

# SOME EXAMPLES OF MODULES OVER NOETHERIAN RINGS

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**1. Introduction.** The purpose of this note is to prove the following result.

**THEOREM 1.** *Let  $n$  be an integer greater than zero. There exists a prime Noetherian ring  $R$  of Krull dimension  $n + 1$  and a finitely generated essential extension  $W$  of a simple  $R$ -module  $V$  such that*

- (i)  $W$  has Krull dimension  $n$ , and
- (ii)  $W/V$  is  $n$ -critical and cannot be embedded in any of its proper submodules.

We refer the reader to [6] for the definition and properties of Krull dimension.

Theorem 1 answers questions of Jategaonkar and Goldie. Let  $R$  be a two-sided Noetherian ring. In [7] Jategaonkar asks whether every finitely generated essential extension of a simple  $R$ -module is artinian, and Goldie [4] asks whether a critical  $R$ -module is necessarily compressible.

The ring  $R$  is the enveloping algebra of a certain finite dimensional metabelian Lie algebra.

Finitely generated, non-artinian essential extensions of simple  $R$ -modules were studied in [8] for the case where  $R$  is a polycyclic group algebra. An example of a 1-critical module which is not compressible was found independently by Goodearl [5]. This example closely resembles our module  $W/V$  for the case  $n = 1$ .

We note that the bounds on Krull dimension are best possible for a prime Noetherian ring  $R$  of Krull dimension  $n + 1$ . For, by [8, Proposition 5.5], a finitely generated essential extension of a simple  $R$ -module can have Krull dimension at most  $n$ , while [6, Proposition 6.8] states that an  $n + 1$ -critical  $R$ -module is isomorphic to a right ideal of  $R$  and so cannot have the property expressed in (ii).

A simplified version of this example (the case  $n = 1$ ) is to appear in [2, Chapter 7]. I am very grateful for the hospitality of the University of Alberta where this work was completed.

**2. The example.** Let  $k$  be a field of characteristic zero and  $\mathcal{L}$  a vector space over  $k$  with basis  $y, x_0, x_1, \dots, x_{n-1}$ .

We make  $\mathcal{L}$  into a Lie algebra by defining

$$\begin{aligned} [x_i x_j] &= 0 & [x_0 y] &= x_0 \\ [x_i y] &= x_i + x_{i-1} & \text{for } i &= 1, \dots, n-1. \end{aligned} \tag{1}$$

Let  $R$  be the universal enveloping algebra of  $\mathcal{L}$ . Then  $R$  is a prime Noetherian ring of Krull dimension  $n + 1$ , by [3, §§2.3 and 3.5].

Let  $I = \sum_{i=0}^{n-1} (y-1)(x_i-1)R$  and  $W = R/I$ . For each non-negative integer  $m$  we set

$$v_m = (y-1)y^m + I \in W.$$

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Then  $v_m = v_0 y^m$  and we have

$$v_0 x_i = v_0 \tag{2}$$

since  $(y-1)x_i - (y-1) \in I$ . Set  $V = v_0 R$ , then  $v_m \in V$  for all  $m$ , and  $V$  is spanned as a vector space by  $\{v_m : m \geq 0\}$ .

LEMMA 1. *The  $R$ -module  $V$  is simple.*

*Proof.* We show by induction that

$$v_m(1-x_0)^m = m!v_0 \tag{3}$$

Suppose that  $v_m(1-x_0)^m = m!v_0$ . Then by (2) and (3),

$$\begin{aligned} v_{m+1}(1-x_0)^{m+1} &= v_m y(1-x_0)^{m+1} \\ &= v_m(y-yx_0)(1-x_0)^m \\ &= v_m(y-x_0y+x_0)(1-x_0)^m \\ &= v_m(1-x_0)y(1-x_0)^m + v_mx_0(1-x_0)^m \\ &= v_m(1-x_0)(y-x_0y+x_0)(1-x_0)^{m-1} + m!v_0 \\ &= v_m(1-x_0)^2y(1-x_0)^{m-1} + 2m!v_0 = \dots \\ &= v_m(1-x_0)^{m+1}y + (m+1)m!v_0 \\ &= m!v_0(1-x_0)y + (m+1)!v_0 \\ &= (m+1)!v_0. \end{aligned}$$

Hence (3) holds for all  $m$ . It follows that if  $v \in V$ ,  $v \neq 0$  then  $v_0 \in vR$ , so  $V$  is simple. Another easy consequence of (3) is that the  $v_i$  form a vector space basis for  $V$ . In order to state the next lemma we introduce some notation. If  $e = (e_0, e_1, \dots, e_{n-1})$  is an  $n$ -tuple of non-negative integers we denote by  $x^e$  the monomial

$$x_0^{e_0}x_1^{e_1} \dots x_{n-1}^{e_{n-1}}.$$

Also, let  $J = (y-1)R$ . Then  $J \supseteq I$  and  $J/I = V$ .

LEMMA 2. (i) *Let  $e = (e_0, e_1, \dots, e_r, 0, \dots, 0)$ . Then the following identity holds in  $R$ .*

$$x^e y = \left( y + \sum_{i=0}^r e_i + \sum_{i=1}^r e_i x_{i-1} x_i^{-1} \right) x^e$$

(ii) *Modulo  $J$  we have*

$$x^e \left( y - 1 - \sum_{i=0}^r e_i \right) \equiv \sum_{i=1}^r e_i x_{i-1} x_i^{-1} x^e.$$

Here the notation  $x_i^{-1}$  is purely symbolic. Thus if  $e_i = 0$  this term does not appear, while if  $e_i > 0$  then  $x_i^{-1} x^e = x^f$  where  $f_j = e_j$  if  $j \neq i$  and  $f_i = e_i - 1$ .

*Proof.* (ii) follows immediately from (i) since  $(y - 1) \in J$ .

(i) The defining relations (1) tell us how  $y$  may be moved to the left past any  $x_i$  and this result records how  $y$  may be moved past any monomial. We use induction on  $r$  and for a fixed  $r$ , induction on the exponent  $e_r$ .

Thus let  $f_i = e_i$  if  $i \neq r$  and  $f_r = e_r + 1$ . Then

$$\begin{aligned} x^f y &= x^e x_r y = x^e (y x_r + x_r + x_{r-1}) \\ &= \left( y + \sum_{i=0}^r e_i + \sum_{i=1}^r e_i x_{i-1} x_i^{-1} \right) x^e x_r + x^e x_r + x^e x_{r-1} \\ &= \left( y + \sum_{i=0}^r e_i + 1 + \sum_{i=1}^r e_i x_{i-1} x_i^{-1} + x_{r-1} x_r^{-1} \right) x^f \\ &= \left( y + \sum_{i=0}^r f_i + \sum_{i=1}^r f_i x_{i-1} x_i^{-1} \right) x^f \end{aligned}$$

as required.

Notice that the module  $R/J = W/V$  has a basis consisting of elements  $x^e + J$  and it is immediate from Lemma 2 that when  $(x^e + J)y$  is written as a linear combination of elements  $x^f + J$ , the exponent sum on each  $x^f$  is the same as on  $x^e$ .

To gain further information from Lemma 2 it is convenient to introduce an ordering on monomials  $x^e$ .

Thus we write  $x^f < x^e$  if for some  $i \geq 0$ ,  $e_i - f_i > 0$  and  $e_{i+1} - f_{i+1} = \dots = e_{n-1} - f_{n-1} = 0$ .

Note that any collection  $\{x^e\}$  of monomials has a unique element which is minimal under this ordering. Also if  $\alpha = \sum \lambda_f x^f$  is a non-zero linear combination of monomials then since  $\text{Supp } \alpha$  is finite there is a unique monomial in  $\text{Supp } \alpha$  which is maximal under this ordering. We denote this monomial by  $\max \alpha$ .

Finally if  $\alpha$  is an arbitrary element of  $R$  and  $\alpha \notin J$  then  $\alpha$  is uniquely representable in the form  $\alpha \equiv \sum \lambda_f x^f \pmod{J}$  and we set  $\max \alpha = \max(\sum \lambda_f x^f)$ .

LEMMA 3. Suppose that  $\alpha = \sum \lambda_f x^f$  and  $\max \alpha = x^e$  where  $e = (e_0, e_1, \dots, e_{n-1})$  satisfies  $e_i > 0$  for some  $i \geq 1$ . If

$$\beta = \alpha \left( y - 1 - \sum_{i=0}^{n-1} e_i \right),$$

then  $\beta \notin J$  and  $\max \beta < \max \alpha$ .

*Proof.* Let

$$\alpha = \sum_{x^f < x^e} \lambda_f x^f + \lambda_e x^e$$

and let  $i$  be the least integer greater than 0 with  $e_i > 0$ .

By Lemma 2  $\max x^e \left( y - 1 - \sum_{i=0}^{n-1} e_i \right) = x^g$  where  $g_{i-1} = e_{i-1} + 1$ ,  $g_i = e_i - 1$  and  $g_j = e_j$  for  $j \neq i, i - 1$ .

Since the monomials  $x^f$  are linearly independent modulo  $J$ , in order to show that  $\beta \notin J$  it suffices to show that  $x^s$  cannot occur in  $\text{Supp } x^f \left( y - 1 - \sum_{i=0}^{n-1} e_i \right)$  for any  $x^f < x^e$  and  $x^f \in \text{Supp } \alpha$ .

Notice that this can only possibly occur if  $\sum_{i=0}^{n-1} e_i = \sum_{i=0}^{n-1} f_i$  and in this case we would have  $x^s = x^h$  where for some  $k$ ,  $h_{k-1} = f_{k-1} + 1$ ,  $h_k = f_k - 1$ ,  $h_l = f_l$ ,  $l \neq k, k - 1$ .

Suppose first that  $k > i$ . Then  $f_k - 1 = e_k$  so  $f_k > e_k$  and  $e_{k+1} - f_{k+1} = \dots = e_{n-1} - f_{n-1} = 0$ . This contradicts the maximality of  $x^e$  in  $\text{Supp } \alpha$ .

Suppose that  $k < i$ . Then  $f_{k-1} + 1 = e_{k-1}$ , and since  $k - 1 < i$  we have  $e_{k-1} = 0$ . Therefore  $f_{k-1} = -1$ , another contradiction.

Hence  $k = i$ , but in this case  $f_{i-1} + 1 = e_{i-1} + 1$ ,  $f_i - 1 = e_i - 1$  and  $f_j = e_j$  if  $j \neq i, i - 1$  and so  $x^f = x^e$ .

We have shown that the term  $x^s$  occurs with non-zero coefficient in  $\beta$ .

To see that  $\max \beta < \max \alpha$  note that if  $x^f < x^e$  then any element  $x^s \in \text{Supp } x^f \left( y - 1 - \sum_{i=0}^{n-1} e_i \right)$  satisfies  $x^s \leq x^f$  by Lemma 2.

LEMMA 4. *The module  $W$  is an essential extension of  $V$ .*

*Proof.* Let  $T$  be a right ideal of  $R$  which strictly contains  $I$ . We must show that  $J \subseteq T$ . If  $T$  contains a non-zero element of  $J$  we are finished since  $J/I$  is simple by Lemma 1.

Hence we may assume that  $T$  contains an element  $\alpha = \sum \lambda_f x^f + r$  where  $r \in J$  and  $\sum \lambda_f x^f \neq 0$ . Among such elements  $\alpha$  choose  $\alpha \in T$  with  $\max \alpha$  minimal, say

$$\alpha = \sum_{x^f < x^e} \lambda_f x^f + \lambda_e x^e + r.$$

If  $e = (e_0, e_1, \dots, e_{n-1})$  and  $e_i > 0$  for some  $i \geq 1$  then Lemma 3 immediately gives a contradiction to the minimality of  $\max \alpha$ .

Therefore  $T$  contains an element of the form  $\lambda_0 x_0^s + \dots + \lambda_t x_0^{s+t} + r$  with  $r \in J$ ,  $\lambda_t \neq 0$ ,  $\lambda_0 \neq 0$   $t \geq 0$ . If  $t$  is chosen minimal then Lemma 2 gives  $t = 0$ .

Hence  $T/I$  contains an element  $x_0^s + r + I$  where  $s \geq 1$  and  $r \in J$ . Therefore

$$(x_0^s + r + I)(y - 1 - s) = (y - 1)x_0^s + r(y - 1 - s) + I = v_0 + r(y - 1 - s) + I \in (J/I) \cap (T/I).$$

By writing  $r$  as a linear combination of the elements  $v_i$ , it is easy to see that this is a non-zero element of  $V$ . Hence  $V \cap (T/I) \neq 0$ .

*Proof of Theorem 1.* It remains to show that  $W/V$  is  $n$ -critical and cannot be embedded in any of its proper submodules.

Let  $kX$  denote the subalgebra of  $R$  which is generated by  $x_0, x_1, \dots, x_{n-1}$ . Then the  $R$ -module  $\bar{W} = W/V$  is free as a  $kX$ -module. We use induction on  $n$  to show that a non-zero  $R$ -module  $\bar{W}$  which is free as a  $kX$ -module has Krull dimension at least  $n$ .

Let  $K = x_0 R$ , a 2-sided ideal of  $R$ , and consider the chain  $\bar{W} > \bar{W}K > \bar{W}K^2 > \dots$ . For  $n = 1$  this chain shows that  $\bar{W}$  has Krull dimension at least 1. Assume  $n > 1$ . Then

$\bar{W}K^m/\bar{W}K^{m+1}$  is a non-zero free  $R/K$ -module for each  $m$ . The ring  $R/K$  has exactly the same defining relations as  $R$  except that the parameter  $n$  has dropped to  $n - 1$ . (This is because  $x_0 = 0$  gives  $[x_1, y] = x_1$  and  $[x_i, y] = x_i + x_{i-1}$  if  $i > 1$ .)

Therefore by induction  $\bar{W}K^m/\bar{W}K^{m+1}$  has Krull dimension at least  $n - 1$  and so  $\bar{W}$  has Krull dimension at least  $n$ .

If we regard  $W/V$  simply as a  $kX$ -module then  $W/V$  is free of rank one. Hence as  $kX$  is a commutative Noetherian domain of Krull dimension  $n$ , it follows that  $W/V$  is  $n$ -critical as a  $kX$ -module and hence also as an  $R$ -module.

Finally, to see that  $W/V = R/J$  cannot be embedded in any proper submodule, notice that by Lemma 2, the only element of  $R/J$  which is annihilated by  $y - 1$  is  $1 + J$ . This completes the proof of Theorem 1.

The case  $n = 1$  of Theorem 1 may be of special interest. In this case  $\mathcal{L}$  has the form

$$\mathcal{L} = kx_0 \oplus ky \quad \text{where} \quad [x_0, y] = x_0$$

and if  $k$  is algebraically closed then  $\mathcal{L}$  is an epimorphic image of any finite dimensional soluble Lie algebra which is not nilpotent [1, p. 71]. Also in this case it is easily seen that the module  $W = R/(y - 1)(x_0 - 1)R$  obtained in Theorem 1 is uniserial, that is every non-zero submodule of  $W$  has a unique maximal submodule. Hence we may state

**THEOREM 2.** *Let  $k$  be an algebraically closed field of characteristic zero and  $\mathcal{L}$  a finite dimensional soluble Lie algebra over  $k$  which is not nilpotent. Let  $R$  be the enveloping algebra of  $\mathcal{L}$ . Then there is a finitely generated (uniserial) essential extension  $W$  of a simple  $R$ -module  $V$  such that*

- (i)  $W$  is not artinian, and
- (ii)  $W/V$  is 1-critical and cannot be embedded in any proper submodule.

REFERENCES

1. W. Bohro, P. Gabriel and R. Rentschler, Primideale in Einhüllenden auflösbarer Lie-Algebren Lecture Notes in Mathematics No. 357, (Springer-Verlag, 1973).
2. A. W. Chatters and C. R. Hajarnavis, *Rings with chain conditions* (Pitman, 1980).
3. J. Dixmier, *Enveloping algebras* (North Holland, 1977).
4. A. W. Goldie, Properties of the idealiser, in *Ring theory*, Ed. R. Gordon (Academic Press, 1972).
5. K. R. Goodearl, Incompressible critical modules, *Comm. Algebra*, **8** (1980), 1845–1852.
6. R. Gordon and J. C. Robson, Krull dimension, *Memoirs Amer. Math. Soc.* **133** (1973).
7. A. V. Jategaonkar, Jacobson’s conjecture and modules over fully bounded Noetherian rings, *J. Algebra* **30** (1974), 103–121.
8. I. M. Musson, Injective modules for group rings of polycyclic groups II, *Quart. J. Math. Oxford* (2), **31** (1980), 449–466.

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