SEMI-NORMAL OPERATORS ON UNIFORMLY SMOOTH
BANACH SPACES

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1. Introduction. In this paper we shall examine the relationship between the
numerical ranges and the spectra for semi-normal operators on uniformly smooth spaces.

Let $X$ be a complex Banach space. We denote by $X^*$ the dual space of $X$ and by
$B(X)$ the space of all bounded linear operators on $X$. A linear functional $F$ on $B(X)$ is
called state if $\|F\| = F(1) = 1$. When $x \in X$ with $\|x\| = 1$, we denote
$$D(x) = \{f \in X^* : \|f\| = f(x) = 1\}.$$ 

Let us set
$$\Pi = \{(x, f) \in X \times X^* : \|f\| = f(x) = \|x\| = 1\}.$$ 

The spatial numerical range $V(T)$ and the numerical range $V(B(X), T)$ of $T \in B(X)$
are defined by
$$V(T) = \{f(Tx) : (x, f) \in U\}$$
and
$$V(B(X), T) = \{F(T) : F \text{ is a state on } B(X)\},$$
respectively.

If $V(T) \subset \mathbb{R}$, then $T$ is called hermitian. An operator $T \in B(X)$ is called hyponormal
(co-hyponormal) if there are hermitian operators $H$ and $K$ such that $T = H + iK$ and
$C = i(HK - KH) \geq 0 \leq 0$.

An operator $T \in B(X)$ is called semi-normal if $T$ is hyponormal or co-hyponormal.
An operator $T$ is called normal if there are hermitians $H$ and $K$ such that $T = H + iK$
and $HK = KH$.

For an operator $T \in B(X)$, the spectrum, the approximate point spectrum, the point
spectrum, the kernel and the dual operator of $T$ are denoted by $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$,
$\text{Ker}(T)$ and $T^*$, respectively.

The following results are well-known:
(1) $\text{co} \ V(T) = V(B(X), T)$, where $\text{co} \ E$ is the closed convex hull of $E$.
(2) $\text{co} \ \sigma(T) \subset \overline{V(T)}$, where $\text{co} \ E$ and $\overline{E}$ are the convex hull and the closure of $E$,
respectively.
(3) $V(T) \subset V(T^*) \subset \overline{V(T)}$.
(4) If $T$ is normal, then $\sigma(T) = \sigma_a(T)$ and $\text{co} \ \sigma(T) = \overline{V(T)} = V(B(X), T)$.

Remark 1. From (3), if $T$ is hyponormal or co-hyponormal, then $T^*$ is co-
hyponormal or hyponormal, respectively.

We set, for $t > 0$:
$$\rho(t) = \sup \{\frac{1}{t}(\|x + y\| + \|x - y\|) - 1 : \|x\| = 1, \|y\| \leq t\}.$$ 

A Banach space $X$ is called uniformly smooth if
$$\frac{\rho(t)}{t} \to 0 \quad \text{as} \quad t \to 0.$$ 

REMARK 2. A Banach space $X$ is uniformly smooth iff $X^*$ is uniformly convex. See [3] for details.

We recall from [1] and [2] the construction of a larger space $X^0$ from a given Banach space $X$. Then the mapping $T \to T^0$ is an isometric isomorphism of $B(X)$ onto a closed subalgebra of $B(X^0)$. Let $\text{Lim}$ be fixed Banach limit on the space of all bounded sequences of complex numbers with the norm $\|\{\lambda_n\}\| = \sup\{|\lambda_n| : n \in \mathbb{N}\}$. Let $\hat{X}$ be the space of all bounded sequences $\{x_n\}$ of $X$. Let $N$ be the subspace of $\hat{X}$ consisting of all bounded sequences $\{x_n\}$ with $\text{Lim} \|x_n\|^2 = 0$. The space $X^0$ is defined as the completion of the quotient space $\hat{X}/N$ with respect to the norm $\|\{x_n\} + N\| = (\text{Lim} \|x_n\|^2)^{1/2}$. Then the following results hold:

$$\sigma(T) = \sigma(T^0), \quad \sigma_\pi(T) = \sigma_\pi(T^0) = \sigma_p(T^0) \quad \text{and} \quad \overline{\sigma_\pi} V(T) = V(T^0).$$


We need the following results.

THEOREM A [2, Theorem 4]. $X$ is uniformly convex iff $X^0$ is uniformly convex.

THEOREM B [5, Lemma 20.3 and Corollary 20.10]. If $H$ is hermitian and $Hx = 0$ with $\|x\| = 1$, then there exists $f \in X^*$ such that $(x, f) \in \Pi$ and $H^*f = 0$.

2. Semi-normal operators on uniformly smooth spaces.

THEOREM 1. Let $X$ be uniformly smooth. Let $T = H + iK$ be semi-normal on $X$.

(1) If $a \in \sigma(H)$, then there is a real number $b$ such that $b \in \sigma(K)$ and $a + ib \in \sigma(T)$.

(2) If $b' \in \sigma(K)$, then there is a real number $a'$ such that $a' \in \sigma(H)$ and $a' + ib' \in \sigma(T)$.

Proof. (1) Since $H$ is hermitian, there exists a sequence $\{x_n\}$ of unit vectors in $X$ such that $(H - a)x_n \to 0$. Since $X^*$ is uniformly convex, by Theorem 3.11 in Mattila [11] it follows that $(H^* - a)f_n \to 0$, where $f_n \in D(x_n)$. Consider the larger space $X^*_{\text{co}}$ of $X^*$. Then $\text{Ker}(H^* - a)$ is a non-zero subspace of $X^*_{\text{co}}$. If $f_0 \in \text{Ker}(H^* - a)$ such that $\|f_0\| = 1$, then by Theorem B there is a sequence $\{\varphi_n\}$ such that $\|\varphi\| = \varphi(f_0) = 1$ and $(H^* - a)\varphi = 0$. We may assume that $C = i(HK - KH) \geq 0$. Then $C^* = i(K^*H^* - H^*K) \geq 0$ and

$$\varphi(C^*f_0) = i\varphi(K^*(H^* - a)f_0) - if_0^*(K^*H - H^*K)f_0 = 0,$$

where $\hat{f}_0$ is the Gel’fand representation of $f_0$. Since, by Theorem A, the space $X^*_{\text{co}}$ is uniformly convex and $C^*_{\text{co}} \geq 0$, it follows that $C_{\text{co}}f_0 = 0$ by Theorem 2.1 in [12]. Therefore, we have that

$$(H^* - a)Kf_0 = 0.$$ 

It is easy to see that $\text{Ker}(H^* - a)$ is invariant for $K^*$. Hence, there exist a real number $b$ and non-zero vector $g_0$ in $\text{Ker}(H^* - a)$ such that $K^*g_0 = bg_0$. It follows that $b \in \sigma_p(K^*)$ and $a + ib \in \sigma_p(T^*)$. And we have that $b \in \sigma(K^*) = \sigma(K)$ and $a + ib \in \sigma(T^*) = \sigma(T)$.

(2) is proved in the same way as (1).

THEOREM 2. Let $X$ be uniformly smooth. Let $T = H + iK$ be semi-normal. Then

$$\co \sigma(T) = \overline{V(T)} = V(B(X), T).$$
Proof. We assume that Re $\sigma(T) \subset \mathbb{R}^+$. Then by Theorem 1 it follows that $\sigma(H) \subset \mathbb{R}^+$. Since $\sigma(H) = V(H) = V(B(X), H)$, it follows that Re $V(B(X), T) \subset \mathbb{R}^+$. Since $\alpha T + \beta$ is semi-normal for every $\alpha, \beta \in \mathbb{C}$, it follows that $\sigma(T) = \overline{V(T)} = V(B(X), T)$.

**Theorem 3.** Let $X$ be uniformly smooth. Let $T = H + iK$ be co-hyponormal on $X$. If $a + ib \in \sigma(T)$, then $a \in \sigma(H)$ and $b \in \sigma(K)$.

**Proof.** If $a + ib \in \sigma(T)$, then $a + ib \in \sigma(T^*)$. Thus there exists $b' \in \mathbb{R}$ such that $a + ib'$ belongs to the boundary of $\sigma(T^*)$. Therefore there exists a sequence $(f_n)$ of unit vectors in $X^*$ such that $(T^* - (a + ib'))f_n \to 0$. Since $X^*$ is uniformly convex and $T^*$ is hyponormal on $X^*$, by Theorem 2.7 in [12] we have that $(H^* - a)f_n \to 0$ and $(K^* - b')f_n \to 0$. It follows that $a \in \sigma(H)$.

$b \in \sigma(K)$ is proved analogously.

**Corollary 4.** Let $X$ be uniformly smooth. Let $T = H + iK$ be co-hyponormal on $X$. Then Re $\sigma(T) = \sigma(H)$ and Im $\sigma(T) = \sigma(K)$.

**Proof.** The proof follows easily from Theorems 1 and 3.

**Problem.** Does Theorem 3 hold for a hyponormal operator?

**Remark 3.** The following theorem holds, which corresponds to Theorem 10.6 in [4]. Let $X$ be uniformly smooth. Then

$$\{ \lambda \in \overline{V(T)} : |\lambda| = \|T\| \} \subset \sigma_n(T).$$

It follows from the uniform convexity of $X^*$ and $\overline{V(T)} = \overline{V(T^*)}$.

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**References**

10. M. Chô and H. Yamaguchi, Bare points of joint numerical ranges for doubly commuting hyponormal operators on strictly $c$-convex spaces, preprint.


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