SEMIGROUPS SATISFYING MINIMAL CONDITIONS II

by T. E. HALL and W. D. MUNN

(Received 2 December, 1977)

In this paper we continue the investigation of minimal conditions on semigroups begun by J. A. Green [3] and taken up by Munn [5]. A unified account of the results in [3] and [5], together with some additional material, is presented in the textbook by Clifford and Preston [1, §6.6]. All terminology and notation not introduced explicitly will be as in [1].

Let $S$ be a semigroup. The relation $\leq$ defined on the set $S/L$ of all $L$-classes of $S$ by the rule that

$$L_a \leq L_b \iff S^1 a \subseteq S^1 b \quad (a, b \in S)$$

is a partial ordering. Similar partial orderings are defined on the sets $S/R$ and $S/J$. (No ambiguity will result from the use of the same symbol $\leq$ for all three partial orderings and for others introduced below.) Following Green [3], we say that $S$ satisfies the condition $ML[M_R, M_J]$ if and only if every nonempty collection of $L$-classes [$R$-classes, $J$-classes] contains a minimal member. It is easy to verify that $ML[M_R, M_J]$ is equivalent to the condition that every strictly descending chain of $L$-classes [$R$-classes, $J$-classes] must be finite.

Now consider the relation $\leq$ defined on the set $S/H$ of all $H$-classes of $S$ as follows:

$$H_a \leq H_b \iff L_a \leq L_b \text{ and } R_a \leq R_b \quad (a, b \in S).$$

Evidently this is again a partial ordering. At the centre of our discussion is the corresponding minimal condition $MH$: every nonempty set of $H$-classes contains a minimal member (equivalently, every strictly descending chain of $H$-classes must be finite).

It is convenient for our purpose to consider three further conditions on $S$, namely $M_L^*, M_R^*$ and $GB$. As in [5; 1, §6.6], we say that $S$ satisfies $M_L^*[M_R^*]$ if and only if, for all $J \in S/J$, the set of all $L$-classes [$R$-classes] of $S$ contained in $J$ contains a minimal member. An element $a \in S$ is said to be group-bound if and only if $a^n$ lies in a subgroup of $S$ for some positive integer $n$. Clearly, every periodic element is group-bound. We say that $S$ itself is group-bound, or that $S$ satisfies the condition $GB$, if and only if each of its elements is group-bound. It should be noted that in an earlier paper [6] (based, in turn, on [2]) group-bound elements of a semigroup were termed “pseudo-invertible”.

The paper is in two sections. In the first of these we examine the interdependence of the seven conditions $ML$, $M_R$, $M_J$, $M_H$, $M_L^*$, $M_R^*$, $GB$. Green [3, Theorem 4; 1, Theorem 6.49] has shown that $ML$ and $M_R$ together imply $M_J$: we extend this result by proving that the conjunction of $ML$ and $M_R$ is logically equivalent to the conjunction of $M_R$, $M_H$, $M_J$, $M_L^*$, $M_R^*$, and $GB$. Green [3, Theorem 4; 1, Theorem 6.49] has shown that $ML$ and $M_R$ together imply $M_J$: we extend this result by proving that the conjunction of $ML$ and $M_R$ is logically equivalent to the conjunction of $M_J$ and $M_H$, to the conjunction of $M_J$ and $GB$, and to the conjunction of $M_R$, $M_L^*$ and $M_R^*$ (Corollary 1.3). Exactly thirteen pairwise inequivalent conditions can be formed from the given seven by taking conjunctions. A complete picture of their interrelationship is provided by a Hasse diagram (1.5).

The second section concerns Schützenberger groups [1, §2.4]. We show that in a semigroup \( S \) satisfying \( M_H \) the Schützenberger group of an arbitrary \( \mathcal{H} \)-class must be a homomorphic image of a subgroup of \( S \) (Theorem 2.1): thus each \( \mathcal{H} \)-class of \( S \) has cardinal not exceeding that of some subgroup of \( S \). We also prove that in a group-bound semigroup whose subgroups are all trivial the relation \( \mathcal{H} \) must itself be trivial (Theorem 2.3). These theorems extend a result of Rhodes [7] for finite semigroups and examples show that they cannot be improved within the context of the conditions studied here.

1. Interdependence of the seven conditions. Let \( X \) and \( Y \) be semigroup conditions. We write \( X \leq Y \) ("\( X \) implies \( Y \)"") if and only if every semigroup satisfying \( X \) also satisfies \( Y \); furthermore, we write \( X = Y \) ("\( X \) is equivalent to \( Y \)"") if and only if \( X \leq Y \) and \( Y \leq X \). With equality of conditions thus defined as logical equivalence, the relation \( \leq \) is readily seen to be a partial ordering of any set of semigroup conditions. The conjunction of a finite family \( (A_1, A_2, \ldots, A_n) \) of semigroup conditions will be denoted by \( A_1 \wedge A_2 \wedge \ldots \wedge A_n \) and is defined as follows: a semigroup satisfies \( A_1 \wedge A_2 \wedge \ldots \wedge A_n \) if and only if it satisfies each of the conditions \( A_i \) \((i = 1, 2, \ldots, n)\). Evidently if \( A, B, C \) are semigroup conditions such that \( A \leq B \) then \( A \wedge C \leq B \wedge C \).

Throughout the remainder of the paper we shall denote the family
\[
(M_L, M_R, M_B, M_H, M_L^*, M_R^*, GB)
\]
by \( \Omega \) and the set of all conjunctions of nonempty subfamilies of \( \Omega \) by \( \Lambda(\Omega) \). Clearly \( \Lambda(\Omega) \) is a finite lower semilattice with respect to \( \leq \), the greatest lower bound of the pair \((A, B)\) being the conjunction \( A \wedge B \). This section is concerned with the structure of \( \Lambda(\Omega) \).

We begin with a restatement of an elementary property of the conditions \( M_L^* \) and \( M_R^* \) [5, Lemma 2.2; 1, Lemma 6.41].

**Lemma 1.1.** Let \( S \) be a semigroup satisfying \( M_L^* \wedge M_R^* \). Then, for all \( a \in S \), \( L_a[R_a] \) is minimal in the set of all \( \mathcal{L} \)-classes \([\mathcal{H} \text{-classes}] \) of \( S \) contained in \( J_a \).

The following theorem establishes various basic relationships between the members of \( \Lambda(\Omega) \).

**Theorem 1.2.**
(i) \( M_L \leq M_L^* \); \( M_R \leq M_R^* \);
(ii) \( M_L \wedge M_R \leq M_f \);
(iii) \( M_f \wedge M_L^* \leq M_L \); \( M_f \wedge M_R^* \leq M_R \);
(iv) \( M_L \wedge M_R^* \leq M_H \); \( M_R \wedge M_L^* \leq M_H \);
(v) \( M_H \leq GB \);
(vi) \( GB \leq M_L^* \wedge M_R^* \).

**Proof.** We note first that the assertions in (i) are immediate consequences of the definitions and that the result in (ii) is due to Green [3, Theorem 4; 1, Theorem 6.49].

(iii) Let \( S \) be a semigroup satisfying \( M_f \) and \( M_L^* \). Consider a nonempty set \( \mathcal{C} \) of \( \mathcal{L} \)-classes of \( S \). Since \( S \) satisfies \( M_f \) there exists \( a \in S \) such that \( L_a \in \mathcal{C} \) and, for all \( x \in S \), if \( L_x \in \mathcal{C} \) and \( J_x \leq J_a \) then \( J_x = J_a \). Suppose that \( b \in S \) is such that \( L_b \in \mathcal{C} \) and \( L_b \leq L_a \). Then
and so $J_b = J_a$. But since $S$ satisfies $M^*_L$ it follows from Lemma 1.1 that $L_a$ is minimal in the set of all $L$-classes contained in $J_a$. Hence $L_a = L_{a'}$. Consequently $L_a$ is minimal in $G$. This shows that $S$ satisfies $M_L$: thus $M_J \land M^*_L \leq M_L$. A similar argument shows that $M_J \land M^*_R \leq M_R$.

(iv) Let $S$ be a semigroup satisfying $M_L$ and $M^*_R$. Consider a sequence $a_1, a_2, a_3, \ldots$ of elements of $S$ such that

$$H_{a_1} \geq H_{a_2} \geq H_{a_3} \geq \ldots$$

We have that $L_{a_1} \geq L_{a_2} \geq L_{a_3} \geq \ldots$ and so, since $S$ satisfies $M_L$, there exists a positive integer $k$ such that the elements $a_k, a_{k+1}, a_{k+2}, \ldots$ are $L$-equivalent. Thus $a_k, a_{k+1}, a_{k+2}, \ldots$ are $R$-equivalent. But $R_{a_k} \geq R_{a_{k+1}} \geq R_{a_{k+2}} \geq \ldots$ and so, by Lemma 1.1, the elements $a_k, a_{k+1}, a_{k+2}, \ldots$ are $R$-equivalent. It follows that $H_{a_k} = H_{a_{k+1}} = H_{a_{k+2}} = \ldots$. This shows that $S$ satisfies $M_R$. Hence $M_L \land M^*_R \leq M_R$. By duality, $M_R \land M^*_L \leq M_L$.

(v) Let $S$ be a semigroup satisfying $M_R$. Consider any element $a \in S$. Since $H_a = H_{a^2} = H_{a^3} = \ldots$ there exists a positive integer $n$ such that $H_{a^n} = H_{a^{n+1}} = H_{a^{n+2}} = \ldots$. But this means that $(a^n, a^{n^2}) \in \mathcal{H}$ and so, by [1, Theorem 2.16], $H_{a^n}$ is a group. Thus $S$ is group-bound. Hence $M_R \leq GB$.

(vi) Let $S$ be a group-bound semigroup. We shall show that $S$ satisfies $M^*_R$. Let $a, b \in S$ be such that $(a, b) \in \mathcal{J}$ and $L_a \leq L_b$. Then there exist elements $u, v, c \in S'$ such that $a = ubv$ and $b = ca$. Thus $a = (uc)av$ and so $a = (uc)^n av^n$ for all positive integers $n$. Now $S'$ is group-bound and so we can choose $n$ such that $n > 1$ and $(uc)^n$ lies in a subgroup of $S'$, with identity element $e$, say. Write $g = (uc)^n$ and let $g^{-1}$ denote the inverse of $g$ in the subgroup $H_e$ of $S'$. Then

$$ea = e(gav^n) = (eg)av^n = gav^n = a$$

and so

$$g^{-1}(uc)^n^{-1} ub = g^{-1}(uc)^{n-1} uca = g^{-1} ga = ea = a.$$ 

Hence $L_a \leq L_b$, from which it follows that $L_a = L_b$. Consequently, $S$ satisfies $M^*_R$. Thus $GB \leq M^*_L \land M^*_R$.

**Corollary 1.3.**

$$M_L \land M_R = M_J \land M_H = M_J \land GB = M_J \land M^*_L \land M^*_R.$$ 

**Proof.** By (ii), $M_L \land M_R \leq M_J$ and, by (i) and (iv), $M_L \land M_R \leq M^*_L \land M_R \leq M_H$. Thus $M_L \land M_R \leq M_J \land M_H$. On the other hand, by (v), (vi) and (iii),

$$M_J \land M_H \leq M_J \land GB \leq M_J \land (M^*_L \land M^*_R) = (M_J \land M^*_L) \land (M_J \land M^*_R) \leq M_L \land M_R.$$ 

This gives the result.

It is straightforward to check that, in view of Theorem 1.2, $\Lambda(\Omega)$ has at most thirteen elements. We proceed to show by means of examples that it has exactly thirteen.
First, we require some further notation. For an arbitrary semigroup $S$ let $S^{\text{opp}}$ denote the semigroup with the same set of elements as $S$ but with the multiplication reversed. Also, for any two semigroups $S$ and $T$ let $S + T$ denote the $0$-direct union of $S^0$ and $T^0$ [1, §6.3].

The next lemma is almost immediate.

**Lemma 1.4.** Let $S$ and $T$ be semigroups and let $X$ be a member of $\Omega$. Then $S + T$ satisfies $X$ if and only if both $S$ and $T$ satisfy $X$.

We now consider four semigroups $S_i (i = 1, 2, 3, 4)$ defined as follows: $S_1$, $S_2$ and $S_3$ are, respectively, an infinite cyclic semigroup, an infinitely descending semilattice and a Croisot-Teissier semigroup of the form $CT(A, \emptyset, p, p)$ [1, §8.2], while $S_4$ is the semigroup with zero 0, nonzero elements the ordered pairs $(i, j)$ of positive integers $i, j$ such that $i < j$, and multiplication of nonzero elements according to the rule that

$$(i, j)(r, s) = \begin{cases} (i, s) & \text{if } j = r, \\ 0 & \text{if } j \neq r \end{cases}$$

[5; 1, §6.6, Example 1].

These semigroups, and four others derived from them, label the rows of the following table, the columns of which are labelled by the members of $\Omega$. The entry in the table corresponding to a semigroup $S$ and a condition $X$ is 1 or 0 according as $S$ satisfies or fails to satisfy $X$. It is a routine matter to check the entries in the first four rows (see [1, Theorem 8.11] for $S_3$): the remaining entries are then easily obtained with the aid of Lemma 1.4 and duality.

<table>
<thead>
<tr>
<th></th>
<th>$M_L$</th>
<th>$M_R$</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M^*_L$</th>
<th>$M^*_R$</th>
<th>GB</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$S_3$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$S_4$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$S_1 + S_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_3 + S_3^{\text{opp}}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_4 + S_2^{\text{opp}}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$S_3 + S_3^{\text{opp}}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

From Theorem 1.2 and Corollary 1.3, together with the above table, duality and the observation that the trivial semigroup satisfies all the members of $\Omega$, we see that $\Lambda(\Omega)$ has the Hasse diagram shown below.
Note that the seven members of $\Omega$ are distinct and that $\Lambda(\Omega)$ can be obtained from the free semilattice on $\Omega$ by imposing precisely the relationships listed in Theorem 1.2.

Remark 1.6. There exist semigroups satisfying none of the conditions in $\Omega$: for example, $S_1 + (S_3 + S_3^{opp})$.

Remark 1.7. By [1, Theorem 6.45] the condition $M_L^* \wedge M_R^*$ on a semigroup $S$ implies that $\emptyset = \emptyset$ on $S$. Now take the Croisot-Teissier semigroup $S_3$ to be such that $\emptyset \neq \emptyset$. Then the examples $S_3$ and $S_3^{opp}$ show that $M_L^* \wedge M_R^*$ is the weakest conjunction of members of $\Omega$ to imply that $\emptyset = \emptyset$, in the sense that any other conjunction implying the condition $\emptyset = \emptyset$ also implies $M_L^* \wedge M_R^*$.

Furthermore, the proof of [5, Theorem 2.3] (see [1, Theorem 6.45]) shows that the condition $M_L^* \wedge M_R^*$ on a semigroup implies that each $[0,\ldots]$ simple principal factor of the semigroup is completely $[0,\ldots]$ simple. Again the examples $S_3$ and $S_3^{opp}$ demonstrate that $M_L^* \wedge M_R^*$ is the weakest conjunction of members of $\Omega$ to imply this condition on principal factors.

Remark 1.8. Let $S$ be a regular semigroup. Then, for all $a$ and $b$ in $S$, $L_a \cong L_b$ if and only if to each idempotent $e \in L_a$ there corresponds an idempotent $f \in L_b$ such that $e \cong f$ [4, Remark 2]. Thus, on $S$, the conditions $M_L$, $M_R$ and $M_H$ are each equivalent to the condition that every nonempty set of idempotents of $S$ contains a minimal member with respect to the usual partial ordering.

2. The Schützenberger group of an $\mathcal{H}$-class. Let $S$ be a semigroup and $H$ an $\mathcal{H}$-class of $S$. Write $T = \{x \in S^1 : Hx \subseteq H\}$. Then $T$ is a subsemigroup of $S^1$ containing the identity 1. Corresponding to each $t \in T$ we define an element $\gamma_t$ of $S_H$ (the full transformation semigroup on $H$) by the rule that $h \gamma_t = ht$ for all $h \in H$. Next, we define $\gamma : T \to S_H$ by setting $\gamma_t = \gamma_t$ for all $t \in T$. Then $\gamma$ is a homomorphism and the image $T\gamma$ is a group of permutations of $H$; moreover, $|T\gamma| = |H|$ and if $H$ is a subgroup of $S$ then $T\gamma \cong H$ [1, §2.4]. We call $T\gamma$ the Schützenberger group of $H$.

The following theorem generalises a result on finite semigroups due to Rhodes (see [7, Proposition 1.1, equivalence of (a) and (c)]).
THEOREM 2.1. Let $S$ be a semigroup satisfying $M_H$ and let $H$ be an $\mathcal{H}$-class of $S$. Then the Schützenberger group of $H$ is a homomorphic image of a subgroup of $S$.

Proof. Let $T, \gamma, (t \in T)$ and $\gamma$ be defined as above. Evidently the $\mathcal{H}$-classes of $S^1$ are just those of $S$, together with $\{1\}$ in the case where $S \neq S^1$. Thus $S^1$ satisfies $M_H$ and hence the set $\{H_t \in S^1 : t \in T\}$ contains a minimal member $H'$, say. Consider an arbitrary element $a$ in $H' \cap T$. Since $H' = H_a \supseteq H_a$ in $S^1$ and $a^2 \in T$, it follows from the minimality of $H'$ that $a$ and $a^2$ are $\mathcal{H}$-equivalent in $S^1$. Therefore $H'$ is a subgroup of $S^1$, by [1, Theorem 2.16], and so $H' \cap T$ is a subsemigroup of $S^1$. Now let $a^{-1}$ denote the inverse of $a$ in $H'$. Then, since $H_t = H$ for all $t \in T$ [1, Lemma 2.21], we have that $H a^{-1} = (Ha^2)a^{-1} = Ha = H$. Hence $a^{-1} \in T$. Thus $H' \cap T$ is a subgroup of $H'$.

Denote the identity of $H'$ by $e$. Then $e \in H' \cap T$. Now consider an arbitrary element $t$ of $T$. Clearly $e t e \in T$ and $e t e \leq H_e = H$ in $S^1$. Hence $e t e \in H'$, by the minimality of $H'$.

Thus $e t e \in H' \cap T$. But since $\gamma$ is a homomorphism and $e^2 = e, \gamma_e$ must be the identity element of the group $T \gamma$. Hence $\gamma_e \gamma = \gamma_{e t e} \in (H' \cap T) \gamma$. Consequently $T \gamma \subseteq (H' \cap T) \gamma$ and so $T \gamma = (H' \cap T) \gamma$. The Schützenberger group $T \gamma$ of $H$ is thus the image under $\gamma$ of a subgroup of $S^1$, namely $H' \cap T$.

The stated result now clearly holds if $S^1 = S$. It also holds if $|T \gamma| = 1$: for $S$ must contain at least one idempotent, by Theorem 1.2(v). We therefore assume that $S^1 \neq S$ and that $|T \gamma| > 1$. To complete the proof it is enough to show that, with these hypotheses, $H' \cap T \subseteq S$. Now $|H' \cap T| > 1$, since $T \gamma = (H' \cap T) \gamma$. Hence $H' \cap H_1 \neq \{1\}$ in $S^1$ and so $H' \subseteq S$. Thus $H' \cap T \subseteq S$, as required.

COROLLARY 2.2. Let $S$ be a semigroup satisfying $M_H$ and let $H$ be an $\mathcal{H}$-class of $S$. Then $S$ has a maximal subgroup $G$ such that $|G| \geq |H|$.

It follows from Corollary 2.2 that if $S$ is a semigroup satisfying $M_H$ and if every subgroup of $S$ is trivial then $\mathcal{H}$ is trivial on $S$. A better result can, however, be obtained directly:

THEOREM 2.3. Let $S$ be a group-bound semigroup in which every subgroup is trivial. Then $\mathcal{H}$ is trivial on $S$.

Proof. Let $H$ be an $\mathcal{H}$-class of $S$ and let $T, \gamma, (t \in T)$ and $\gamma$ be defined as before. Let $t \in T$. Since $S^1$ is a group-bound semigroup in which every subgroup is trivial there exists a positive integer $n$ such that $t^n$ is an idempotent of $S^1$. Hence $t^n, t^{n+1} = t^{n+1}$ and $t^{2n-1} = t^n$, from which it follows that $t^{n+1} \in H_n$. But $H_n$ is a group and so $t^{n+1} = t^n$. Thus $\gamma_t = g_t \gamma$, and so, since $T \gamma$ is a group, $\gamma$ is the identity of $T \gamma$. Consequently $|T \gamma| = 1$.

But $|T \gamma| = |H|$ and therefore $|H| = 1$.

We conclude by showing that, in a certain sense, the results of Theorems 2.1 and 2.3 are best possible.

EXAMPLES 2.4. Let $(T, \cdot)$ be a semigroup, let $(H, *)$ be a group and let $\phi : T \rightarrow H$ be a surjective homomorphism. We assume that the sets $T, H$ and $\{0\}$ are pairwise disjoint and we write $S = T \cup H \cup \{0\}$. By means of the following rules we extend the binary operation...
on $T$ to an operation on $S$:

$$s0 = 0s = 0, \quad gh = 0, \quad gt = g \ast t \phi, \quad tg = (t \phi) \ast g$$

for all $s \in S$, all $g, h \in H$ and all $t \in T$. It is straightforward to verify that $S$ is a semigroup with $H$ and $\{0\}$ as two of its $\mathcal{H}$-classes, the remaining $\mathcal{H}$-classes being precisely those of $T$. Clearly $H$ is also a $\mathcal{J}$-class of $S$.

By making particular choices for $T, H$ and $\phi$ in this construction we obtain three examples ((a), (b), (c) below).

(a) Let $K$ be a nontrivial finite group and let 1 denote its identity. For all positive integers $n$ let $K^{(n)}$ denote the direct product of $n$ copies of $K$ and for all positive integers $m, n$ with $m \leq n$ define a homomorphism $\phi^n_m : K^{(m)} \rightarrow K^{(n)}$ by the rule that

$$(k_1, \ldots, k_m)\phi^n_m = (k_1, \ldots, k_m, 1, \ldots, 1)$$

for all $k_1, \ldots, k_m \in K$. Take $T = \bigcup_{n=1}^{\infty} K^{(n)}$ and define a multiplication on $T$ by setting

$$st = (s\phi^n_m)(t\phi^n_m),$$

where $s \in K^{(m)}, t \in K^{(n)}$ and $p = \max\{m, n\}$, the product on the right-hand side being computed in $K^{(p)}$. By [1, Theorem 4.11], $T$ is a semilattice of groups. Its $\mathcal{H}$-classes are just the groups $K^{(n)}$, each of which is finite.

Next, take $H$ to be the group consisting of all infinite sequences $(k_1, k_2, k_3, \ldots)$ of elements of $K$ with at most finitely many entries different from 1, under componentwise multiplication. Finally, define $\phi : T \rightarrow H$ by the rule that

$$(k_1, \ldots, k_n)\phi = (k_1, \ldots, k_n, 1, 1, \ldots)$$

for all positive integers $n$ and all $k_1, \ldots, k_n \in K$. Then $\phi$ is a surjective homomorphism (and $\phi \circ \phi^{-1}$ is the least group congruence on $T$).

In this case the semigroup $S$ has only finite subgroups, but possesses an infinite $\mathcal{H}$-class, namely $H$. Clearly $S$ is periodic and so satisfies the condition GB.

(b) Take $H$ to be a nontrivial group, take $T$ to be the free semigroup $\mathcal{F}_H$ on $H$ and take $\phi : T \rightarrow H$ to be any surjective homomorphism. Then $S$ has exactly one subgroup, namely $\{0\}$. Thus every subgroup of $S$ is trivial but $S$ has a nontrivial $\mathcal{H}$-class, namely $H$. In this case $S$ satisfies $M_L^R$ and $M_R^R$.

(c) Let $H$ be a nontrivial group, let $U$ be a Baer-Levi semigroup [1, §8.1] and let $T$ denote the direct product $U \times H$ $[U^{\text{opp}} \times H]$. Define $\phi$ to be the projection of $T$ onto $H : (u, h) \mapsto h$ for all $(u, h) \in T$. It is readily seen that $T$ has no subgroups. Hence, as in (b), the only subgroup of $S$ is the trivial subgroup $\{0\}$, while $S$ has a nontrivial $\mathcal{H}$-class, namely $H$. It is also easy to verify that since $U$ $[U^{\text{opp}}]$ satisfies $M_R$ and $M_L$ $[M_L$ and $M_J]$ the same is true for $T$ and so also for $S$.

Remark 2.5. By Theorem 2.1, the condition $M_H$ on a semigroup $S$ implies that the Schützenberger group of each $\mathcal{H}$-class of $S$ is a homomorphic image of a subgroup of $S$. Examples 2.4(a) and (c) show that $M_H$ is the weakest member of $\Lambda(\Omega)$ to imply this condition on the Schützenberger groups of $\mathcal{H}$-classes.
By Theorem 2.3, the condition GB on a semigroup $S$ with no nontrivial subgroups implies that $\mathcal{H}$ is trivial on $S$. Examples 2.4(b) and (c) show that GB is the weakest member of $\Lambda(\Omega)$ to imply that $\mathcal{H}$ is trivial on semigroups with no nontrivial subgroups.

REFERENCES

1. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Surveys of the Amer. Math. Soc. 7 (Providence, R.I., 1961 (vol. I) and 1967 (vol. II)).