A CLASS OF MAXIMAL ORDERS INTEGRAL
OVER THEIR CENTRES

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1. Introduction. In a recent paper [1], Brown, Hajarnavis and MacEacharn have considered non-commutative Noetherian local rings of finite global dimension which are integral over their centres. For such a ring $R$ they have shown:

(i) $R$ is a prime ring whose Krull and global dimensions coincide;

(ii) $R = \bigcap R_p$ where $p$ runs through the set of rank one primes of the centre of $R$, and each $R_p$ is hereditary;

(iii) the centre of $R$ is a Krull domain.

We shall show that each $R_p$ in (ii) above is in fact a principal right and left ideal ring. We deduce that the above ring $R$ is a maximal order (defined below), and that if $R$ is in addition a PI-ring with centre $Z$ then it is a maximal $Z$-order in the sense of Fossum [5].

Our result covers the case where $R$ is a local Noetherian ring of finite global dimension finitely generated as a module over its centre, which has previously been discussed in [7], and indeed our proof is somewhat easier than that given there. However, let $D$ be a division ring which is locally finite dimensional, but not finite dimensional, over its centre. Then the localization of the polynomial ring $D[X_1, \ldots, X_n]$ at the maximal ideal generated by $X_1, \ldots, X_n$ is a local Noetherian ring of global dimension $n$ which is integral, but not finitely generated, over its centre. The reader will find further details in [1, 7.1].

Throughout, all rings will be assumed to have an identity, and Noetherian will mean left and right Noetherian. A ring $R$ with Jacobson radical $J$ is called semilocal (respectively local) if $R/J$ is semisimple (respectively simple) Artinian. For a right $R$-module $M$, $M^{\oplus s}$ denotes a direct sum of $s$ copies of $M$.

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2. The Main Theorem. We shall first briefly define maximal orders (in the sense of Asano). Details may be found in [6].

Let $R$ be a ring with a simple Artinian (right and left) quotient ring $Q$. A subset $I$ of $Q$ is called a right $R$-ideal if:

(i) $I$ is a right $R$-submodule of $Q$;

(ii) $I$ contains a unit of $Q$;

(iii) $UI \subseteq R$ for some unit $u$ of $Q$.

Left $R$-ideals and (two-sided) $R$-ideals are defined in the obvious fashion. Clearly any non-zero ideal of $R$ is an $R$-ideal. If $I$ is an $R$-ideal, write

$O_r(I) = \{q \in Q \mid Iq \subseteq I\}$

$O_l(I) = \{q \in Q \mid qI \subseteq I\}$.
Then $R$ is called a maximal order if $O_r(I) = R = O_l(I)$ for each $R$-ideal $I$ of $Q$. We note by [6, 3.1], $R$ is a maximal order precisely when $O_r(I) = R = O_l(I)$ for each non-zero (ordinary) ideal $I$ of $R$.

We begin with the following lemma, whose proof may be found in, for example, [4, 10.2].

**Lemma 1.** Let $R$ be any ring, $J$ its Jacobson radical, and $P$ and $Q$ finitely generated (f.g.) projective right $R$-modules. If $P/PJ$ is an $R/J$-module direct summand of $Q/QJ$, then $P$ is a direct summand of $Q$.

Since a local ring has a unique simple right module (up to isomorphism), it follows that such a ring has a unique f.g. projective indecomposable right module. We shall, however, wish to apply Lemma 1 to certain semilocal localizations of a local ring, and thus require:

**Lemma 2.** Let $R$ be a right Noetherian ring of finite right global dimension, and suppose that $R$ has a unique f.g. projective indecomposable right module $P$. Let $S = R_\mathcal{T}$ be the classical localization of $R$ at a right Ore set $\mathcal{T}$ of regular elements. Suppose that $S$ is semilocal. Then $S$ has a unique f.g. projective indecomposable right module, namely $P \otimes_R S$.

**Proof.** Let $Q$ be a f.g. projective indecomposable right $S$-module. We can write $Q = q_1S + \ldots + q_tS$ with each $q_i \in Q$. Let

$$K = q_1R + \ldots + q_tR,$$

and form an $R$-projective resolution

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to K \to 0.$$

Each $P_i$ can be chosen finitely generated, and hence is a direct sum of copies of $P$. Since $K \otimes_R S \cong Q$ and $RS$ is flat, we have an exact sequence of $S$-modules

$$0 \to P_n \otimes_R S \to \ldots \to P_0 \otimes_R S \to Q \to 0.$$

As $Q$ is $S$-projective an easy induction on the length of this resolution shows that there are integers $k$ and $l$ such that

$$(P \otimes_R S)^{\oplus k} \oplus Q \cong (P \otimes_R S)^{\oplus l}.$$

If $J$ is the Jacobson radical of $S$, we obtain

$$
\frac{(P \otimes_R S)^{\oplus k}}{(P \otimes_R S)^{\oplus k} \cdot J} \oplus \frac{Q}{QJ} \cong \frac{(P \otimes_R S)^{\oplus l}}{(P \otimes_R S)^{\oplus l} \cdot J}.
$$

Comparing the simple modules occurring, we must therefore have

$$
\frac{Q}{QJ} \cong \frac{(P \otimes_R S)^{\oplus (l-k)}}{(P \otimes_R S)^{\oplus (l-k)} \cdot J}.
$$

From Lemma 1 and the indecomposability of $Q$ we deduce $Q \cong P \otimes_R S$, as required.
We fix some notation. For the remainder of the paper, \( R \) will be a local Noetherian ring of finite global dimension integral over its centre \( Z \). Further, \( \mathcal{B} \) will denote the set of rank one primes of \( Z \). We can now prove:

**Proposition 3.** For each \( p \in \mathcal{B} \), \( R_p \) is a principal left and right ideal ring.

**Proof.** By the result quoted in the introduction, \( R_p \) is certainly a hereditary Noetherian prime ring, and is semilocal by [3, 2.2]. Let \( I \) be a non-zero right ideal of \( R_p \). We are to prove that \( I \) is principal, and so we may assume that \( I \) is essential as a right ideal of \( R_p \). By Lemma 2, \( R_p \) has a unique f.g. projective indecomposable right module \( Q \), and so \( I \cong Q^\oplus s \) for some \( s \). Also, \( R \cong Q^\oplus t \) for some \( t \). Since the uniform dimensions of \( IR \) and \( R_R \) are equal, we have \( s = t \) and \( I \) is right principal.

We are in a position to obtain our main result.

**Theorem 4.** \( R \) is a maximal order.

**Proof.** We have \( R = \bigcap_{p \in \mathcal{B}} R_p \) by [1, 6.7], and by Proposition 3 each \( R_p \) is a principal left and right ideal ring. If now \( I \) is a non-zero ideal of \( R \) and \( q \) lies in the quotient ring of \( R \),

\[
qI \subset I \Rightarrow qIR_p \subset IR_p \quad \text{for each} \quad p \in \mathcal{B} \Rightarrow q \in \bigcap_{p \in \mathcal{B}} R_p
\]

since \( IR_p \) is an invertible ideal of \( R_p \). Thus \( R \) is a maximal order by [6, 3.1].

Theorem 4 fails should the requirement that \( R \) be local be weakened to one of semilocality. To see this, let \( S \) be the ring of integers localized at 2 and, using the usual notation, put

\[
T = \begin{bmatrix} S & 2S \\ S & S \end{bmatrix}.
\]

Then \( T \) is a semilocal hereditary Noetherian prime ring finitely generated over its centre. However, \( T \) is not a maximal order. For if \( I \) is the ideal

\[
egin{bmatrix} 2S & 2S \\ S & S \end{bmatrix}
\]

of \( T \), and

\[
q = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}
\]

then \( qI \subset I \) and \( q \) lies in the quotient ring of \( T \), yet \( q \notin T \).

We recall the definition of a maximal \( C \)-order from [5]. Let \( C \) be a Krull domain with quotient field \( K \), and \( Q \) a finite dimensional central simple \( K \)-algebra. A \( C \)-order is, in the sense of Fossum, a subring \( T \) of \( Q \) satisfying:

(i) \( C \subset T \);
(ii) \( K \cdot T = Q \);
(iii) \( T \) is integral over \( C \).

A \( C \)-order is called maximal if it is not properly contained in any \( C \)-order in \( Q \).
Suppose that $R$ is, in addition to our previous assumptions, a PI-ring. Proposition 1.5 of [2] now guarantees that $R$ is a maximal Z-order.

In particular we note that, by [6, 4.2 p. 147], for each $p \in B$ there is a unique prime ideal of $R$ lying over $p$. Each $R_p$ is thus a local ring. Presumably this last statement remains valid without the additional PI hypothesis, but we have been unable to confirm this.

REFERENCES