

REPRESENTATIONS OF INFINITE SOLUBLE GROUPS

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1. Introduction. The purpose of this paper is to study the following two questions.

(1) When does the group algebra of a soluble group have infinite dimensional irreducible modules?

(2) When is the group algebra of a torsion free soluble group primitive?

In relation to the first question, Roseblade [13] has proved that if G is a polycyclic group and k an absolute field then all irreducible kG -modules are finite dimensional. Here we prove a converse.

THEOREM A. *Let G be a finitely generated hyperabelian group which is not polycyclic-by-finite. If k is any field, then kG has an infinite dimensional irreducible module.*

We remark that B. A. F. Wehrfritz has recently and independently proved a result similar to Theorem A [17].

Recently some progress has been made on the second question. For instance, Roseblade has solved the problem for polycyclic groups in [14], while in [2] Brown has treated the case of torsion free metanilpotent groups whose Fitting subgroup has infinite rank.

Here we study the finite rank case. In many situations a reduction to the case where the group G has an abelian normal subgroup A with G/A infinite cyclic, seems possible, for example [4, Theorem 4.6]. By a result of Passman [11, 9.3.2] the group algebra kG will always be primitive if $\Delta(G)=1$, and the field k is large enough. Thus we are concerned to find techniques which will work over any field.

We can adapt the definition of a plinth given on [11, p. 547] to this situation. If A is a torsion free abelian subgroup of a group G , and A has finite rank, we say A is a *plinth* in G if there is a subgroup G_0 containing A such that

- (i) $A \trianglelefteq G_0$,
- (ii) $|G : G_0| < \infty$ and $G_0/C_{G_0}(A)$ is abelian, and
- (iii) G_0 and all its subgroups of finite index act rationally irreducibly on A .

If Γ is a group of automorphisms of the abelian group A , we say A is a Γ -*plinth* if A is a plinth in the split extension $G = A \rtimes \Gamma$.

Suppose that A is an $\langle x \rangle$ -plinth and $G = \langle A, x \rangle$. The problem of deciding when kG is primitive here splits into two cases. Suppose that A_0 is a finitely generated subgroup of A of maximum rank. If $A_0^{\langle x \rangle}/A_0$ is finite then we may assume $A_0 = A_0^{\langle x \rangle}$. In this case G is locally polycyclic and the methods of [14] can be applied, at least if k is non-absolute (see Corollary 4.6). We study the contrary case where $A_0^{\langle x \rangle}/A_0$ is infinite.

THEOREM B. *Suppose that A is an $\langle x \rangle$ -plinth such that $G = \langle A, x \rangle$ is not locally polycyclic. If k is any field, then kG is primitive.*

To prove Theorem B, we need a variant of Bergman's Theorem [1], [11, Chapter 9],

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for these groups. We prove that if A is an $\langle x \rangle$ -plinth and I is a non-zero ideal of kA such that $I^x = I$, then kA/I is algebraic (Corollary 4.3). We next look for maximal ideals M of kA such that

$$\bigcap_{n \in \mathbb{Z}} M^{x^n} = 0.$$

This is closely related to the Ergodic Conjecture of Farkas and Passman [4] and is known to imply primitivity.

We set $M^\dagger = A \cap (1 + M)$.

THEOREM C. *Suppose that A is an $\langle x \rangle$ -plinth such that $A = A_0^{\langle x \rangle}$ for a finitely generated subgroup A_0 of A and suppose $G = \langle A, x \rangle$ is not polycyclic-by-finite. If M is a maximal ideal of kA , then $\bigcap_{n \in \mathbb{Z}} M^{x^n} = 0$ if and only if A/M^\dagger is infinite.*

If A/A_0 is a p -group and k is an absolute field of characteristic p , then kA has no maximal ideal M with A/M^\dagger infinite. However, in this case we deduce Theorem B from a result of Irving, [8, Theorem 4.2].

Examples of groups satisfying the hypotheses of Theorems B and C, may be constructed as follows. Let x be an algebraic number and A_0 a \mathbb{Z} -lattice in $\mathbb{Q}(x)$, A the $\mathbb{Z}\langle x \rangle$ -module generated by A_0 , and G the split extension of A by x . If we assume that x^n does not lie in any of the proper subfields of $\mathbb{Q}(x)$ for $n \geq 1$, then A will be an $\langle x \rangle$ -plinth. If in addition we ensure that x is not a unit in the ring of algebraic integers of $\mathbb{Q}(x)$ then $G = \langle A, x \rangle$ is not locally polycyclic.

We remark that Irving has shown that for any integer $r > 1$ the group algebra of the group presented by $\langle a, x \mid a^x = a^r \rangle$ is primitive over any field, [8, Theorem 5.2].

As it turns out, Theorem C provides more information about x -invariant ideals of kA . Given any non-zero x -invariant ideal I of kA , it is easily seen that \sqrt{I} is a semiprime x -invariant ideal of kA , where \sqrt{I}/I is the nil-radical of kA/I . Thus we may assume that I is semiprime. In general, a semiprime ideal of kA is an infinite intersection of prime ideals. However, Theorem C restricts the kinds of maximal ideals M which can contain I to those with A/M^\dagger finite.

We prove the following result, which is a sharper version of Corollary 4.3.

THEOREM D. *Let A be an $\langle x \rangle$ -plinth and I a non-zero x -invariant semi-prime ideal of kA . Suppose that $A = A_0^{\langle x \rangle}$ where A_0 is finitely generated, and A is not finitely generated. Then $|A : I^\dagger| < \infty$, and I is a finite intersection of maximal ideals.*

Theorem A is proved in §3 of this paper; Theorems B–D in §4. In §2, we prove some preliminary results about abelian groups which are needed later.

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2. Abelian Groups of Finite Rank. Throughout this section A will denote a torsion free abelian group of finite rank and A_0 a finitely generated subgroup of maximum rank.

LEMMA 2.1. *If M is a maximal ideal of kA then kA/M is algebraic over k .*

Proof. Since kA is generated by elements which are integral over kA_0 , kA is integral over kA_0 . Hence by [10, Theorems 44 and 47], $M_0 = M \cap kA_0$ is a maximal ideal of kA_0 . By the Nullstellensatz kA_0/M_0 is finite dimensional and the result follows since kA/M is algebraic over kA_0/M_0 .

LEMMA 2.2. *Every maximal ideal M of kA satisfies $|A : M^\dagger| < \infty$ if and only if k is an absolute field of characteristic p , and A/A_0 is a finite extension of a p -group.*

Proof. Suppose the stated conditions hold, and that M is a maximal ideal of kA . By Lemma 2.1 kA/M is an absolute field and A/M^\dagger is periodic, and so is a finite extension of a p -group. As k has characteristic p , $|A : M^\dagger| < \infty$.

To see the necessity of the conditions, let \tilde{k} be the algebraic closure of k and $\phi : A \rightarrow (\tilde{k})^*$ a homomorphism from A to the multiplicative group $(\tilde{k})^*$ of \tilde{k} , and let F be the subfield of \tilde{k} generated by k and $\phi(A)$. Then any element of F may be written as a finite sum $\sum_{x \in A} \lambda_x \phi(x)$ where the coefficients λ_x belong to k . We may regard F as a kA -module. Suppose that V is a non-zero submodule of F and $\alpha \in V, \alpha \neq 0$. Then $\alpha^{-1} = \sum_{x \in A} \lambda_x \phi(x) \in F$. For all x in this finite sum $\alpha \phi(x) \in V$. Therefore $1 = \alpha \alpha^{-1} = \sum \lambda_x \alpha \phi(x) \in V$. Hence F is an irreducible kA -module and $F \cong kA/M$ for some maximal ideal M of kA . Clearly $M^\dagger = \text{Ker } \phi$.

Now suppose ζ is an element of k which is not a root of unity and let a be a non-trivial element of A . The map $\langle a \rangle \rightarrow k^*$ given by $a \rightarrow \zeta$ extends to a homomorphism $\phi : A \rightarrow (k)^*$, since $(k)^*$ is divisible. As $\text{Ker } \phi \cap \langle a \rangle = 1$, $A/\text{Ker } \phi$ is infinite.

Finally suppose that $A/A_0 \cong P \oplus Q$ where P is a p -group and Q is an infinite p' -group. Since Q has finite rank, we see that either Q involves some $C_{q^n}, q \neq p$ or Q involves a direct sum of cyclic groups of prime orders p_i for infinitely many primes p_i . Since $(\tilde{k})^*$ contains a copy of each of these groups this yields a homomorphism $\phi : A \rightarrow \tilde{k}^*$ such that $A/\text{Ker } \phi$ is infinite.

LEMMA 2.3. *Let x be an automorphism of A such that $A = A_0^{(x)}$. If C is a subgroup of A with $A/C \cong C_{p^n}$, then there is an element a of A_0 whose conjugates $\{a^{x^r} \mid r \in \mathbb{Z}\}$ lie in infinitely many distinct cosets of C in A .*

Proof. Let $A_0 = \langle a_1, a_2, \dots, a_n \rangle$. There is no integer s such that $(a_i^{x^r})^{p^s} \in C$ for all $i = 1, \dots, n$ and $r \in \mathbb{Z}$, since A/C has unbounded exponent, and $A = A_0^{(x)}$ is generated by the conjugates $a_i^{x^r}$. Since there are only finitely many a_i , there is an element $a = a_i$ whose conjugates have unbounded order modulo C .

We introduce some notation. Let B be a subgroup of the abelian group A and p a prime. Let \sqrt{B} be the isolator of B in A , that is \sqrt{B}/B is the torsion subgroup of A/B . Similarly we let ${}^p\sqrt{B}/B$ and ${}^{p'}\sqrt{B}/B$ denote the p -torsion and p' -torsion subgroups of A/B respectively. We say B is isolated, p -isolated or p' -isolated according as $B = \sqrt{B}, B = {}^p\sqrt{B}$ or $B = {}^{p'}\sqrt{B}$.

LEMMA 2.4. *Let x be an automorphism of A such that $A = A_0^{(x)}$. Suppose ${}^p\sqrt{A_0}/A_0$ is*

infinite. Then A has a subgroup B containing A_0 such that either

- (1) $B^x \subseteq B$ and B/B^x is a finite p -group, or
- (2) $B \subseteq B^x$ and B^x/B is a finite p -group.

If in addition $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}\langle x \rangle$ -module, then $\bigcap_{n \in \mathbb{Z}} B^{x^n} = 1$.

Proof. Let $A_1 = \sqrt[p]{A_0}$ and $A_2 = \bigcup_{n \geq 0} A_0 A_0^x A_0^{x^2} \dots A_0^{x^n}$. Then $A_2^x \subseteq A_2$. If $a \in A_1$, then $a^m \in A_0$ for some integer m prime to p , so $(a^x)^m \in A_0^x \subseteq A_2 \subseteq A_1 A_2$. Since $A_1 A_2$ is p' -isolated we have $a^x \in A_1 A_2$. Therefore $(A_1 A_2)^x \subseteq A_1 A_2$. If this inclusion is strict, then 1) holds with $B = A_1 A_2$.

If $(A_1 A_2)^x = A_1 A_2$, then since $A_0^{x^{-1}}$ is finitely generated, there is an integer r such that

$$A_0^{x^{-1}} \subset A_1 A_0 A_0^x \dots A_0^{x^r} = B.$$

Since B is p' -isolated this gives $B^{x^{-1}} \subset B$ as before. The inclusion is strict since $\sqrt[p]{A_0}/A_0$ is infinite, and $B/B^{x^{-1}}$ is a finite p -group, so 2) holds.

Finally, suppose that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}\langle x \rangle$ -module and let $C = \bigcap_{n \in \mathbb{Z}} B^{x^n}$. Since $C^x = C$, either $C = 1$ or C has rank n .

If C has rank n then $C_0 = C \cap A_0$ has finite index in A_0 . But then $A_0^{(x)}/C_0^{(x)}$ has finite exponent and is finite. This implies A/C is finite and so A/B is finite. This is inconsistent with either (1) or (2).

We remark that if $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}\langle x \rangle$ -module, the conclusion of Lemma 2.4 is precisely what is required to apply Theorem 4.2 of Irving's paper [8]. This shows that kG is primitive, where $G = \langle A, x \rangle$ and k is any field of characteristic p .

3. Infinite Dimensional Irreducible Modules

LEMMA 3.1. *Let G be a polycyclic-by-finite group, k an absolute field and M a finitely generated kG -module of infinite dimension over k . Then there are elements $m \in M$ and $x \in G$ such that mkX is infinite dimensional where $X = \langle x \rangle$.*

Proof. Without loss of generality M is faithful for G and cyclic, and $M \cong kG/I$ where I is a right ideal of kG . Since G must be infinite, it has a torsion free abelian normal subgroup $A = \langle a_1, \dots, a_n \rangle$. If $I \cap k\langle a_j \rangle = 0$ for some j then we can take $x = a_j$ and $m = 1 + I \in M$. Otherwise for all j , $k\langle a_j \rangle / (I \cap k\langle a_j \rangle)$ has finite dimension, and so $kA / (I \cap kA)$ is finite dimensional. Hence, $T = \{a \in A \mid a - 1 \in I \cap kA\}$ satisfies $|A : T| < \infty$ since k is absolute (see [11, 12.3.8]). Therefore, $|A : \text{core}_G(T)| < \infty$ and since M is faithful A is finite, a contradiction.

Proof of Theorem A. Since G is finitely generated, and the class of polycyclic-by-finite groups is finitely presented, we can suppose that G is not polycyclic-by-finite but that every proper homomorphic image of G is polycyclic-by-finite by [12, Lemma 6.17]. Let A be a non-trivial abelian normal subgroup of G . Then G/A is polycyclic-by-finite and we may treat A as a $\mathbb{Z}(G/A)$ -module.

Case 1. A has infinite rank. We can suppose that $A = \langle a \rangle^G$, where $a \in A$, $a \neq 1$. Hence,

A is a cyclic $\mathbb{Z}G$ -module. Suppose that A is \mathbb{Z} -torsion-free. By [11, Theorem 12.2.7] there is a free abelian subgroup A_0 of A such that A/A_0 is a π -torsion group where π is a finite set of primes. If $p \notin \pi$, then $A^p \cap A_0 = A_0^p$ and so $A_0/A_0^p = A_0/(A^p \cap A_0) \cong (A_0A^p)/A^p \subseteq A/A^p$. Now as A is \mathbb{Z} -torsion free and has infinite rank, A_0 also has infinite rank. Therefore, A_0/A_0^p is an infinite elementary abelian p -group and hence so is A/A^p . Therefore, G/A^p is not polycyclic-by-finite. Our assumption on homomorphic images gives $A^p = 1$, a contradiction since A is \mathbb{Z} -torsion free.

Hence, for some prime p , $A_p = \{a \in A \mid a^p = 1\} \neq 1$. By assumption, G/A_p is polycyclic-by-finite. Hence A_p is an infinite elementary abelian p -group. We can regard A_p as an infinite dimensional \mathbb{F}_pG/A -module. By Lemma 3.1 there are elements $a \in A_p$ and $x \in G$ such that $a\mathbb{F}_pX$ is infinite dimensional, where $X = \langle x \rangle$. It follows that the elements $\{ax^n \mid n \in \mathbb{Z}\}$ are linearly independent over \mathbb{F}_p .

Hence $\langle a \rangle^X \cong \bigoplus_{n \in \mathbb{Z}} a^{x^n}$, and $H = \langle a, x \rangle \cong C_p \sim C_\infty \leq G$. Now by [11, Lemma 9.2.8] kH is primitive for any field k . Therefore, kH has an infinite dimensional irreducible module and so does kG .

Case 2. A has finite rank. Clearly A is not finitely generated. We first reduce to the case where $G = \langle A, x \rangle$ is abelian-by-(infinite cyclic). Let W be an irreducible $\mathbb{Q}G$ -submodule of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $B = A \cap W$. Then B is finitely generated as a $\mathbb{Z}G$ -module but not as an abelian group since G/B is polycyclic-by-finite. We can suppose that $A = B$, so that A is rationally irreducible. By passing to a subgroup of finite index we can suppose $G/C_G(A) = \langle x_1, \dots, x_r \rangle$ is abelian. It follows that there is an element a of A and $x = x_i$ such that $\langle a \rangle^{\langle x \rangle}$ is not finitely generated as an abelian group. Now the group $\langle a, x \rangle$ is finitely generated and metabelian but not polycyclic-by-finite so we can suppose that $G = \langle a, x \rangle$ and $A = \langle a \rangle^G$. As before $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}\langle x \rangle$ -module. Let A_0 be a finitely generated subgroup of A of maximum rank such that $A = A_0^{\langle x \rangle}$. Let k be a field of characteristic $p \geq 0$. There are now two subcases depending on the structure of A/A_0 .

Suppose first that $\sqrt[p]{A_0}/A_0$ is infinite. Then by Lemma 2.4 and [8, Theorem 4.2] kG is primitive.

Finally suppose $\sqrt[p]{A_0}/A_0$ is finite. Since A/A_0 involves only finitely many primes, there is a maximal ideal M of kA such that $A/M \dagger \cong C_{q^n}$ where $q \neq p$, by Lemma 2.2. Let $V = kA/M$ and $W = V \otimes_{kA} kG$. We claim that W is an irreducible kG -module. Let $C = M \dagger$ and using Lemma 2.3 choose an element $a \in C$ whose conjugates $\{a^{x^r} \mid r \in \mathbb{Z}\}$ lie in infinitely many distinct cosets of C in A . Then for all $r \geq 1$, $C^{\langle x^r \rangle}/C$ is infinite and so $C^{\langle x^r \rangle} = A$. Suppose that W' is a non-zero submodule of W and that $w = v_0 + v_1x + \dots + v_r x^r \in W'$ where $v_i \in V$, with $v_0 \neq 0$ and $v_r \neq 0$. If $r \geq 1$, then as $C^{\langle x^r \rangle} = A$ there is an element c of C such that either $x^r c x^{-r} \notin C$ or $x^{-r} c x^r \notin C$. In the first case $v_0 c = v_0$ and $v_r x^r c \neq v_r x^r$ while in the second $v_0 x^{-r} c x^r \neq v_0$ and $v_r x^r (x^{-r} c x^r) = v_r x^r$. A standard ‘shortest length’ argument shows that $W' \cap V \neq 0$. Hence, $V \subseteq W'$ since V is an irreducible kA -module. However, W is generated by the conjugates of V under x so $W' = W$. This completes the proof of Theorem A.

REMARKS. (1) Part of the motivation for Theorem A lies in obtaining a converse to the result of Jategaonkar [9] that a finitely generated abelian-by-polycyclic group is

residually finite. For suppose that the group G satisfies the hypothesis of Theorem A, then there is an irreducible $F_p G$ -module A such that the split extension $A \triangleleft G$ is finitely generated but not residually finite.

(2) We might ask, as in [3, Questions 2 and 9] for general conditions under which all irreducible kG -modules are finite dimensional over k . At present only special cases are known. For countable locally finite groups with no elements of order p see [5], [7] and for soluble groups over \mathbb{C} , see [15]. If G is a finitely generated linear group, then by [16, Theorem 10.16], G is either soluble-by-finite or contains a non-cyclic free subgroup. Since the group algebra of a free group is primitive, it follows from Theorem A that if all the irreducible kG -modules are finite dimensional then G is polycyclic-by-finite.

Finally, we remark that if G is a finitely generated infinite p -group, and k a field of characteristic p , it is not known whether $J(kG) = \omega kG$, see discussion on page 415 of [11]. If this is the case then the only irreducible kG -module is the trivial one.

4. Primitive Group Algebras. Throughout this section A will denote a torsion-free abelian group of finite rank and I an ideal of kA , $I \neq kA$. We first outline how the results of Bergman [1] may be extended to this situation. With minor changes we can follow the proof given in [11, Chapter 9] using log subgroups. A *log subgroup* for I is a subgroup W of A such that for all $\alpha \in I$, $\alpha \neq 0$ there exist $x, y \in \text{Supp } \alpha$ with $xy^{-1} \in W$ and $xy^{-1} \neq 1$. Whenever these subgroups appear as kernels of valuations as in [11, Lemma 9.3.5] they are necessarily isolated in A since the value group is torsion-free. Thus we can work with isolated log subgroups.

LEMMA 4.1. *Let A be torsion-free abelian of finite rank and $\mathcal{F} = \{F_\alpha \mid \alpha \in I\}$ a collection of finite non-empty subsets of A . Let $\mathcal{W} = \mathcal{W}(\mathcal{F})$ be the set of all isolated subgroups W of A such that $W \cap F_\alpha \neq \emptyset$ for all α . Then every member of \mathcal{W} contains a minimal member, and \mathcal{W} has only finitely many minimal members.*

Proof. Note that if \bar{A} is any proper torsion-free image of A , then \bar{A} has smaller rank than A . Hence we can use induction on $\text{rank}(A)$. If $\text{rank}(A) = 0$ then $A = \langle 1 \rangle$ while if $\text{rank}(A) = 1$ the only isolated subgroups of A are $\langle 1 \rangle$ and A and the result is trivial in these cases. If $1 \in F_\alpha$ for all α , then $\langle 1 \rangle$ is the unique minimal member of \mathcal{W} . Suppose that for some β , $1 \notin F_\beta = \{a_1, a_2, \dots, a_n\}$. Let $\mathcal{W}_i = \{W \in \mathcal{W} \mid a_i \in W\}$. Then we have $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \dots \cup \mathcal{W}_n$ and it suffices to show that each \mathcal{W}_i has only finitely many minimal members and that any member of \mathcal{W}_i contains a minimal member.

Consider \mathcal{W}_1 . Let $A_1 = \sqrt{\langle a_1 \rangle}$ the isolator of $\langle a_1 \rangle$ in A , and let $\bar{}$ denote the natural homomorphism $A \rightarrow A/A_1 = \bar{A}$. If $W \in \mathcal{W}_1$ then $W \cap F_\alpha \neq \emptyset$ for all α and so $\bar{W} \cap \bar{F}_\alpha \neq \emptyset$. Moreover, if $W_1, W_2 \in \mathcal{W}_1$ with $\bar{W}_1 = \bar{W}_2$, then since $a_1 \in W_1, W_2$ and the W_i are isolated, $A_1 \subseteq W_1 \cap W_2$ and hence $W_1 = W_2$.

Conversely, if $\bar{W} \in \mathcal{W}(\bar{\mathcal{F}})$ and W denotes the full inverse image of \bar{W} under $\bar{}$ then W is isolated in A , since \bar{W} is isolated in \bar{A} . Hence $\bar{}$ gives a one-one correspondence between members of \mathcal{W}_1 and $\mathcal{W}(\bar{\mathcal{F}})$ and the result follows by induction.

Let I be an ideal of kA . We define the rank of I by

$$\text{rank}(I) = \text{rank}(A) - \max\{\text{rank}(B) \mid I \cap kB = 0\}.$$

The proof of the next theorem and the corollary may now be adapted from [11, Chapter 9].

THEOREM 4.2. *Let A be a torsion-free abelian group of finite rank, and $I \neq kA$ an ideal of kA . Then any isolated log subgroup for I contains a minimal isolated log subgroup, and I has only finitely many minimal isolated log subgroups W_1, W_2, \dots, W_m . Furthermore, $\text{rank}(I) = \min\{\text{rank}(W_i)\}$. Finally if α is any automorphism of A , then $W_1^\alpha, W_2^\alpha, \dots, W_m^\alpha$ are the minimal isolated log subgroups for I^α .*

COROLLARY 4.3. *Let A be torsion-free abelian of finite rank, and Γ a group of automorphisms of A . Suppose that A is a Γ -plinth. If I is a Γ -invariant ideal of kA , then either $I = 0$ or kA/I is algebraic over k , that is given $a \in A$, there is a polynomial $f(t) \in k[t]$ such that $f(a) \in I$.*

We remark that in the situation of the corollary, any prime ideal of kA containing I must be a maximal ideal.

In proving Theorem B, we may assume that $A = A_0^{(x)}$ where A_0 is finitely generated.

LEMMA 4.4. *Let A_0 be a finitely generated subgroup of A of maximum rank. Let x be an automorphism of A such that no power of x centralises A and $A_1 = A_0^{(x)}$, $G = A \triangleleft \langle x \rangle$ and $G_1 = A_1 \triangleleft \langle x \rangle$.*

(i) *Suppose there is a maximal ideal M of kA_1 such that $\bigcap_{n \in \mathbb{Z}} M^{x^n} = 0$, then there is a maximal ideal N of kA such that $\bigcap_{n \in \mathbb{Z}} N^{x^n} = 0$.*

(ii) *If kG_1 is primitive then kG is primitive.*

Proof. (i) Choose a maximal ideal N of kA containing M , by Zorn's Lemma. Then $M = N \cap kA_1$. If $I = \bigcap_{n \in \mathbb{Z}} N^{x^n} \neq 0$, then $I \cap kA_1 \neq 0$, since kA is a domain which is integral over kA_1 . However, $I \cap kA_1 = \bigcap_{n \in \mathbb{Z}} M^{x^n} = 0$. Hence $I = 0$.

(ii) This follows by a similar argument using the fact that any non-zero ideal of kG has non-zero intersection with kA , [11, Lemma 7.4.9].

As in [4] the primitivity problem for torsion free soluble groups of finite rank may often be reduced to the abelian-by-(infinite cyclic) case.

In the next result and the corollary, A will denote the Zaleskii subgroup of the group G (see [11, section 9.1]). If I is a non-zero ideal of kG then $I \cap kA \neq 0$.

LEMMA 4.5. *If G is a torsion-free soluble group of finite rank such that $\Delta(G) = 1$, then $\Delta(\langle A, x \rangle) = 1$ for some $x \in G$.*

Proof. This is essentially the same as the proof of [4, Lemma 4.5]. We merely indicate the modifications that must be made. First we note that for G torsion-free soluble of finite rank, G is nilpotent-by-finite if and only if G is f.c. hypercentral. The crucial point here is that if $B = \Delta(G)$ and B_0 is a finitely generated subgroup of G such that B/B_0 is torsion, then $G_0 = C_G(B_0)$ has finite index in G and G_0 centralizes B since extraction of roots in B is unique. It now follows as in Lemma 4.2 of [4] that $H = G/C_G(A)$ is abelian-by-finite and we may suppose this group is actually abelian. The action of G on A by conjugation induces a representation of H as matrices with rational entries. If the result is

false, then by Lemmas 3.2 and 4.4 of [4] each matrix in H has a complex eigenvalue which is a root of unity.

let $V_0 = A \otimes_{\mathbb{Z}} \mathbb{Q}$, $V = A \otimes_{\mathbb{Z}} \mathbb{C}$. By Lemma 4.3 of [4] there is a finitely generated subgroup H_0 of H with H/H_0 torsion such that H_0 acts trivially on a non-zero subspace W of V . We can assume that

$$W = V^{H_0} = \{v \in V \mid vh = v \text{ for all } h \in H_0\}.$$

Then W is a submodule of V . Also by choosing a basis of \mathbb{C} over \mathbb{Q} we see that $W = W_0 \otimes_{\mathbb{Q}} \mathbb{C}$ where $W_0 = V_0^{H_0}$. If $H_1 = C_H(W_0)$ then H/H_1 is isomorphic to a periodic group of matrices over the rational numbers. Hence H/H_1 is finite, see [12, Part I, p. 85], so there is an integer $k > 0$ such that $G^k \subseteq C_G(W_0)$. The proof may now be completed as in [4, Lemma 4.5].

COROLLARY 4.6. *Let G be a torsion-free soluble group of finite rank with $\Delta(G) = 1$ and suppose that G is locally polycyclic. If k is a non-absolute field, then kG is primitive.*

Proof. By Lemma 4.5 there is an element x in G such that $H = \langle A, x \rangle$ satisfies $\Delta(H) = 1$. It suffices to show that kH is primitive. Since k is non-absolute there is an irreducible kA -module V which is faithful for A . It is easy to see that the induced module $V^H = V \otimes_{kA} kH$ is irreducible since A is self-centralising in H . Let $I = \text{ann}_{kH}(V^H)$. If $I \neq 0$, then $I \cap kA$ is a non-zero x -invariant ideal of kA . Since H is locally polycyclic there is a finitely generated x -invariant subgroup A_0 of A such that A/A_0 is torsion. Then by the argument of Lemma 4.4 $J = I \cap kA_0 \neq 0$ and J is a non-zero x -prime ideal of kA_0 with $J^\dagger = 1$. This contradicts [14, Theorem D].

If $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}\langle x \rangle$ -module we can extend Lemma 2.3.

LEMMA 4.7. *Let x be an automorphism of A such that $A = A_0^{(x)}$ for a finitely generated subgroup A_0 . Suppose that A itself is not finitely generated and that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}\langle x \rangle$ -module. If B is a subgroup of A with A/B infinite then there is an element a of A_0 whose conjugates $\{a^{x^r} \mid r \in \mathbb{Z}\}$ lie in infinitely many distinct cosets of B in A .*

Proof. By Lemma 2.3, it is enough to show that A/B has an image isomorphic to C_p^∞ for some prime p . If \sqrt{B}/B is infinite, we observe that \sqrt{B}/B can involve only finitely many primes, so this follows from the structure of Černikov groups, [6, Theorem 19.2]. If \sqrt{B}/B is finite we can suppose that $B = \sqrt{B}$. If in addition A/A_0B is infinite the result follows again since A/A_0 is Černikov.

Suppose that A/A_0B is finite. Then A/B is finitely generated and since B is isolated

$$A = B \times C \text{ with } C \cong A/B$$

by [6, Corollary 25.3]. Since A/B is infinite, $\text{rank}(B) < \text{rank}(A)$, and so $\bigcap_{n \in \mathbb{Z}} B^{x^n} = 1$. In fact, some finite intersection is trivial. To see this note that each finite intersection $B^{x^{n_1}} \cap B^{x^{n_2}} \cap \dots \cap B^{x^{n_r}}$ is isolated in A and that A has the minimum condition on isolated subgroups. Therefore $B^{x^{n_1}} \cap B^{x^{n_2}} \cap \dots \cap B^{x^{n_r}} = 1$ for finitely many conjugates of B . Hence A embeds in $A/B^{x^{n_1}} \times A/B^{x^{n_2}} \times \dots \times A/B^{x^{n_r}}$ which is finitely generated. This contradiction shows that A/A_0B cannot be finite if B is isolated.

Proof of Theorem C. We have $A = A_0^{\langle x \rangle}$ where A_0 is a finitely generated subgroup of A , and A is an $\langle x \rangle$ -plinth. Let M be a maximal ideal of kA . We must show that $\bigcap_{n \in \mathbb{Z}} M^{x^n} = 0$ if and only if $A/M\uparrow$ is infinite.

If $|A : M\uparrow| = n < \infty$, let $B = \{a^n \mid a \in A\}$ then $B \subseteq M\uparrow$, and since B is characteristic $\bigcap_{n \in \mathbb{Z}} M^{x^n}$ contains ωkB and so is non-zero. Now suppose that $A/M\uparrow$ is infinite and using Lemma 4.7 choose an element a of A whose conjugates $\{a^{x^r} \mid r \in \mathbb{Z}\}$ lie in infinitely many distinct cosets of $M\uparrow$ in A . If $I = \bigcap_{n \in \mathbb{Z}} M^{x^n} \neq 0$, then by Corollary 4.3, there is a non-zero polynomial $f(t) \in k[t]$ such that $f(a) \in I$. Since $I^x = I$ we have $f(a^{x^r}) \in I$ for all $r \in \mathbb{Z}$. However, since the elements $\{a^{x^r} \mid r \in \mathbb{Z}\}$ lie in infinitely many distinct cosets of $M\uparrow$, they represent infinitely many distinct elements of the field kA/M . In other words the non-zero polynomial $f(t)$ has infinitely many distinct roots in some extension field. This contradiction shows that $I = 0$.

Proof of Theorem B. We have a group G of the form $G = \langle A, x \rangle$ where A is an $\langle x \rangle$ -plinth, and G is not locally polycyclic. We have to show that kG is primitive for all fields k .

By Lemma 4.4, we may assume that $A = A_0^{\langle x \rangle}$ where A_0 is a finitely generated subgroup of A . Let k be a field of characteristic $p \geq 0$. If $\sqrt[p]{A_0/A_0}$ is infinite then kG is primitive by Lemma 2.4 and [8, Theorem 4.2]. On the other hand, if $\sqrt[p]{A_0/A_0}$ is finite, then since A/A_0 is infinite, it is not a finite extension of a p -group. Therefore, by Lemma 2.2 there is a maximal ideal M of kA such that $A/M\uparrow$ is infinite. Hence, $\bigcap_{n \in \mathbb{Z}} M^{x^n} = 0$ by Theorem C and again kG is primitive.

Proof of Theorem D. We have an $\langle x \rangle$ -plinth A and a non-zero x -invariant semiprime ideal I of kA . We have to show that $A/I\uparrow$ is finite and that I is a finite intersection of maximal ideals.

Corollary 4.3 shows that kA/I is algebraic so any prime ideal containing I is maximal. Let a be an element of A and $f(t) \in k[t]$ a polynomial such that $f(a) \in I$. For a maximal ideal M containing I we have $|A : M\uparrow| < \infty$ by Theorem C. Let n be the least integer such that $a^n - 1 \in M$, then $\Phi_n(a) \in M$ where $\Phi_n(t)$ is the n th cyclotomic polynomial. Note that n is prime to the characteristic of k . If $g(t)$ is a polynomial of least degree such that $g(a) \in M$, then $g(t)$ is irreducible, and $g(t)$ divides $\Phi_n(t)$ and $f(t)$ since $\Phi_n(a)$ and $f(a)$ belong to I . Therefore, for each such integer n , $\Phi_n(t)$ and $f(t)$ have a common factor. Since the cyclotomic polynomials $\Phi_n(t)$ are relatively prime we obtain only finitely many integers n as M ranges over all maximal ideals containing I . Hence, $a^m - 1 \in I$, for some integer m so that $A/I\uparrow$ is periodic. If $I_0 = I\uparrow \cap A_0$ then A_0/I_0 is finite, and so $A_0^{\langle x \rangle}/I_0^{\langle x \rangle}$ has finite exponent so is finite. However, $I_0^{\langle x \rangle} \subseteq I\uparrow$ since the latter is $\langle x \rangle$ -invariant and contains I_0 . Therefore, $A/I\uparrow$ is finite and $\dim_k kA/I$ is finite. Therefore, there are only finitely many maximal ideals containing I , and I is their intersection.

REFERENCES

1. G. M. Bergman, The logarithmic limit set of an algebraic variety, *Trans. Amer. Math. Soc.* **157** (1971), 459–469.

2. K. A. Brown, Primitive group rings of soluble groups, *Arch. Math. (Basel)* **36** (1981), 404–413.
3. D. R. Farkas, Group rings, an annotated questionnaire, *Comm. Alg.* **8** (1980), 585–602.
4. D. R. Farkas and D. S. Passman, Primitive Noetherian group rings, *Comm. Alg.* **5** (1978), 301–315.
5. D. R. Farkas and R. L. Snider, Group algebras whose simple modules are injective, *Trans. Amer. Math. Soc.* **194** (1974), 241–248.
6. L. Fuchs, *Abelian groups* (Pergamon, 1960).
7. B. Hartley, Injective modules over group rings, *Quart. J. Math.* **28** (1977), 1–29.
8. R. S. Irving, Some more primitive group rings, *Israel J. Math.* **37** (1980), 331–350.
9. A. V. Jategaonkar, Integral group rings of polycyclic-by-finite groups, *J. Pure and Applied Algebra* **4** (1974), 337–343.
10. I. Kaplansky, *Commutative rings* (University of Chicago Press, 1974).
11. D. S. Passman, *The algebraic structure of group rings* (Wiley-Interscience, 1977).
12. D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, parts 1 and 2 (Springer-Verlag, 1972).
13. J. E. Roseblade, Group rings of polycyclic groups, *J. Pure and Applied Algebra* **3** (1974), 307–328.
14. J. E. Roseblade, Prime ideals in group rings of polycyclic groups, *Proc. London Math. Soc.* **36** (1978), 385–447.
15. R. L. Snider, Soluble groups whose representations are finite dimensional, *Abstracts Amer. Math. Soc.* (1980), Abstract 775-A6, page 250.
16. B. A. F. Wehrfritz, *Infinite linear groups* (Springer-Verlag, 1973).
17. B. A. F. Wehrfritz, Groups whose irreducible representations have finite degree I, *Math. Proc. Cambridge Philos. Soc.* **90** (1981), 411–421.

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