ON IMAGES AND INVERSE IMAGES OF WEIERSTRASS POINTS†

by R. F. LAX

(Received 29 November, 1976)

1. Introduction. The classical theory of Weierstrass points on a compact Riemann surface is well-known (see, for example, [3]). Ogawa [6] has defined generalized Weierstrass points. Let $Y$ denote a compact complex manifold of (complex) dimension $n$. Let $E$ denote a holomorphic vector bundle on $Y$ of rank $q$. Let $J^k(E)$ ($k = 0, 1, \ldots$) denote the holomorphic vector bundle of $k$-jets of $E$ [2, p. 112]. Put $r_k(E) = \text{rank } J^k(E) = q \cdot (n + k)!/n!k!$. Suppose that $\Gamma(E)$, the vector space of global holomorphic sections of $E$, is of dimension $\gamma(E) > 0$. Consider the trivial bundle $Y \times \Gamma(E)$ and the map

$$j_k = j^E_k: Y \times \Gamma(E) \to J^k(E),$$

which at a point $Q \in Y$ takes a section of $E$ to its $k$-jet at $Q$. Put $\mu = \min(\gamma(E), r_k(E))$.

Definition ([6]). Let $W_k(E)$ denote the reduced closed analytic subspace of $Y$ defined by the vanishing of the exterior power $\Lambda^* j_k$. We call $W_k(E)$ the space of $k$th order Weierstrass points of $E$.

Note that if $r_k(E) \leq \gamma(E)$, then the points of $W_k(E)$ are those $Q \in Y$ such that the map $j_{k,Q}: \Gamma(E) \to J^k_Q(E)$ is not surjective and if $r_k(E) \geq \gamma(E)$, then the points of $W_k(E)$ are those $Q \in Y$ such that $j_{k,Q}$ is not injective.

Suppose that $X$ is a compact complex manifold and $f: X \to Y$ is a holomorphic map. Let $P \in X$ and put $Q = f(P)$. Then we have the following commutative diagram of $\mathbb{C}$-linear maps.

$$\begin{align*}
\Gamma(Y, E) & \xrightarrow{\alpha} J^k_Q(E) \\
\downarrow \varphi & \downarrow \psi \\
\Gamma(X, f^*E) & \xrightarrow{\beta} J^k_{f,P}(f^*E)
\end{align*}$$

Here $\alpha$ and $\beta$ are the maps $j^E_{k,Q}$ and $j^{f*E}_{k,P}$ respectively, and $\varphi$ and $\psi$ are induced by composition with $f$ (cf. [2, p. 113]). We will use this diagram to show that under certain conditions on $f$ and $E$, we have $f(W_k(f^*E)) \subseteq W_k(E)$ or $f^{-1}(W_k(E)) \subseteq W_k(f^*E)$ as sets.

† This research was supported by NSF Grant MCS 76-06002.

2. An inverse image theorem.

(2.1) Theorem. With notation as in §1, suppose that the following three conditions are satisfied:

(i) \( \gamma(E) \leq n_k(E) \),
(ii) \( \gamma(f^*E) \leq n_k(f^*E) \),
(iii) \( f \) is surjective.

Then \( f^{-1}(W_k(E)) \subseteq W_k(f^*E) \) as sets.

Proof. Let \( P \in X \), \( Q = f(P) \) and suppose that \( Q \in W_k(E) \). Since \( f \) is surjective, the map \( \varphi \) in diagram (*) is injective. Hence, if \( \beta \) is injective, then \( \alpha \) must be injective. But, since \( Q \in W_k(E) \), the map \( \alpha \) is not injective. Thus, \( \beta \) is not injective and \( P \in W_k(f^*E) \).

(2.2) We may apply this theorem in the following situation. Let \( Y \) be a compact Riemann surface of genus \( g \) and put \( X = Y^n \). Let \( T_Y \) (resp. \( T_X \)) denote the complex analytic cotangent bundle on \( Y \) (resp. on \( X \)). Then \( \gamma(T_Y) = g \) and \( \gamma(T_X) = \frac{1}{2} \dim H^1(X, \mathbb{C}) = ng \), using the Kunneth formula and standard facts about Kahler manifolds [4, p. 124].

Let \( \pi_i : X \to Y \) denote projection on the \( i \)th factor. It follows that we have an injection \( 0 \to \pi_i^*T_Y \to T_X \) and hence that \( \gamma(\pi_i^*T_Y) \leq \gamma(T_X) = ng \). Suppose \( g \geq 3 \). Now, \( \gamma(T_Y) = r_{g-1}(T_Y) \) and it is not hard to see that \( \gamma(\pi_i^*T_Y) \leq ng \leq r_{g-1}(\pi_i^*T_Y) \). Hence, by our theorem,

\[
\pi_i^{-1}(W_{g-1}(T_Y)) \subseteq W_{g-1}(\pi_i^*T_Y).
\]

Note that the points of \( W_{g-1}(T_Y) \) are the classical Weierstrass points of \( Y \). Also, since we have the injection \( 0 \to \pi_i^*T_Y \to T_X \), we obtain

\[
\pi_i^{-1}(W_{g-1}(T_Y)) \subseteq W_{g-1}(T_X)
\]

by arguing as in the proof of 2.1 and noting that \( \gamma(T_X) \leq r_{g-1}(T_X) \).

(2.3) Remark. With notation as in 2.2, we have shown in [5] that

\[
W_{g-1}(\Lambda^nT_X) = \bigcup_{i=1}^n \pi_i^{-1}(W_{g-1}(T_Y)).
\]

Thus, \( W_{g-1}(\Lambda^nT_X) \subseteq W_{g-1}(T_X) \). Is there any reason for this relationship?

(2.4) We give an example here to show that the inverse image of a Weierstrass point is not always a Weierstrass point. Let \( f : X \to Y \) be a smooth (i.e. unramified) 2-sheeted cover with both \( X \) and \( Y \) hyperelliptic, the latter of genus \( g \) \((\geq 2)\). Such a covering is explicitly constructed in [7, pp. 188 and 203]. By the Riemann–Hurwitz formula, \( X \) has
Let \( T_X \) (resp. \( T_Y \)) denote the canonical bundle on \( X \) (resp. on \( Y \)). Since \( f \) is an unramified cover, it follows from [1, VI, 4.9] that \( f^*T_Y = T_X \). Now, \( W_{g-1}(T_Y) \) consists of \( 2g+2 \) points, while \( W_{g-1}(T_X) \subseteq W_{2g-2}(T_X) \), which contains only \( 4g \) points. Thus \( f^{-1}(W_{g-1}(T_Y)) \) cannot be a subset of \( W_{g-1}(T_X) \). Note that condition (ii) of 2.1 is violated.

### 3. An image theorem.

(3.1) **Theorem.** With notation as in §1, suppose that the following three conditions are satisfied:

(i) \( \gamma(E) \geq r_k(E) \),

(ii) \( \gamma(f^*E) \geq r_k(f^*E) \),

(iii) \( f \) is a local biholomorphism at \( P \in X \).

Then \( P \in W_k(f^*E) \) implies \( f(P) \in W_k(E) \).

**Proof.** Put \( Q = f(P) \). Since \( f \) is a local biholomorphism at \( P \), the map \( \psi \) in diagram (*) is onto. Now, if \( Q \notin W_k(E) \), then \( \alpha \) is onto; hence \( \beta \) must be onto and \( P \notin W_k(f^*E) \).

(3.2) **Corollary.** Let \( X \) (resp. \( X' \)) be a compact Riemann surface of genus \( g \) (resp. \( g' \)) and canonical bundle \( T \) (resp. \( T' \)). Let \( f : X \to X' \) be a smooth cover. Then \( f(W_k(T)) \subseteq W_k(T') \) for \( k \leq g'-1 \). In particular, if \( X \) is hyperelliptic, then \( X' \) is hyperelliptic.

**Proof.** Since \( f \) is unramified, \( f^*T' = T \). The first statement is then an easy consequence of the theorem. The second statement uses the fact that \( X \) is hyperelliptic if and only if \( X \) has a hyperelliptic Weierstrass point ([3, p. 228]), which is equivalent to \( W_1(T) \) being nonempty.

(3.3) **Remark.** As has been pointed out to me by R. D. M. Accola and A. Nobile, 3.2 may be proved by elementary methods and without the assumption that \( f \) is unramified. The idea is as follows. Let \( L \) (resp. \( L' \)) denote the field of meromorphic functions on \( X \) (resp. on \( X' \)). Then if \( P \in W_k(T) \) and \( h \in L \) has a pole of order at most \( k+1 \) at \( P \) and no other poles, then \( \text{Tr}_{L.K}(h) \), the trace of \( h \), will be an element of \( L' \) with pole of order at most \( k+1 \) at \( f(P) \) and no other poles.

### References


**LOUISIANA STATE UNIVERSITY,**
**BATON ROUGE, LOUISIANA 70803,**
**U.S.A.**