $b$ being repeated $n$ times in each case; consequently we have $T_S^{nb} = T_S^{nb}$ for all $n \geq 1$.

If $S$ is a t. family for $E'$, let us say temporarily that $S^b$ (resp. $S^{bb}$) is the bounded-polar (resp. bounded-bipolar) family of $S$, and that $T_S^b$ (resp. $T_S^{bb}$) is the bounded-polar (resp. bounded-bipolar) topology of $T_S$. It is clear that $\{S^0: S \in S^b\}$ forms a neighbourhood base at $0$ for the bounded-polar topology $T_S^b$, and that $\{B^0: B \in S^{bb}\}$ forms a neighbourhood base at $0$ for the bounded-bipolar topology $T_S^{bb}$. If $S_1$ and $S_2$ are two t. families for $E'$ with $S_1 \subset S_2$, then $S_2^b \subset S_1^b$.

**Lemma 2.1.** For a t. family $S$ for $E'$, we have:

(a) $S \subset S^{bb}$;

† This research was supported by an N.R.C. grant.
(b) $T_S \subseteq T_S^{bb}$;
(c) $S^b = S^{3b}$ and $T_S^b = T_S^{3b}$.

The proof is straightforward and will be omitted. We shall see that the inclusion in (a) may be strict.

In the sequel we denote by $\beta(E, E')$ the strong topology on $E$, i.e., the topology of uniform convergence on all $\sigma(E', E)$-bounded subsets of $E'$, and by $\beta^*(E, E')$ the topology of uniform convergence on all $\beta(E', E)$-bounded subsets of $E'$. As usual, $\tau(E, E')$ denotes the Mackey topology on $E$. Clearly,

$$\tau(E, E') \leq \beta^*(E, E') \leq \beta(E, E')$$

and

$$\tau(E', E) \leq \beta^*(E', E) \leq \beta(E', E).$$

It is not hard to see that each $\sigma(E, E')$-bounded subset of $E$ is $\beta^*(E, E')$-bounded, and that dually each $\sigma(E', E)$-bounded subset of $E'$ is $\beta(E', E)$-bounded.

**Examples.** (1) If $S_f$ is the family of all finite subsets of $E$, then we have that $T_{S_f} = \sigma(E', E)$; $T_{S_f}^{bb} = \beta(E, E')$; $T_{S_f}^{bb} = \beta^*(E', E)$. Therefore we conclude that $S_f \neq S_f^{bb}$ and $T_{S_f} \neq T_{S_f}^{bb}$, in general.

(2) If $S_\beta$ is the family of all $\beta(E, E')$-bounded subsets of $E$, then we have $T_{S_\beta} = \beta^*(E', E)$, $S_\beta$ is the family of all $\sigma(E', E)$-bounded subsets of $E'$ and $T_{S_\beta} = \beta(E, E')$. Therefore we conclude that $S_\beta = S_\beta^{bb}$ and $T_{S_\beta} = T_{S_\beta}^{bb}$.

(3) If $S_\sigma$ is the family of all $\sigma(E, E')$-bounded subsets of $E$, then we have $T_{S_\sigma} = \beta(E', E)$, $T_{S_\sigma}^{bb} = \beta^*(E', E')$ and $S_\sigma^{bb}$ is the family of all $\sigma(E, E')$-bounded subsets of $E$.

(4) Let $S_c$ be the family of all $T$-compact convex circled subsets of $E$ and let $c(E', E)$ be the topology on $E'$ of uniform convergence on $S_c$. Then $\sigma(E', E) = \alpha(E', E) \leq \tau(E', E)$; furthermore we have $T_{S_c} = c(E', E)$, $T_{S_c}^{bb} = \beta(E, E')$ and $T_{S_c}^{bb} = \beta^*(E', E)$.

**Definition 2.2.** Let $(E, T)$ be a locally convex space and $S$ a t. family for $E'$. Then $E$ is said to be

1. **$S$-barrelled** if each member in $S^b$ is $T$-equicontinuous;
2. **countably $S$-barrelled** if each member of $S^b$ which is the countable union of $T$-equicontinuous subsets of $E$ is $T$-equicontinuous;
3. **$\sigma$-$S$-barrelled** if each member in $S^b$ which is a countable set is $T$-equicontinuous.

If $S$ is the family of all finite subsets of $E$, then $E$ is $S$-barrelled (resp. countably $S$-barrelled, $\sigma$-$S$-barrelled) if and only if it is barrelled (resp. countably barrelled, $\sigma$-barrelled) under the usual terminology of [4] and [6] (resp. [3], [2]). $\sigma$-barrelled spaces are also called $\omega$-barrelled by Levin and Saxon [5]. Clearly each $\beta(E', E)$-bounded set is in $S^b$ for any t. family for $E'$. Hence $E$ is quasibarrelled (or countably quasibarrelled or $\sigma$-evaluable) if $E$ is $S$-barrelled (or countably $S$-barrelled or $\sigma$-$S$-barrelled).

If $S$ is the family of all $\sigma(E, E')$-bounded subsets of $E$, then $E$ is $S$-barrelled (resp. countably $S$-barrelled, $\sigma$-$S$-barrelled) if and only if $E$ is quasibarrelled (resp. countably...
quasibarrelled, \(\sigma\)-evaluable) under the usual terminology of [4] [6] (resp. [3], [2]). Here we call \(\sigma\)-evaluable spaces \(\sigma\)-infrabarrelled.

If \((E, T)\) is a locally convex Riesz space and if \(S\) is the family of all order-bounded subsets of \(E\), then \(E\) is \(S\)-barrelled if and only if it is order-infrabarrelled under the usual definition of [11].

As a consequence of Lemma 2.1, we have the following result.

**Lemma 2.3.** Let \(S\) be a t. family for \(E'\). \(E\) is \(S\)-barrelled (resp. countably \(S\)-barrelled, \(\sigma\)-\(S\)-barrelled) if and only if \(E\) is \(S^{bb}\)-barrelled (resp. countably \(S^{bb}\)-barrelled, \(\sigma\)-\(S^{bb}\)-barrelled).

In particular, \((E, T)\) is barrelled if and only if each \(\sigma(E, E')\)-closed convex circled subset of \(E\) which absorbs all \(\beta(E, E')\)-bounded subsets of \(E\) is a \(T\)-neighbourhood of 0.

Using a standard argument, for instance, see Schaefer [6] and Köthe [4, p. 396], it is easily seen that \(E\) is \(S\)-barrelled if and only if each closed convex circled subset of \(E\) which absorbs all members of \(S\) is a \(T\)-neighbourhood of 0, and that \(E\) is countably \(S\)-barrelled if and only if for any sequence \((V_n)\) of closed convex circled \(T\)-neighbourhoods of 0, if \(V = \bigcap_{n=1}^{\infty} V_n\) absorbs all members in \(S\) then \(V\) is a \(T\)-neighbourhood of 0.

In order to give a dual characterization of the \(\sigma\)-\(S\)-barrelledness, we require the following terminology. Let \(S\) be a t. family for \(E'\). By an \(S\)-absorbent sequence (of closed sets) in \(E\) we mean a sequence \(\{V_n: n \geq 1\}\) of (closed) convex circled sets in \(E\) for which the following two conditions are satisfied:

(i) \(V_n \subset V_{n+1}\) for all \(n \geq 1\);

(ii) each member in \(S\) is absorbed by some \(V_n\).

If \(S\) is the family of all finite subsets of \(E\), then \(\{V_n: n \geq 1\}\) is an \(S\)-absorbent sequence if and only if it is an absorbent sequence in \(E\) in the sense of [2]; and if \(S\) is the family of all \(\sigma(E, E')\)-bounded subsets of \(E\), then \(\{V_n: n \geq 1\}\) is a \(\sigma\)-absorbent sequence if and only if it is a bounded-absorbent sequence in the sense of [2].

**Proposition 2.4.** Let \(S\) be a t. family for \(E'\). Then \(E\) is \(\sigma\)-\(S\)-barrelled if and only if for any \(S\)-absorbent sequence \(\{V_n: n \geq 1\}\) in \(E\), the sequence \(\{f_n: n \geq 1\}\) is equicontinuous, where \(f_n \in V_n^0\) for all \(n \geq 1\).

**Proof.** Necessity. For any \(S \in S\) there exists \(\lambda > 0\) and \(n_0 > 1\) such that \(S \subset \lambda V_n\) for all \(n \geq n_0\). For each \(n \geq 1\), let \(f_n \in V_n^0\). Then \(|f_n(x)| \leq \lambda\) for all \(x \in S\) and \(n \geq n_0\). Since \(S\) is \(\sigma(E, E')\)-bounded, there exists \(\mu > 0\) with \(|f_n(x)| \leq \mu\) for all \(x \in S\) and \(n = 1, \ldots, n_0 - 1\). Thus \(\sup \{|f_n(x)|: x \in S, n \geq 1\} \leq \max(\lambda, \mu) < \infty\) and so \(\{f_n: n \geq 1\} \in S^b\). Hence by hypothesis \(\{f_n: n \geq 1\}\) is equicontinuous.

Sufficiency. Let \(\{h_n: n \geq 1\}\) be a \(T_S\)-bounded sequence in \(E'\). For each \(k \geq 1\), we define

\[V_k = \{x \in E: |h_n(x)| \leq 1\ \text{for all}\ n \geq k\}.
\]

Then \(\{V_n: n \geq 1\}\) is an \(S\)-absorbent sequence in \(E\). As \(h_k \in V_k^0\) for all \(k \geq 1\), we conclude from the hypothesis that \(\{h_k: k \geq 1\}\) is equicontinuous. This shows that \(E\) is \(\sigma\)-\(S\)-barrelled.
If $S_1$ and $S_2$ are two t. families for $E'$ such that $S_1 \subset S_2$, then the following implications hold:

\[
S_1\text{-barrelledness} \Rightarrow \text{countably } S_1\text{-barrelledness} \Rightarrow \sigma\text{-}S_1\text{-barrelledness} \downarrow \downarrow \downarrow
\]
\[
S_2\text{-barrelledness} \Rightarrow \text{countably } S_2\text{-barrelledness} \Rightarrow \sigma\text{-}S_2\text{-barrelledness}.
\]

Therefore it is natural to ask under what conditions on $E$ (or $E'$) the corresponding converse implications hold. We have the following result.

**Theorem 2.5.** Let $S_1$ and $S_2$ be two t. families for $E'$ such that $S_1 \subset S_2$. Then $(E, T)$ is $\sigma\text{-}S_1\text{-barrelled}$ (resp. countably $S_1\text{-barrelled}$, $S_1\text{-barrelled}$) if and only if the following two conditions hold:

(i) $E$ is $\sigma\text{-}S_2\text{-barrelled}$ (resp. countably $S_2\text{-barrelled}$, $S_2\text{-barrelled}$);

(ii) each $S_1\text{-absorbent}$ sequence of closed sets in $E$ is $S_2\text{-absorbent}$.

**Proof.** Suppose that $E$ is $\sigma\text{-}S_1\text{-barrelled}$ and that $\{V_n: n \geq 1\}$ is an $S_1\text{-absorbent}$ sequence of closed sets in $E$ which is not $S_2\text{-absorbent}$. Then there exists $B \in S_2$ such that $B \subset nV_n$ is false for all natural numbers $n \geq 1$. For each $n \geq 1$, let $x_n$, in $B$, be such that $x_n \notin nV_n$. As $V_n$ is closed convex and circled, the bipolar theorem ensures that there exists $f_n \in V_n^0$ such that

\[
|f_n(x_n)| > n.
\]

As $E$ is $\sigma\text{-}S_1\text{-barrelled}$ and $\{V_n: n \geq 1\}$ is an $S_1\text{-absorbent}$ sequence of closed sets in $E$, it follows from Proposition 2.4 that $\{f_n: n \geq 1\}$ is $T\text{-equicontinuous}$ sequence, and hence that $\{f_n: n \geq 1\}$ is $T_{S_2}\text{-bounded}$; consequently $\{f_n: n \geq 1\}$ must be absorbed by $B^0$, contrary to the inequality (1). Therefore the conditions are necessary. We show that the conditions are also sufficient.

Let $\{f_n: n \geq 1\}$ be a $T_{S_1}\text{-bounded}$ sequence in $E'$. For each $k \geq 1$, let

\[
V_k = \{x \in E: |f_n(x)| \leq 1 \text{ for all } n \geq k\}.
\]

The $T_{S_1}\text{-boundedness}$ of $\{f_n: n \geq 1\}$ ensures that $\{V_n: n \geq 1\}$ is an $S_1\text{-absorbent}$ sequence of closed sets in $E$, and hence $\{V_n: n \geq 1\}$ is $S_2\text{-absorbent}$ by the hypotheses. On the other hand, since $E$ is assumed to be $\sigma\text{-}S_2\text{-barrelled}$ and since $f_n \in V_n^0$ for all $n \geq 1$, it follows from Proposition 2.4 that $\{f_n: n \geq 1\}$ is $T\text{-equicontinuous}$, and hence that $E$ is $\sigma\text{-}S_2\text{-barrelled}$.

The necessity part of the proof for countably $S_1\text{-barrelled}$ and $S_1\text{-barrelled}$ spaces is similar and so is omitted. The sufficiency part for all cases can be handled as follows. Observe that $S_1^b \supset S_2^b$. To show that (ii) implies $S_1^b = S_2^b$, let $A \in S_1^b$ and $A \notin S_2^b$. Then there is $B \in S_2$, a sequence $\{x_n\} \subset B$ and a sequence $\{f_n\} \subset A$ such that $|f_n(x_n)| > n$ for all $n \geq 1$. Since $V_n = \{x \in E: |f_m(x)| \leq 1 \text{ for } m \geq n\}$ is an $S_1\text{-absorbent}$ sequence of closed sets in $E$, it follows by (ii) that it is also $S_2\text{-absorbent}$. Hence there exist $n$ and $\lambda$ such that $B \subset \lambda V_n$, a contradiction.

**Remark.** $E$ is $\sigma\text{-barrelled}$ (resp. countably barreled, barreled) if and only if it is $\sigma\text{-infrabarrelled}$ (resp. countably quasibarrelled, infrabarrelled) and each absorbent sequence of closed sets in $E$ is bounded-absorbent.
COROLLARY 2.6. Let $S_1$ and $S_2$ be two $t$ families for $E'$ such that $S_1 \subseteq S_2$. Then:

(a) $E$ is countably $S_1$-barrelled if and only if it is countably $S_2$-barrelled as well as $\sigma$-$S_1$-barrelled;

(b) $E$ is $S_1$-barrelled if and only if it is $\sigma$-$S_1$-barrelled as well as $S_2$-barrelled.

Proof. If $E$ is countably $S_1$-barrelled, then it is obvious that $E$ is countably $S_2$-barrelled as well as $\sigma$-$S_1$-barrelled. Conversely, if $E$ is countably $S_2$-barrelled and if $E$ is $\sigma$-$S_1$-barrelled, then by Theorem 2.5, each $S_1$-absorbent sequence of closed sets in $E$ is $S_2$-absorbent. We conclude from Theorem 2.5 again that $E$ is countably $S_1$-barrelled. This proves the assertion (a). The proof of (b) is similar.

REMARK. $E$ is countably barrelled if and only if it is $\sigma$-barrelled and countably quasi-barrelled. $E$ is barrelled if and only if it is countably barrelled and quasibarrelled.

COROLLARY 2.7. Let $S_1$ and $S_2$ be two $t$ families for $E'$ such that $S_1 \subseteq S_2$. Then the following assertions hold.

(a) Let $E$ be countably $S_2$-barrelled. Then $E$ is countably $S_1$-barrelled if and only if each $S_1$-absorbent sequence of closed sets in $E$ is $S_2$-absorbent.

(b) Let $E$ be $S_2$-barrelled. Then $E$ is $S_1$-barrelled if and only if $E$ is $\sigma$-$S_1$-barrelled, and this is the case if and only if each $S_1$-absorbent sequence of closed sets in $E$ is $S_2$-absorbent.

Proof. (a) follows from Theorem 2.5 and Corollary 2.6 (a), while (b) follows from Corollary 2.6 (b) and the assertion (a) of this corollary.

Let $E$ be a locally convex space. A convex circled $\sigma(E, E')$-bounded subset $B$ of $E$ is said to be infracomplete if the normed space $E(B) = \bigcup_n nB$ equipped with the norm $\|x\|_B = \inf \{\lambda \geq 0 : x \in \lambda B\}$ ($x \in E(B)$) is complete. It is clear that every convex circled $\sigma(E, E')$-bounded and $\tau(E, E')$-sequentially complete subset of $E$ is infracomplete. By the Banach–Mackey theorem, we see that every infracomplete subset $B$ of $E$ is $\beta(E, E')$-bounded (see [4, §20, 11(3)]).

Levin and Saxon [5] say that a locally convex space $E$ has the property (C) (resp. the property (S)) if every $\sigma(E', E)$-bounded subset of $E'$ is $\sigma(E', E)$-relatively countably compact (resp. $E'$ is $\sigma(E', E)$-sequentially complete). As a consequence of the result mentioned above ([4, §20, 11(3)]), we obtain the following result which gives a connection between $\sigma$-barrelledness and the property (S).

PROPOSITION 2.8. For a $\sigma$-infrabarrelled locally convex space $E$, the following statements are equivalent:

(a) $E$ is $\sigma$-barrelled;

(b) $E$ has the property (C);

(c) $E$ has the property (S);

(d) each $\sigma(E', E)$-bounded, $\sigma(E', E)$-closed subset of $E'$ is $\sigma(E', E)$-sequentially complete.
Proof. The implications (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) are obvious. Finally, if the statement (d) holds, then by the Banach–Mackey theorem each \(\sigma(E', E)\)-bounded subset of \(E'\) is \(\beta(E', E)\)-bounded, and thus the implication (d) \Rightarrow (a) follows.

Consider the vector space \(m\) of all bounded sequences with the Mackey topology \(\tau(m, l_1)\). Levin and Saxon have shown in [5, Proposition 6] that \((m, \tau(m, l_1))\) is a Mackey space with the property (S) but not property (C). According to this result and Proposition 2.8, we conclude that Mackey spaces are, in general, not \(\sigma\)-infrabarrelled spaces.

As another consequence of the Banach–Mackey theorem, we have the following result.

**Proposition 2.9.** Let \(E\) be a locally convex space for which every \(\sigma(E', E)\)-bounded closed set is \(\sigma(E', E)\)-sequentially complete (equivalently, \(E\) has the property (S)). Then the following assertions hold.

1. If \(E\) is infrabarrelled (in particular, bornological) then it is barrelled.
2. If \(E\) is countably infrabarrelled then it is countably barrelled.

Proof. According to the Banach–Mackey theorem each \(\sigma(E', E)\)-bounded subset of \(E'\) is \(\beta(E', E)\)-bounded, and the result follows.

Since metrizable locally convex spaces are infrabarrelled, part (1) of the preceding result is a generalization of Saxon [7, Theorem 2.7]. The following corollary is now immediate.

**Corollary 2.10.** Let \(E\) be a locally convex space in which every \(\sigma(E, E')\)-bounded closed set is \(\tau(E, E')\)-sequentially complete (in particular, \(E\) is either \(\tau(E, E')\)-sequentially complete or quasi-complete). Then the following assertions hold.

1. If \(E\) is \(\sigma\)-infrabarrelled then \(E\) is \(\sigma\)-barrelled and a fortiori has the property (S).
2. If \(E\) is countably infrabarrelled (resp. barrelled) then it is countably barrelled (resp. barrelled).

3. The hereditary property. Saxon, Levin [8] and Valdivia [9] have shown independently that a vector subspace with countable codimension of a barrelled space is barrelled. Also Saxon and Levin [5] have shown that a vector subspace with countable codimension of a \(\sigma\)-barrelled space is \(\sigma\)-barrelled. The same is true for countably barrelled spaces as shown by Webb [12]. We give a different and direct proof of this fact.

**Theorem 3.1.** Let \(M\) be a countable codimensional vector subspace of a countably barrelled space \(E\). Then \(M\) is countably barrelled when furnished with the relative topology.

**Proof.** In our proof we consider three cases.

1. \(M\) is dense in \(E\). In this case, the topological dual \(M'\) of \(M\) can be canonically identified with \(E'\). Let \(S\) be a \(\sigma(M', M)\)-bounded subset of \(M'\) and let \(\{S_n : n \geq 1\}\) be a sequence of equicontinuous subsets of \(M'\) for which \(S = \bigcup_{n=1}^{\infty} S_n\). Since \(M\) is dense in \(E\), it follows from [5, Lemma 2] that \(S\) is \(\sigma(E', E)\)-bounded. Further we show that each \(S_n\) is an equicontinuous subset of \(E'\).
In fact, let $S^0_n$ denote the polar of $S_n$ taken in $E$. Since $S_n$ is an equicontinuous subset of $M'$, $S^0_n \cap M$ is a 0-neighbourhood in $M'$; then there exists an open 0-neighbourhood $U_n$ in $E$ such that $U_n \cap M \subset S^0_n \cap M$. The density of $M$ ensures that $U_n \subset \overline{U_n \cap M} \subset S^0_n$, and hence $S_n$ is an equicontinuous subset of $E'$.

Now the countable barrelledness of $E$ implies that $S$ is an equicontinuous subset of $E'$ and surely an equicontinuous subset of $M'$. This shows that $M$ is countably barrelled.

(b) $M$ is closed in $E$. Let $N$ be any algebraic complement to $M$ in $E$. Since countably barrelled spaces are $\sigma$-barrelled, it follows from [7, Theorem 1.1] that $N$ is a topological complement and has the strongest locally convex topology. Hence $N$ is closed in $E$, and $E/N$ are topologically isomorphic. Since $E$ is countably barrelled, by [3, Corollary 14], $E/N$ is countably barrelled and therefore $M$ must be countably barrelled.

(c) General case. Since $\overline{M}$ is a closed vector subspace of $E$ with countable codimension, it follows from (b) that $\overline{M}$ is countably barrelled. As $M$ is dense in $\overline{M}$, we conclude from (a) that $M$ is countably barrelled. This completes the proof of the theorem.

**Corollary 3.2.** Let $E$ be a $\sigma$-barrelled (DF)-space. Then any vector subspace $M$ of $E$ with countable codimension is a countably barrelled (DF)-space.

**Proof.** By Corollary 2.6, $E$ is a countably barrelled (DF)-space, and hence $M$ is a countably barrelled space by the preceding theorem. Since $E$ has a countable fundamental system of bounded sets, and since $M$ is a subspace, it follows that $M$ contains a countable fundamental system of bounded subsets of $M$. Therefore $M$ is a countably barrelled (DF)-space.

The preceding result was proved by Valdivia [10, Theorem 3] in the special case when $E$ is barrelled.

**4. Various types of absorbent sequences.** Let $E$ be a vector space. By an increasing sequence of sets in $E$ we mean a sequence $\{V_n : n \geq 1\}$ of convex circled subsets of $E$ such that $V_n \subset V_{n+1}$ for all $n \geq 1$. Let $\{V_n : n \geq 1\}$ be an increasing sequence of sets in $E$. It is clear that $\{nV_n : n \geq 1\}$ is an increasing sequence of sets in $E$, and that if $E$ is a locally convex space then $\{V_n : n \geq 1\}$ is also an increasing sequence of sets in $E$, where $V_n$ is the closure of $V_n$. An increasing sequence $\{V_n : n \geq 1\}$ of sets in $E$ is called an increasing sequence of $(P)$ sets in $E$ if each $V_n$ has the property $(P)$; for instance, $\{V_n : n \geq 1\}$ is an increasing sequence of closed (resp. complete, compact, metrizable etc.) sets in $E$ if each $V_n$ is closed (resp. complete, compact, metrizable etc.).

It is known from §2 that the concept of $S$-absorbent sequences is useful for studying the relationship between various types of barrelledness. It is not hard to give an example of an increasing sequence of sets in $E$ which is not $S$-absorbent. Therefore it is interesting to find some sufficient and necessary condition to ensure that increasing sequences are $S$-absorbent.

**Proposition 4.1.** Let $S$ be a $t.$ family for $E'$ and suppose that $\{V_n : n \geq 1\}$ is an increasing sequence of closed sets in $E$. Then it is an $S$-absorbent sequence if and only if for any $f_n \in V_n^0$ ($n \geq 1$), the sequence $\{f_n : n \geq 1\}$ is $T_S$-bounded.
Proof. Suppose that \( \{ V_n : n \geq 1 \} \) is \( S \)-absorbent and that \( \{ f_n : n \geq 1 \} \) is not \( T_S \)-bounded for some \( f_n \in V_n^0 \) (\( n \geq 1 \)). Then there exists \( B \in S \) such that \( \{ f_n : n \geq 1 \} \subseteq k^2 B^0 \) is false for all natural numbers \( k \geq 1 \). For each \( k \geq 1 \), there exists \( n_k \) such that \( f_{n_k} \notin k^2 B^0 \). On the other hand, since \( \{ V_n : n \geq 1 \} \) is \( S \)-absorbent, there exists \( \lambda > 0 \) and \( n_0 \geq 1 \) such that

\[
V_n^0 \subseteq V_{n_0}^0 \subseteq \lambda B^0 \quad \text{for all} \quad n \geq n_0,
\]

it then follows that \( f_n \in \lambda B^0 \) for all \( n \geq n_0 \), which contradicts the fact that \( f_{n_0} \notin k^2 B^0 \). Therefore the condition is necessary.

Conversely, if \( \{ V_n : n \geq 1 \} \) is not an \( S \)-absorbent sequence, then there exists \( B \in S \) such that \( B \subseteq n V_n \) is false for all natural numbers \( n \geq 1 \). For each \( n \), let \( x_n \in B \setminus (n V_n) \) and let \( f_n \) in \( V_n^0 \), be such that \( |f_n(x_n)| > n \). Then the sequence \( \{ f_n : n \geq 1 \} \) is not \( T_S \)-bounded. This completes the proof.

In the sequel we always assume that \( E \) is a locally convex space and that \( S \) is a topologizing family for \( E' \). If \( S_1 \) is another topologizing family for \( E' \) such that \( S \subseteq S_1 \), then each \( S_1 \)-absorbent sequence in \( E \) must be \( S \)-absorbent. The converse is true for \( S_1 = S^{bb} \) as the following result shows.

Corollary 4.2. \( \{ V_n : n \geq 1 \} \) is an \( S \)-absorbent sequence of closed sets in \( E \) if and only if it is an \( S^{bb} \)-absorbent sequence.

Proof. This follows from Proposition 4.1 and Lemma 2.1.

The preceding result was proved by De Wilde and Houet [2, Theorem 1] in the case when \( S \) is the family of all finite subsets of \( E \).

Corollary 4.3. Let \( S_1 \) and \( S_2 \) be two t. families for \( E' \) such that \( S_1 \subseteq S_2 \). Then the following statements are equivalent:

(i) each \( T_{S_1} \)-bounded subset of \( E' \) is \( T_{S_2} \)-bounded;

(ii) each \( S_1 \)-absorbent sequence of closed sets in \( E \) is \( S_2 \)-absorbent.

Proof. The implication (i) \( \Rightarrow \) (ii) follows from Proposition 4.1, while the implication (ii) \( \Rightarrow \) (i) has been observed in Theorem 2.5.

When \( S_1 \) is the family of all finite subsets of \( E \) and \( S_2 \) is the family of all \( \sigma(E, E') \)-bounded subsets of \( E \), then the implication (i) \( \Rightarrow \) (ii) in the preceding result was proved by Valdivia [9, Theorem 6] in the case when \( E \) is barrelled, and was proved by De Wilde and Houet [2, Corollary 1] in the case when \( E \) is \( \sigma \)-barrelled.

By making use of Theorem 2.5, for a \( \sigma \)-barrelled space \( E \), each \( \sigma(E', E) \)-bounded subset of \( E' \) is \( \beta(E', E) \)-bounded.

Corollary 4.4. Let \( S_1 \) and \( S_2 \) be two t. families for \( E' \) such that \( S_1 \subseteq S_2 \), and let \( E \) satisfy one of the equivalent conditions (i) and (ii) of Corollary 4.3. If \( S_2 \) has a sequence \( \{ B_n : n \geq 1 \} \) such that each member of \( S_1 \) is absorbed by some \( B_n \), then the saturated hull (\([6] \), p. 81) of \( S_2 \) contains a countable fundamental subfamily.

Proof. For each \( n \), let \( V_n \) be the closed convex circled hull of \( \bigcup_{j=1}^{n} B_j \). Then \( V_n \) is in the saturated hull of \( S_2 \), and \( \{ V_n : n \geq 1 \} \) is an \( S_1 \)-absorbent sequence of closed sets in \( E \), so by
the hypothesis, \( \{V_n : n \geq 1\} \) is \( S_2 \)-absorbent. Consequently \( \{nV_n : n \geq 1\} \) is a countable fundamental subfamily of the saturated hull of \( S_2 \) because a member of the saturated hull of \( S_2 \) is either a subset, scalar multiple or an absolute convex hull of a finite number of elements of \( S_2 \).

REMARK. If \( E \) is a countably barrelled space with a sequence \( \{B_n : n \geq 1\} \) of bounded sets such that \( \bigcup_{n=1}^{\infty} B_n \) is absorbing, then \( E \) is a \( (DF) \)-space.

Corollary 4.4 was proved by Valdivia [9, Corollary 2.6] in the case when \( E \) is barrelled. A trivial modification of De Wilde and Houet's argument in [2] yields the following more general result, but for completeness we shall give the entire proof.

**Theorem 4.5.** Let \( E \) be a \( \sigma \)-\( S \)-barrelled space and \( \{V_n : n \geq 1\} \) an \( S \)-absorbent sequence in \( E \). Then

\[
\bigcup_{m=1}^{\infty} V_m \subset (1 + \varepsilon) \bigcup_{m=1}^{\infty} V_m \quad \text{for all} \quad \varepsilon > 0.
\]

**Proof.** If \( x \notin (1 + \varepsilon) \bigcup_{m}^{\infty} V_m \) for some \( \varepsilon > 0 \), then \( x \notin \bigcup_{m}^{\infty} V_m \) for all \( m \geq 1 \), and thus, for any \( m \geq 1 \), there exists \( f_m \in V_m^0 \) such that \( f_m(x) > 1 + \varepsilon \). Since \( E \) is \( \sigma \)-\( S \)-barrelled, by Proposition 2.4, \( \{f_m : m \geq 1\} \) has a \( \sigma(E', E) \)-cluster point \( f \), say, in \( E' \); hence \( f(x) \geq 1 + \varepsilon \). On the other hand, since \( V_n \) is increasing and \( f_n \in V_n^0 \), it follows that \( f \in V_n^0 \) for all \( n \geq 1 \) or, equivalently \( f \in \bigcap_{n \geq 1} V_n^0 = \left( \bigcup_{n \geq 1} V_n \right)^0 \). However the inequality \( f(x) \geq 1 + \varepsilon \) shows that \( x \notin \bigcup_{m}^{\infty} V_m \).

This completes the proof.

REMARKS. (1) As De Wilde in [1, p. 212] pointed out, the condition in Theorem 4.5 that \( E \) be \( \sigma \)-\( S \)-barrelled can be replaced by the following condition: \( \{V_n : n \geq 1\} \) is an \( S \)-absorbent sequence in \( E \) such that for each \( f_n \in V_n^0 \) \( (n \geq 1) \), the sequence \( \{f_n : n \geq 1\} \) is equicontinuous.

(2) According to the preceding theorem, Corollaries 2.a–2.d in [2] hold for a \( \sigma \)-\( S \)-barrelled space.

ACKNOWLEDGEMENT. The authors are grateful to the referee for many helpful comments.

REFERENCES


Prof. T. Husain

**DEPARTMENT OF MATHEMATICS**

**McMASTER UNIVERSITY**

**HAMILTON, ONTARIO**

**CANADA**

Dr. Yau-Chuen Wong

**DEPARTMENT OF MATHEMATICS**

**UNITED COLLEGE**

**CHINESE UNIVERSITY OF HONG KONG**

**HONG KONG**