SOME EXAMPLES OF MODULES OVER NOETHERIAN RINGS

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1. Introduction. The purpose of this note is to prove the following result.

THEOREM 1. *Let n be an integer greater than zero. There exists a prime Noetherian ring R of Krull dimension $n+1$ and a finitely generated essential extension W of a simple R-module V such that*

(i) *W has Krull dimension n, and*

(ii) *W/V is n-critical and cannot be embedded in any of its proper submodules.*

We refer the reader to [**6**] for the definition and properties of Krull dimension.

Theorem 1 answers questions of Jategaonkar and Goldie. Let R be a two-sided Noetherian ring. In [**7**] Jategaonkar asks whether every finitely generated essential extension of a simple R-module is artinian, and Goldie [**4**] asks whether a critical R-module is necessarily compressible.

The ring R is the enveloping algebra of a certain finite dimensional metabelian Lie algebra.

Finitely generated, non-artinian essential extensions of simple R-modules were studied in [**8**] for the case where R is a polycyclic group algebra. An example of a 1-critical module which is not compressible was found independently by Goodearl [**5**]. This example closely resembles our module W/V for the case $n=1$.

We note that the bounds on Krull dimension are best possible for a prime Noetherian ring R of Krull dimension $n+1$. For, by [**8**, Proposition 5.5], a finitely generated essential extension of a simple R-module can have Krull dimension at most n, while [**6**, Proposition 6.8] states that an $n+1$-critical R-module is isomorphic to a right ideal of R and so cannot have the property expressed in (ii).

A simplified version of this example (the case $n=1$) is to appear in [**2**, Chapter 7]. I am very grateful for the hospitality of the University of Alberta where this work was completed.

2. The example. Let k be a field of characteristic zero and $\mathscr{L}$ a vector space over k with basis $y, x_0, x_1, \ldots, x_{n-1}$.

We make $\mathscr{L}$ into a Lie algebra by defining

$$\begin{gathered}[x_i x_j]=0 \qquad [x_0 y]=x_0 \\ [x_i y]=x_i+x_{i-1} \quad \text{for} \quad i=1,\ldots,n-1.\end{gathered} \tag{1}$$

Let R be the universal enveloping algebra of $\mathscr{L}$. Then R is a prime Noetherian ring of Krull dimension $n+1$, by [**3**, §§2.3 and 3.5].

Let $I=\sum_{i=0}^{n-1}(y-1)(x_i-1)R$ and $W=R/I$. For each non-negative integer m we set

$$v_m=(y-1)y^m+I\in W.$$

Then $v_m = v_0 y^m$ and we have

$$v_0 x_i = v_0 \tag{2}$$

since $(y-1)x_i - (y-1) \in I$. Set $V = v_0 R$, then $v_m \in V$ for all m, and V is spanned as a vector space by $\{v_m : m \geq 0\}$.

LEMMA 1. *The R-module V is simple.*

Proof. We show by induction that

$$v_m(1-x_0)^m = m!v_0 \tag{3}$$

Suppose that $v_m(1-x_0)^m = m!v_0$. Then by (2) and (3),

$$\begin{aligned} v_{m+1}(1-x_0)^{m+1} &= v_m y(1-x_0)^{m+1} \\ &= v_m(y - yx_0)(1-x_0)^m \\ &= v_m(y - x_0 y + x_0)(1-x_0)^m \\ &= v_m(1-x_0)y(1-x_0)^m + v_m x_0(1-x_0)^m \\ &= v_m(1-x_0)(y - x_0 y + x_0)(1-x_0)^{m-1} + m!v_0 \\ &= v_m(1-x_0)^2 y(1-x_0)^{m-1} + 2m!v_0 = \dots \\ &= v_m(1-x_0)^{m+1} y + (m+1)m!v_0 \\ &= m!v_0(1-x_0)y + (m+1)!v_0 \\ &= (m+1)!v_0. \end{aligned}$$

Hence (3) holds for all m. It follows that if $v \in V$, $v \neq 0$ then $v_0 \in vR$, so V is simple. Another easy consequence of (3) is that the v_i form a vector space basis for V. In order to state the next lemma we introduce some notation. If $e = (e_0, e_1, \dots, e_{n-1})$ is an n-tuple of non-negative integers we denote by x^e the monomial

$$x_0^{e_0} x_1^{e_1} \dots x_{n-1}^{e_{n-1}}.$$

Also, let $J = (y-1)R$. Then $J \supseteq I$ and $J/I = V$.

LEMMA 2. (i) *Let $e = (e_0, e_1, \dots, e_r, 0, \dots, 0)$. Then the following identity holds in R.*

$$x^e y = \left(y + \sum_{i=0}^{r} e_i + \sum_{i=1}^{r} e_i x_{i-1} x_i^{-1}\right) x^e$$

(ii) *Modulo J we have*

$$x^e\left(y - 1 - \sum_{i=0}^{r} e_i\right) \equiv \sum_{i=1}^{r} e_i x_{i-1} x_i^{-1} x^e.$$

Here the notation x_i^{-1} is purely symbolic. Thus if $e_i = 0$ this term does not appear, while if $e_i > 0$ then $x_i^{-1} x^e = x^f$ where $f_j = e_j$ if $j \neq i$ *and* $f_i = e_i - 1$.

Proof. (ii) follows immediately from (i) since $(y-1)\in J$.

(i) The defining relations (1) tell us how y may be moved to the left past any x_i and this result records how y may be moved past any monomial. We use induction on r and for a fixed r, induction on the exponent e_r.

Thus let $f_i=e_i$ if $i\neq r$ and $f_r=e_r+1$. Then

$$\begin{aligned}x^fy=x^ex_ry&=x^e(yx_r+x_r+x_{r-1})\\&=\left(y+\sum_{i=0}^{r}e_i+\sum_{i=1}^{r}e_ix_{i-1}x_i^{-1}\right)x^ex_r+x^ex_r+x^ex_{r-1}\\&=\left(y+\sum_{i=0}^{r}e_i+1+\sum_{i=1}^{r}e_ix_{i-1}x_i^{-1}+x_{r-1}x_r^{-1}\right)x^f\\&=\left(y+\sum_{i=0}^{r}f_i+\sum_{i=1}^{r}f_ix_{i-1}x_i^{-1}\right)x^f\end{aligned}$$

as required.

Notice that the module $R/J=W/V$ has a basis consisting of elements x^e+J and it is immediate from Lemma 2 that when $(x^e+J)y$ is written as a linear combination of elements x^f+J, the exponent sum on each x^f is the same as on x^e.

To gain further information from Lemma 2 it is convenient to introduce an ordering on monomials x^e.

Thus we write $x^f<x^e$ if for some $i\geq 0$, $e_i-f_i>0$ and $e_{i+1}-f_{i+1}=\dots=e_{n-1}-f_{n-1}=0$.

Note that any collection $\{x^e\}$ of monomials has a unique element which is minimal under this ordering. Also if $\alpha=\sum\lambda_fx^f$ is a non-zero linear combination of monomials then since $\operatorname{Supp}\alpha$ is finite there is a unique monomial in $\operatorname{Supp}\alpha$ which is maximal under this ordering. We denote this monomial by $\max\alpha$.

Finally if α is an arbitrary element of R and $\alpha\notin J$ then α is uniquely representable in the form $\alpha\equiv\sum\lambda_fx^f \bmod J$ and we set $\max\alpha=\max(\sum\lambda_fx^f)$.

LEMMA 3. *Suppose that $\alpha=\sum\lambda_fx^f$ and $\max\alpha=x^e$ where $e=(e_0,e_1,\dots,e_{n-1})$ satisfies $e_i>0$ for some $i\geq 1$. If*

$$\beta=\alpha\left(y-1-\sum_{i=0}^{n-1}e_i\right),$$

then $\beta\notin J$ and $\max\beta<\max\alpha$.

Proof. Let

$$\alpha=\sum_{x^f<x^e}\lambda_fx^f+\lambda_ex^e$$

and let i be the least integer greater than 0 with $e_i>0$.

By Lemma 2 $\max x^e\left(y-1-\sum_{i=0}^{n-1}e_i\right)=x^g$ where $g_{i-1}=e_{i-1}+1$, $g_i=e_i-1$ and $g_j=e_j$ for $j\neq i,\ i-1$.

Since the monomials x^f are linearly independent modulo J, in order to show that $\beta \notin J$ it suffices to show that x^g cannot occur in $\operatorname{Supp} x^f\left(y-1-\sum_{i=0}^{n-1} e_i\right)$ for any $x^f < x^e$ and $x^f \in \operatorname{Supp} \alpha$.

Notice that this can only possibly occur if $\sum_{i=0}^{n-1} e_i = \sum_{i=0}^{n-1} f_i$ and in this case we would have $x^g = x^h$ where for some k, $h_{k-1} = f_{k-1}+1$, $h_k = f_k - 1$, $h_l = f_l$, $l \neq k$, $k-1$.

Suppose first that $k > i$. Then $f_k - 1 = e_k$ so $f_k > e_k$ and $e_{k+1} - f_{k+1} = \dots e_{n-1} - f_{n-1} = 0$. This contradicts the maximality of x^e in $\operatorname{Supp} \alpha$.

Suppose that $k < i$. Then $f_{k-1} + 1 = e_{k-1}$, and since $k-1 < i$ we have $e_{k-1} = 0$. Therefore $f_{k-1} = -1$, another contradiction.

Hence $k = i$, but in this case $f_{i-1} + 1 = e_{i-1} + 1$, $f_i - 1 = e_i - 1$ and $f_j = e_j$ if $j \neq i$, $i-1$ and so $x^f = x^e$.

We have shown that the term x^g occurs with non-zero coefficient in β.

To see that $\max \beta < \max \alpha$ note that if $x^f < x^e$ then any element $x^g \in \operatorname{Supp} x^f\left(y-1-\sum_{i=0}^{n-1} e_i\right)$ satisfies $x^g \leq x^f$ by Lemma 2.

Lemma 4. *The module W is an essential extension of V.*

Proof. Let T be a right ideal of R which strictly contains I. We must show that $J \subseteq T$. If T contains a non-zero element of J we are finished since J/I is simple by Lemma 1.

Hence we may assume that T contains an element $\alpha = \sum \lambda_f x^f + r$ where $r \in J$ and $\sum \lambda_f x^f \neq 0$. Among such elements α choose $\alpha \in T$ with $\max \alpha$ minimal, say

$$\alpha = \sum_{x^f < x^e} \lambda_f x^f + \lambda_e x^e + r.$$

If $e = (e_0, e_1, \dots, e_{n-1})$ and $e_i > 0$ for some $i \geq 1$ then Lemma 3 immediately gives a contradiction to the minimality of $\max \alpha$.

Therefore T contains an element of the form $\lambda_0 x_0^s + \dots + \lambda_t x_0^{s+t} + r$ with $r \in J$, $\lambda_t \neq 0$, $\lambda_0 \neq 0$ $t \geq 0$. If t is chosen minimal then Lemma 2 gives $t = 0$.

Hence T/I contains an element $x_0^s + r + I$ where $s \geq 1$ and $r \in J$. Therefore

$$(x_0^s + r + I)(y-1-s) = (y-1)x_0^s + r(y-1-s) + I = v_0 + r(y-1-s) + I \in (J/I) \cap (T/I).$$

By writing r as a linear combination of the elements v_i, it is easy to see that this is a non-zero element of V. Hence $V \cap (T/I) \neq 0$.

Proof of Theorem 1. It remains to show that W/V is n-critical and cannot be embedded in any of its proper submodules.

Let kX denote the subalgebra of R which is generated by $x_0, x_1, \dots, x_{n-1}$. Then the R-module $\bar{W} = W/V$ is free as a kX-module. We use induction on n to show that a non-zero R-module $\bar{W}$ which is free as a kX-module has Krull dimension at least n.

Let $K = x_0 R$, a 2-sided ideal of R, and consider the chain $\bar{W} > \bar{W}K > \bar{W}K^2 > \dots$. For $n = 1$ this chain shows that $\bar{W}$ has Krull dimension at least 1. Assume $n > 1$. Then

$\bar{W}K^m/\bar{W}K^{m+1}$ is a non-zero free R/K-module for each m. The ring R/K has exactly the same defining relations as R except that the parameter n has dropped to $n-1$. (This is because $x_0=0$ gives $[x_1y]=x_1$ and $[x_iy]=x_i+x_{i-1}$ if $i>1$.)

Therefore by induction $\bar{W}K^m/\bar{W}K^{m+1}$ has Krull dimension at least $n-1$ and so $\bar{W}$ has Krull dimension at least n.

If we regard W/V simply as a kX-module then W/V is free of rank one. Hence as kX is a commutative Noetherian domain of Krull dimension n, it follows that W/V is n-critical as a kX-module and hence also as an R-module.

Finally, to see that $W/V=R/J$ cannot be embedded in any proper submodule, notice that by Lemma 2, the only element of R/J which is annihilated by $y-1$ is $1+J$. This completes the proof of Theorem 1.

The case $n=1$ of Theorem 1 may be of special interest. In this case $\mathscr{L}$ has the form

$$\mathscr{L}=kx_0\oplus ky \quad \text{where} \quad [x_0y]=x_0$$

and if k is algebraically closed then $\mathscr{L}$ is an epimorphic image of any finite dimensional soluble Lie algebra which is not nilpotent [**1**, p. 71]. Also in this case it is easily seen that the module $W=R/(y-1)(x_0-1)R$ obtained in Theorem 1 is uniserial, that is every non-zero submodule of W has a unique maximal submodule. Hence we may state

THEOREM 2. *Let k be an algebraically closed field of characteristic zero and $\mathscr{L}$ a finite dimensional soluble Lie algebra over k which is not nilpotent. Let R be the enveloping algebra of $\mathscr{L}$. Then there is a finitely generated (uniserial) essential extension W of a simple R-module V such that*

(i) *W is not artinian, and*
(ii) *W/V is 1-critical and cannot be embedded in any proper submodule.*

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