GRADED RINGS OF COHOMOLOGICAL DIMENSION 2

Q. S. WU
Institute of Mathematics, Fudan University, Shanghai, 200433, China
e-mail: cwu@math.washington.edu, qswu@fudan.edu.cn

and J. J. ZHANG
Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA
e-mail: zhang@math.washington.edu

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Abstract. Let $A$ be a noetherian connected graded ring with a balanced dualizing complex $R$. If $A$ has cohomological dimension and Krull dimension 2, then

1. $R$ is Auslander;
2. $\text{Kdim} M = \text{Kdim} M$ for all noetherian graded $A$-modules $M$.

In particular, if $A$ is AS-Gorenstein of injective and Krull dimension 2, then

3. $A$ is Auslander-Gorenstein;
4. $A$ is 2-pure with a self-injective artinian quotient ring;
5. $A$ has a residue complex.

(1,3,4) generalize a result of Levasseur [7, 5.13] and (5) generalizes a result of Ajitabh [1, 3.12].

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0. Introduction. Auslander property is closely related to other properties such as catenarity, localizability, and existence of nice dimension function (see [16]). The Auslander property is proved for some classes of rings such as AS-regular rings of global dimension no more than 3 [2, 3, 7] and the Sklyanin algebras [11, 12]. It is an open question whether or not every noetherian AS-Gorenstein ring is Auslander-Gorenstein. Levasseur [7, 5.13] proved that a noetherian AS-Gorenstein ring $A$ of injective dimension 2 is Auslander-Gorenstein if $A$ has Gelfand-Kirillov dimension 2 and $A$ is 2-homogeneous (i.e., 2-pure) with a self-injective artinian quotient ring. The main object of this note is to generalize Levasseur’s result. The following is proved in Section 3.

Theorem 0.1. Let $A$ be a noetherian connected graded ring with a balanced dualizing complex $R$. Suppose $A$ has cohomological dimension 2. Then the following are equivalent:

1. $R$ is Auslander.
2. $\text{Kdim} A = 2$ where $\text{Kdim}$ denotes the Krull dimension.
3. there is a graded dimension function $\text{Kdim}$ such that $\text{Kdim} A < 3$.
4. for any chain of graded primes $P = P \subseteq P \subset A$, $\text{Kdim} A/P \leq 1$.
5. the canonical dimension $\text{Cdim}$ is a graded dimension function.
6. $\text{Cdim}$ is exact.
7. $\text{Cdim} M = \text{Kdim} M$ for all noetherian graded $A$-modules $M$.

The definitions of the Auslander property and Gorenstein property are given next.
Definition 0.2. A balanced dualizing complex $R$ over $A$ is Auslander if for every noetherian graded $A$-module $M$ and for every graded $A$-submodule $N \subseteq \text{Ext}^i(M, R)$, $\text{Ext}^j(N, R) = 0$ for all $j < i$.

Definition 0.3. (1) A noetherian connected graded ring $A$ is called AS-Gorenstein (where AS stands for Artin and Schelter) if $A$ has injective dimension $d < \infty$ and $\text{Ext}^i(k, A) = 0$ for $i \neq d$ and $\text{Ext}^d(k, A) = k(e)$ for some $e \in \mathbb{Z}$, where $k$ is viewed as either left or right trivial $A$-module.

(2) A noetherian (graded) ring $A$ is called (graded) Auslander-Gorenstein if $A$ has finite injective dimension and for every noetherian (graded) $A$-module $M$ and for every (graded) $A$-submodule $N \subseteq \text{Ext}^i(M, A)$, $\text{Ext}^j(N, A) = 0$ for all $j < i$.

We refer to [13, 14, 15, 16] for other definitions and notations.

By [7, 6.3] every connected graded Auslander-Gorenstein ring is AS-Gorenstein. By [5, 0.1], a noetherian graded ring is Auslander-Gorenstein if and only if it is graded Auslander-Gorenstein. By [14, 4.14], AS-Gorenstein rings have balanced dualizing complexes and in this case the injective dimension of $A$ is its cohomological dimension. By [15, Section 0], a connected graded ring is Auslander-Gorenstein if and only if it is AS-Gorenstein and its balanced dualizing complex is Auslander.

In addition to Theorem 0.1, we prove the following for AS-Gorenstein rings in Section 4.

Theorem 0.4. Let $A$ be a noetherian AS-Gorenstein ring of injective dimension 2. Then the following are equivalent (and equivalent to the conditions in Theorem 0.1):

1. $A$ is Auslander-Gorenstein.
2. $A$ has a self-injective artinian quotient ring.
3. $A$ has an artinian quotient ring.
4. For every minimal prime ideal $P \subseteq A$, $K\dim A/P = 2$.
5. $A$ has a residue complex.
6. $A$ is 2-pure, i.e., for every nonzero left (right) ideal $I \subseteq A$, $K\dim I = 2$.

We are unable to show that every noetherian connected graded ring of cohomological dimension 2 has Krull dimension 2. The following partial result is proved in Section 2.

Proposition 0.5. Let $A$ be a noetherian connected graded ring with a balanced dualizing complex. If $A$ has cohomological dimension 2, then

1. $K\dim A_A \leq 2$, where $A^e = A \otimes A^{op}$,
2. there are only finitely many graded prime ideals $P$ with $K\dim A/P \geq 2$.

1. Balanced dualizing complexes. Let $k$ be a base field and $A$ a connected graded ring over $k$, i.e., $A = k \oplus A_1 \oplus A_2 \oplus \cdots$. Unless otherwise stated we are working on connected graded rings over $k$ and their graded modules. If $A$ is left (or right) noetherian, then each $A_i$ is finite dimensional over $k$. Let $\mathfrak{m}$ denote the maximal graded ideal $A_{\geq 1}$ of $A$. The $\mathfrak{m}$-torsion submodule of $M$, denoted by $\Gamma_\mathfrak{m}(M)$, is the union of all finite dimensional submodules of $M$. The left (right) cohomological dimension of $A$ is defined to be the cohomological dimension of $\Gamma_\mathfrak{m}$ applying to the left (right) graded $A$-modules.
Our main tool is the balanced dualizing complex introduced by Yekutieli [14]. A theorem of Van den Bergh [13, 6.3] states as follows: A noetherian connected graded ring \( A \) has a balanced dualizing complex if and only if \( A \) satisfies left and right \( \chi \)-condition and has finite left and right cohomological dimension. Both \( \chi \)-condition and having finite cohomological dimension are checked for a large class of noetherian graded rings [4, 16].

From now on \( A \) is a noetherian connected graded ring with a balanced dualizing complex \( R \). It follows from [13, 4.8] that the left cohomological dimension of \( A \) is equal to the right cohomological dimension of \( A \), which is denoted by \( \text{cd} A \).

Let \( M \) be a graded \( A \)-module. The graded vector space dual of \( M \) is \( M^0 \hat{\otimes}_n \text{Hom}_k(M, n, k) \). The local duality theorem states the following: for any \( X \in D(\text{Gr} A) \),

\[
R\Gamma_m(X) = R\text{Hom}_A(X, R) \tag{E1.1}
\]

(see [14, 4.18] and [13, 5.1]).

A dualizing complex \( R \) induces a convergent spectral sequence: for any noetherian module \( M \)

\[
E^{p,q}_2 := \text{Ext}^p(\text{Ext}^q(M, R), R) \Rightarrow \text{Ext}^{p+q}(M)
\]

where \( \text{Ext}^{p,q}(M) = 0 \) if \( p \neq q \) and \( \text{Ext}^{0,0}(M) = M \) [16, 1.7].

Suppose \( A \) has cohomological dimension \( d \) with a balanced dualizing complex \( R \). It follows from the local duality theorem (E1.1) that, for all \( i > 0 \) and \( i < -d \),

\[
\text{Ext}^i(M, R) = 0
\]

for all left (or right) \( A \)-modules \( A \). Hence we may choose \( R \) to be a complex in \( D^b_{\text{fg}}(\text{Gr} A) \) of the form

\[
R = \cdots 0 \rightarrow R^{-d} \rightarrow \cdots \rightarrow R^{-1} \rightarrow R^0 \rightarrow 0 \cdots
\]

where each \( R^q \) is injective as left and as right \( A \)-module.

**Lemma 1.3.** Let \( M \) be a noetherian graded \( A \)-module.

1. \( \text{Ext}^0(M, R) = \Gamma_m(M) \).
2. If \( \dim_k M < \infty \), then \( \text{Ext}^q(M, R) = 0 \) for all \( q \neq 0 \) and \( \text{Ext}^0(M, R) = M' \).
3. If \( \dim_k M = \infty \), then \( \dim_k \text{Ext}^q(M, R) = \infty \) for some \( q < 0 \).

**Proof.** (1.2) Follow from the local duality (E1.1).

(3) Suppose that \( \text{Ext}^q(M, R) \) is finite dimensional for all \( q < 0 \). By (1) \( \text{Ext}^0(M, R) \) is finite dimensional. Hence, by (2), \( \text{Ext}^q(\text{Ext}^0(M, R), R) \) is finite dimensional for all \( p, q \). It follows from the spectral sequence (E1.2) that \( M \) is finite dimensional. \( \square \)

If \( \text{cd} A = 0 \), then the short exact sequence \( 0 \rightarrow m \rightarrow A \rightarrow A/m \rightarrow 0 \) induces an exact sequence

\[
0 \rightarrow \Gamma_m(m) \rightarrow \Gamma_m(A) \rightarrow A/m \rightarrow R^1 \Gamma_m(m) = 0.
\]
Hence \(1 \in \Gamma_m(A)\), so \(A = \Gamma_m(A)\) is finite dimensional. Next we consider the case \(\text{cd}A = 1\). Let \(\text{GKdim}\) denote the Gelfand-Kirillov dimension.

**Lemma 1.4.** Let \(A\) and \(B\) be noetherian graded rings.

1. If \(\text{GKdim}A = 1\), then \(A\) is PI (PI=satisfying a polynomial identity) and \(\text{Kdim}A = 1\).
2. If \(\text{Kdim}A = 1\), then \(\text{GKdim}A = 1\).
3. If \(M\) is a graded \((A,B)\)-module noetherian on both sides, then the following are equivalent:
   a. \(\text{Kdim}M_B = 1\)
   b. \(\text{GKdim}M_B = 1\)
   c. \(1 \leq \text{GKdim}M_B < 2\)
   d. \(\text{GKdim}_AM = 1\).

**Proof.** (1) By [10], if \(\text{GKdim}A = 1\), then \(A\) is PI. By [8, 6.4.8 and 13.10.6] \(\text{Kdim}A = \text{GKdim}A = 1\).

(2) We may assume \(A\) is prime. By a graded version of Goldie’s theorem [9, I.1.6], there is a homogeneous regular element \(x \in A\) of degree \(d > 0\). Hence \(\text{Kdim} A/xA = 0\), which implies that \(A/xA\) is finite dimensional. Hence \(\text{GKdim}A = 1\).

(3) By noetherian induction we may assume \(M\) is critical and \(\text{ann}(A,M) = Q\) and \(\text{ann}(M_B) =: P\) are graded prime ideals such that \(A\) is \(Q\)-torsionfree and \(M_B\) is \(B/P\)-torsionfree. Hence \(\text{Kdim} M_B = \text{Kdim} B/P\) and \(\text{GKdim}M_B = \text{GKdim}B/P\). Now (a) and (b) are equivalent by (1,2), and (b) and (c) are equivalent by Bergman’s gap theorem [6, 2.5]. (b) and (d) are equivalent because \(\text{GKdim}\) is symmetric [6, 5.4].

**Lemma 1.5.** If \(\text{cd}A = 1\), then \(A\) is PI of \(\text{Kdim}1\). More generally, if \(M\) is a noetherian graded \(A\)-module such that (a) \(\text{Ext}^{-i}(M,R) = 0\) for all \(i > 1\) and (b) \(f : M(-d) \to M\) is an injective map for some \(d > 0\), then \(M/f(M)\) is finite dimensional.

**Proof.** We may assume that \(A\) is prime [4, 8.5] and not \(k\). By [9, I.1.6], there is a homogeneous regular element \(x \in A\) of degree \(d > 0\). Then there is a short exact sequence

\[
0 \to A(-d) \to A \to A/xA \to 0.
\]

By Lemma 1.4, it suffices to show \(\text{GKdim}A = 1\). To prove \(\text{GKdim}A = 1\), it suffices to prove that \(A/xA\) is finite dimensional over \(k\).

We now consider the general case when \(0 \to M(-d) \to M \to M/f(M) \to 0\) is exact and show that \(M/f(M)\) is finite dimensional. By the long exact sequence we see that

\[
\text{Ext}^{-i}(M/f(M),R) = 0 \quad \text{for all } i > 1, \quad \text{and}
\]

\[
0 \to \text{Ext}^{-1}(M/f(M),R) \to \text{Ext}^{-1}(M,R) \to \text{Ext}^{-1}(M(-d),R) \to \text{Ext}^{0}(M/f(M),R)
\]

is exact. Let \(f^n(M)\) be the image of the map \(f^n : M(-dn) \to M\). Then \(f^n(M)/f^{n+1}(M) \cong (M/f(M))(-dn)\). The short exact sequence

\[
0 \to f^n(M)/f^{n+1}(M) \to M/f^{n+1}(M) \to M/f^n(M) \to 0
\]

yields a long exact sequence.
0 \to \text{Ext}^{-1}(M/f^n(M), R) \to \text{Ext}^{-1}(M/f^{n+1}(M), R) \\
\text{Ext}^{-1}(f^n(M)/f^{n+1}(M), R) \to \text{Ext}^0(M/f^n(M), R).

Suppose $M/f(M)$ is infinite dimensional. By Lemma 1.3(3), $\text{Ext}^{-1}(M/f(M), R)$, and hence $\text{Ext}^{-1}(f^n(M)/f^{n+1}(M), R)$, is infinite dimensional. Since $\text{Ext}^0(M/f^n(M), R)$ is finite dimensional, the injective map

$$\text{Ext}^{-1}(M/f^n(M), R) \to \text{Ext}^{-1}(M/f^{n+1}(M), R)$$

is not surjective. This shows that $\{\text{Ext}^{-1}(M/f^n(M), R) | n \geq 0\}$ is an ascending chain of submodules of $\text{Ext}^{-1}(M, R)$, which contradicts to the fact that $\text{Ext}^{-1}(M, R)$ is noetherian.

Let $H^iR$ be the $i$-th cohomology of $R$. By definition, $H^iR$ is noetherian on both sides for all $i$.

**Lemma 1.6.** Suppose $\text{cd}A = d \geq 2$ and $M$ is a noetherian graded $A$-module.

1. $H^{-d}R$ is noetherian on both sides and $\text{Ext}^{-d}(M, R) = \text{Hom}(M, H^{-d}R)$.
2. Every nonzero left (right) submodule of $H^{-d}R$ has $\text{Kdim} \geq 2$.
3. If $A$ is prime such that, for every nonzero graded prime $P$, $\text{Kdim} A/P \leq 1$, then $H^{-d}R$ is $A$-torsionfree on both sides. Namely, $\text{Ext}^{-d}(M, R) = 0$ for all $A$-torsion modules $M$.

**Proof.** (1) Clear.

2. For any $\alpha$, the maximal submodule of $H^{-d}R$ of Krull dimension at most $\alpha$ is a graded submodule. Hence it suffices to consider graded submodules. Suppose $N$ is a right graded submodule of $H^{-d}R$ of $\text{Kdim} < 2$. Since $H^{-d}R$ is noetherian on both sides, we may further assume that (a) $N$ is a critical subbimodule of $H^{-d}R$, (b) $P := \text{ann}(N_A)$ is prime, and (c) $N$ is $A/P$-torsionfree [8, 4.3.5(iii)]. Hence $\text{Kdim} A/P = \text{Kdim} N_A < 2$, and then $\text{Kdim} A/P \leq 1$. By (1) and Lemma 1.3(2), $H^{-d}R$ contains no finite dimensional submodule. Thus $\text{Kdim} A/P = 1$. By Lemma 1.4, $A/P$ is PI, whence there is a central regular elements $x \in A/P$ of degree $d > 0$. The short exact sequence

$$0 \to N(-d) \to N \to N/Nx \to 0$$

yields a long exact sequence

$$\text{Ext}^{-d}(N/Nx, R) \to \text{Ext}^{-d}(N, R) \to \text{Ext}^{-d}(N(-d), R) \to \text{Ext}^{-d+1}(N/Nx, R).$$

Since $N/Nx$ is finite dimensional, two ends are zero [Lemma 1.3(2)]. Since $\text{Ext}^{-d}(N, R)$ is left bounded, a graded version of Nakayama lemma implies that $\text{Ext}^{-d}(N, R)$ is zero. This contradicts to the fact $N \subset H^{-d}R$.

3. Suppose $H^{-d}R$ is not $A$-torsionfree. Since $H^{-d}R$ is noetherian on both sides, there is a critical submodule $N \subset H^{-d}R$ such that (a) $N_A$ is $A$-torsion, (b) $P := \text{ann}(N_A)$ is a nonzero graded prime ideal, and (c) $N$ is $A/P$-torsionfree. By the hypothesis, $\text{Kdim} N_A = \text{Kdim} A/P \leq 1$. This contradicts to (2).

If $P$ is a graded prime ideal of $A$, let $E(A/P)$ denote the graded injective hull of $A/P$ and $E_{A/P}$ denote the graded injective hull of a graded uniform right ideal of $A/P$.
Lemma 1.7. Let $I$ be the minimal injective resolution of $R$ as right $A$-module complex.

1. $I^n = A'$.
2. Let $B^{-i}$ be the image of $I^{-i-1} \to I^{-i}$. Then $I^{-1}/B^{-1}$ is locally finite.
3. If $A$ is prime and $Kdim A \geq 2$, then $I^{-1}$ does not contain any shift of the injective module $E_A$.
4. If $P$ is a graded prime ideal of $A$ such that $Kdim A/P \geq 2$, then $I^{-1}$ does not contain any shift of the injective module $E_{A/P}$.

Proof. (1) By local duality (E1.1), $\text{Ext}^0 (k, A') = \Gamma_{m} (k)' = k$. Thus $I^n = A' \oplus J$ where $J$ is $m$-torsionfree. Suppose $I^n \neq A'$, namely, $J \neq 0$. Then the inclusion $J \hookrightarrow I^n$ induces a nonzero element in $\text{Ext}^0 (J, R)$. Since $J$ is $m$-torsionfree, by local duality $\text{Ext}^0 (J, R) = \Gamma_{m} (J) = 0$, a contradiction.

(2) We have an exact sequence
$$0 \to H^{-1} R \to I^{-1}/B^{-1} \to I^n.$$ By (1) $I^n = A'$ is locally finite. By the definition of dualizing complex, $H^{-1} R$ is noetherian and hence locally finite. The statement follows.

(3) Let $Q_{gr}(A)$ be the graded quotient ring of $A$. It is clear that $E(A)$ is isomorphic to $Q_{gr}(A)$ as left and as right graded $A$-module. If $E(A)$ is locally finite, then the Hilbert series of $E(A) = Q_{gr}(A)$ is periodic. Hence $A$ has $GKdim 1$, which contradicts to the hypothesis. Therefore $E(A)$ is not locally finite. Since $E(A)$ is a finite direct sum of shifts of $E_A$, $E_A$ is not locally finite.

Suppose $E_A$ is a submodule of $I^{-1}$. Since $A$ is prime and $B^{-1}$ is $A$-torsion, $E_A$ does not intersect with $B^{-1}$. Hence $E_A$ is a submodule of $I^{-1}/B^{-1}$. By (2), $E_A$ is locally finite, a contradiction.

(4) Let $R_{A/P}$ be the dualizing complex over $A/P$. By [16, 3.2 and 4.16], there is a morphism $\phi : R_{A/P} \to R$ in $D(\text{Gr} A^c)$ such that
$$R_{A/P} \cong \text{RHom}_{A}(A/P, R) \cong \text{RHom}_{A^c}(A/P, R). \tag{E1.8}$$

Note that if $M$ is an injective $A$-module, then $\text{Hom}(A/P, M)$ is an injective $A/P$-module. Hence $I_{A/P} := \text{Hom}(A/P, M)$ is a complex of injective $A/P$-modules. The minimality of the complex $I$ implies that $I_{A/P}$ is a minimal complex of injective $A/P$-modules. By (E1.8), $I_{A/P}$ is a minimal injective resolution of $R_{A/P}$. Suppose $I^{-1}$ contains injective $A$-module $E_{A/P}$. Then $I^{-1}_{A/P}$ contains injective $A/P$-module $E_{A/P}$, which contradicts with (3). \hfill \square

2. Rings with cohomological dimension 2. By Lemma 1.5, if $\text{cd} A = 1$ then $A$ is PI of $Kdim 1$, which implies that $R$ is Auslander. In this section we will discuss the case $\text{cd} A = 2$. If $Kdim A = 1$, then $A$ is PI [Lemma 1.4]. By [16, 5.13 and 5.14], $\text{cd} A = 1$. So $\text{cd} A = 2$ implies $Kdim A \geq 2$. Next is a consequence of Lemma 1.7(4).

Proposition 2.1. Suppose $\text{cd} A = 2$. Then there are only finitely many graded primes $P \subset A$ with $Kdim A/P \geq 2$. As a consequence $A$ has a finite classical Krull dimension.
Proof. Let \( I \) be the minimal injective resolution of \( R \) as right \( A \)-module complex. Then \( I^2 \) is the injective hull of \( H^{-2}R \). Since \( H^{-2}R \) is noetherian, its Goldie rank is finite and \( I^2 \) is a finite direct sum of indecomposable injectives. By Lemma 1.7(4) and [16, 1.11], if \( P \) is prime with \( \dim A/P \geq 2 \), then \( E_{A/P} \) is a direct summand of \( I^2 \). If \( P \neq P' \), then \( E_{A/P} \neq E_{A/P'} \). Therefore the number of graded primes \( P \) with \( \dim A/P \geq 2 \) is no more than the Goldie rank of \( H^{-2}R \).

The last assertion follows from [9, 1.1.9]. \( \square \)

Lemma 2.2. Let \( \text{cd} A = 2 \) and let \( M \) be a noetherian graded right \( A \)-module.

1. If \( \text{Hom}(M, A/P) \neq 0 \) for a graded prime \( P \) with \( \dim A/P \geq 2 \), then \( \text{Ext}^{-2}(M, R) \neq 0 \).

2. Let \( A \) be prime. If \( M \) is \( A \)-torsion and every factor of \( M \) with \( \dim \geq 2 \) is \( A \)-faithful, then \( \text{Ext}^{-2}(M, R) = 0 \).

3. Let \( A \) be prime such that, for every nonzero graded prime \( P \), \( \dim A/P \leq 1 \). If \( M \) is not \( A \)-torsion, then \( \text{Ext}^{-2}(M, R) \) is not \( A \)-torsion.

Proof. (1) Since \( \text{Ext}^{-2}(M, R) = \text{Hom}(M, H^{-2}R) \), it suffices to show that \( A/P \) as right \( A \)-module embeds to a finite direct sum of shifts of \( H^{-2}R \). By Lemma 1.7(1,4), \( E_{A/P} \) does not appear in \( I^0 \) and \( I^{-1} \). Since \( E_{A/P} \) appears in the minimal injective resolution of \( R \) [16, 1.11], it must appear in \( I^2 \). Therefore \( H^{-2}R \) contains a shift of a graded uniform right ideal of \( A/P \), and hence \( A/P \) embeds to a finite direct sum of shifts of \( H^{-2}R \).

(2) Let \( T \) be the \( A \)-torsion submodule of \( H^{-2}R \) as right \( A \)-module. Then \( T \) is a submodule and hence noetherian on both sides. Since \( T_A \) is \( A \)-torsion, the right annihilator of \( T \) is not zero. Suppose \( \text{Ext}^{-2}(M, R) \neq 0 \). Then there is a nonzero map \( f: M \rightarrow T \). Thus the image of \( f \) has \( \dim \geq 2 \) and is not \( A \)-faithful.

(3) First we may replace \( M \) by a factor of \( M \) and assume \( M \) is a graded uniform right ideal of \( A \). Then replace \( M \) by a finite direct sum of shifts of \( M \) we may assume \( M \) is an essential right ideal of \( A \). The exact sequence

\[
0 \rightarrow M \rightarrow A \rightarrow A/M \rightarrow 0
\]
yields a long exact sequence

\[
0 \rightarrow \text{Ext}^{-2}(A/M, R) \rightarrow \text{Ext}^{-2}(A, R) \rightarrow \text{Ext}^{-2}(M, R) \rightarrow \cdots
\]

Since \( A/M \) is \( A \)-torsion, \( \text{Ext}^{-2}(A/M, R) = 0 \) [Lemma 1.6(3)]. Since \( \text{Ext}^{-2}(A, R) = H^{-2}R \) is \( A \)-torsionfree [Lemma 1.6(3)], \( \text{Ext}^{-2}(M, R) \) is not \( A \)-torsion. \( \square \)

Lemma 2.3. Suppose \( \text{cd} A \leq 2 \) and \( M \) is a noetherian graded \( A \)-module.

1. Let \( E^{p,q}(M) = \text{Ext}^p(\text{Ext}^q(M, R), R) \). Then \( E^{p,q}(M) = 0 \) for \( (p, q) = (-2, 0), (-1, 0), (-1, -2), (0, -2) \) and \( (-2, -1) \).

2. If \( \text{Ext}^{-2}(M, R) \neq 0 \), then \( \dim \text{Ext}^{-2}(M, R) \geq 2 \).

Proof. (1) We have a convergent spectral sequence (E1.2) and the \( E_2 \)-page of (E1.2) is

\[
\begin{array}{ccc}
E^{-2,0}(M) & E^{-1,0}(M) & E^{0,0}(M) \\
E^{-2,-1}(M) & E^{-1,-1}(M) & E^{0,-1}(M) \\
E^{-2,-2}(M) & E^{-1,-2}(M) & E^{0,-2}(M)
\end{array}
\]
with other terms being zero. The boundary maps in the $E_r$-page have degree $(r, r - 1)$. For $(p, q) = (-2, 0), (-1, 0), (-1, -2), (0, -2), \ E^{p,q}(M) = E^{p,q}_2 = E^{p,q}_\infty = 0.

At $(p, q) = (-2, -1)$, we have $E^{-2,-1}_3 = E^{-2,-1}_\infty = 0$. Therefore $E^{-2,-1}(M)$ is a submodule of $E^{0,0}(M)$ which is finite dimensional [Lemma 1.3(1)]. Let $N = \text{Ext}^{-1}(M, R)$. Then (E1.2) for $N$ shows that $E^{0,-2}(N) = 0$, which implies that $E^{-2}(N) = E^{-2,-1}(M)$ contains no nonzero $M$-torsion submodule. Therefore $E^{-2,-1}(M)$ must be zero.

(2) Let $N = \text{Ext}^{-2}(M, R)$. By (1) $\text{Ext}^{p}(N, R) = 0$ for $p = 0, -1$. If $\text{Kdim} N < 2$, by Lemma 1.6(2) $\text{Ext}^{-2}(N, R) = 0$. By (E1.2) $N = 0$. $\square$

**Proposition 2.4.** Suppose $\text{cd}A = 2$. Then the following conditions are equivalent:

1. for any chain of graded primes $P^* \subseteq P \subseteq A$, $\text{Kdim} A/P \leq 1$.
2. $R$ is Auslander.
3. $\text{Kdim} A = 2$.
4. $\text{Kdim} \text{Ext}^{-1}(M, R) \leq 1$ for all noetherian graded module $M$.

**Proof.** (1) $\Rightarrow$ (2) By [16, 4.18] we may assume $A$ is prime. By Lemma 1.3(1), for any noetherian graded $A$-module $M$, $\text{Ext}^{0}(M, R)$ is finite dimensional and by Lemma 1.3(2) $\text{Ext}^{q}(N, R) = 0$ for all $N \subseteq \text{Ext}^{0}(M, R)$ and for all $q < 0$. By the definition of cd, for all $N \subseteq \text{Ext}^{0}(M, R)$ and for all $q < -\text{cd}A$, $\text{Ext}^{q}(N, R) = 0$. Hence it remains to show that $\text{Ext}^{-2}(N, R) = 0$ for all $N \subseteq \text{Ext}^{-1}(M, R)$. By Lemma 1.6(3), it suffices to show that $\text{Ext}^{-1}(M, R)$ is $A$-torsion, which follows from Lemmas 2.2(3) and 2.3(1).

(2) $\Rightarrow$ (3) By [16, 4.14], $\text{Kdim} A \leq \text{cd}A = 2$. So $\text{Kdim} A = 2$.

(3) $\Rightarrow$ (1) Clear, or see Lemma 3.3(2).

(3) $\Rightarrow$ (4) Again we may assume $A$ is prime. By the proof of (1) $\Rightarrow$ (2), $\text{Ext}^{-1}(M, R)$ is $A$-torsion. Hence $\text{Kdim} \text{Ext}^{-1}(M, R) \leq \text{Kdim} A - 1 \leq 1$.

(4) $\Rightarrow$ (2) By the prove of (1) $\Rightarrow$ (2), it remains to show that $\text{Ext}^{-2}(N, R) = 0$ for all $N \subseteq \text{Ext}^{-1}(M, R)$. Since $\text{Kdim} \text{Ext}^{-1}(M, R) \leq 1$, $\text{Kdim} N \leq 1$ and by Lemma 1.6(2), $\text{Ext}^{-2}(N, R) = 0$. $\square$

**Remark 2.5.** We can also prove the following, but tedious proofs are omitted.

Suppose $\text{cd}A = 2$. Then the following are equivalent:

1. $\text{Kdim} A = 2$.
2. $\text{Kdim} M \otimes_A P \leq 1$ for every graded primes $P$ with $\text{Kdim} A/P = 2$ and for all noetherian graded $A$-module $M$ with $\text{Kdim} M = 1$.
3. $\text{Kdim} \text{Ext}^{-1}(M, R) \leq 1$ for every noetherian graded $A$-module $M$ with $\text{Kdim} M = 1$.

**Lemma 2.6.** Suppose $\text{cd}A = 2$. Let $M$ be a graded $A$-bimodule noetherian on both sides.

1. If $\text{Kdim} M = 1$, then $\text{Ext}^{-2}(M, R) = 0$ and $\text{Kdim} \text{Ext}^{-1}(M, R) = 1$.
2. If $\text{Kdim} M > 1$, then $\text{Kdim} \text{Ext}^{-2}(M, R) \geq 2$ and $\text{Kdim} \text{Ext}^{-1}(M, R) \leq 1$.

**Proof.** (1) By Lemma 1.6(2), $\text{Ext}^{-2}(M, R) = 0$. By Lemma 1.3(3), $\text{Kdim} \text{Ext}^{-1}(M, R) > 0$. So it remains to show $\text{Kdim} \text{Ext}^{-1}(M, R) \leq 1$. By noetherian
induction we may assume the bimodule $M$ is right $A/P$-torsionfree for some graded ideal $P$ and $\text{Kdim} A/P = 1$. Hence $\text{Ext}^{-1}(M, R)$ is a noetherian left $A/P$-module, which has $\text{Kdim} \leq 1$.

(2) To prove $\text{Kdim} \text{Ext}^{-2}(M, R) \geq 2$ we may replace $M$ by a factor of $M$ and assume that every proper bimodule factor of $M$ has $\text{Kdim}$ no more than 1. Pick a nonzero subbimodule $M' \subset M$ such that $M'$ is right $A/P$-torsionfree for some graded prime ideal $P$. Let $\overline{M} = M/M'$. Then $\text{Kdim} \overline{M} \leq 1$. The short exact sequence

$$0 \to M' \to M \to \overline{M} \to 0$$

yields an exact sequence

$$0 \to \text{Ext}^{-2}(\overline{M}, R) \to \text{Ext}^{-2}(M, R) \to \text{Ext}^{-2}(M', R) \to \text{Ext}^{-1}(\overline{M}, R).$$

By (1) $\text{Ext}^{-2}(\overline{M}, R) = 0$ and $\text{Kdim} \text{Ext}^{-1}(\overline{M}, R) \leq 1$. By Lemma 2.2(1), $\text{Ext}^{-2}(M', R) \neq 0$ and by Lemma 2.3(2), $\text{Kdim} \text{Ext}^{-2}(M', R) > 1$. Hence $\text{Kdim} \text{Ext}^{-2}(M, R) \geq 2$.

\textbf{Proposition 2.7.} Suppose $\text{cd} A = 2$.

(1) Let $M$ be a graded $A$-bimodule noetherian on both sides. If $M \supset M_1 \supset M_2 \supset \cdots$ is a descending chain of subbimodules, then $\text{Kdim} M_i/M_{i+1} \leq 1$ for $i \gg 0$. As a consequence $\text{Kdim} M \leq 2$.

(2) $\text{Kdim} A \leq 2$.

\textbf{Proof.} (2) is a special case of (1).

(1) Let $N_i = M/M_i$ and let $M'_i = M_i/M_{i+1}$. Suppose $\text{Kdim} M'_i > 1$ for all $i$. The exact sequence

$$0 \to M'_i \to N_{i+1} \to N_i \to 0$$

yields an exact sequence

$$0 \to \text{Ext}^{-2}(N_i, R) \to \text{Ext}^{-2}(N_{i+1}, R) \to \text{Ext}^{-2}(M'_i, R) \to \text{Ext}^{-1}(N_i, R) \to \cdots.$$ 

By Lemma 2.6, $\text{Kdim} \text{Ext}^{-1}(N_i, R) \leq 1$ and $\text{Kdim} \text{Ext}^{-2}(M'_i, R) \geq 2$. Hence the injective map $\text{Ext}^{-2}(N_i, R) \to \text{Ext}^{-2}(N_{i+1}, R)$ is not surjective for all $i$. Thus $(\text{Ext}^{-2}(N_i, R) \mid i \geq 0)$ is an ascending chain of submodules of $\text{Ext}^{-2}(M, R)$, which contradicts to the fact $\text{Ext}^{-2}(M, R)$ is noetherian.

Proposition 0.5 follows from Corollary 2.1 and Proposition 2.7.

\textbf{Remark 2.8.} (1) We don’t know whether or not $\text{Kdim} A = 2$ implies $\text{GKdim} A = 2$.

(2) If $A$ is PI or FBN or more generally satisfies similar submodule condition, then $R$ is Auslander and $\text{Kdim} A = \text{GKdim} A = \text{cd} A$ [16, 5.13 and 5.14].

3. Graded dimension function.

\textbf{Definition 3.1.} A \textbf{graded dimension function} is a map $\partial$ sending finitely generated right (and left) graded modules over a connected graded ring to the set of $-\infty$, all real numbers, and all ordinals $\geq \omega$ satisfying the following properties:
\(\partial 0 = -\infty\), and \(\partial M = 0\) if and only if \(M\) is finite dimensional;

(d1) If \(M\) is infinite dimensional, then \(\partial M \geq 1\);

(d2) \(\partial M = \max\{\partial (M/N), \partial N\}\) for any \(N \subseteq M\) (this property is called **exactness**);

(d3) If \(x\) is a regular element of \(M = A/P\) for some prime ideal \(P \subseteq A\), then \(\partial M \geq \partial (M/xM) + 1\);

(d4) \(\partial M = \partial (M(1))\) where \(M(1)\) is the degree shift of \(M\).

The graded dimension function is slightly different from the dimension function in the ungraded case (see [8, Sect. 6.8.4] or [16, 2.4]) because we add some natural conditions (d0), (d1) and (d4) concerning graded modules. Given a finitely generated graded module \(M\), \(K\dim M\) in the graded module category is equal to \(K\dim M\) in the ungraded module category. Using this fact one can easily check that Krull dimension is also a graded dimension function. For Gelfand-Kirillov dimension, (d0,d1,d3,d4) are clear and the next lemma shows that (d2) is true.

**Lemma 3.2.** Let \(A\) be a right noetherian connected ring and let \(M\) be a finitely generated right graded \(A\)-module. Then

1. \(\GKdim M = \lim \log_k \left(\sum_{i \leq n} \dim_k M_i\right)\).
2. \(\GKdim M = \max\{\GKdim M/N, \GKdim N\}\) for any graded submodule \(N \subseteq M\).
3. If \(f : M(-d) \to M\) is an injective map for some integer \(d > 0\), then \(\GKdim M/f(M) = \GKdim M - 1\).

**Proof.** (1) This follows from the facts that \(A\) is a finitely generated algebra and that \(M\) is a finitely generated right \(A\)-module.

(2) By [8, 8.3.2(ii)], \(\GKdim M \geq \max\{\GKdim M/N, \GKdim N\}\). Let \(\alpha\) be any number bigger than \(\max\{\GKdim M/N, \GKdim N\}\). Then there is a constant \(C\) such that

\[
\sum_{i \leq n} \dim_k (M/N)_i \leq C\alpha^n \quad \text{and} \quad \sum_{i \leq n} \dim_k N_i \leq C\alpha^n
\]

for \(n \gg 0\). Thus \(\sum_{i \leq n} \dim_k M_i \leq 2C\alpha^n\) for \(n \gg 0\). By (1) \(\GKdim M \leq \alpha\).

(3) By [8, 8.3.5], \(\GKdim M/f(M) \leq \GKdim M - 1\). To prove the opposite inequality let \(\alpha\) be any number bigger than \(\GKdim M/f(M)\). Let \(\dim_k (M/f(M))_i = h(i)\). Then there is a constant \(C\) such that \(\sum_{j \leq i} h(j) \leq C\alpha^i\) for all \(i > 0\). Let \(\dim_k M_i = g(i)\). Then

\[
g(i) = h(i) + h(i-d) + h(i-2d) + \cdots \leq \sum_{j \leq i} h(j) \leq C\alpha^i,
\]

whence \(\sum_{j \leq i} g(j) \leq C'\alpha^{i+1}\) for some constant \(C'\) and for \(i > 0\). Therefore \(\GKdim M \leq \alpha + 1\).

**Lemma 3.3.** Let \(\partial\) be a graded dimension function.

1. If \(\partial(A_+) < 2\), then \(A\) is PI of \(K\dim \leq 1\).
2. If \(\partial(A_+) < 3\), then, for any chain of graded primes \(P' \subset P \subseteq A\), \(K\dim A/P \leq 1\).

**Proof.** (1) We may assume \(A\) is prime and not \(k\). Since \(A\) is noetherian, by [9, 1.1.6], there is a homogeneous regular element \(x \in A\) of positive degree. Then \(\partial(A/xA) \leq \partial A - 1 < 1\). Hence \(\partial(A/xA) = 0\) and \(A/xA\) is finite dimensional. Therefore \(\GKdim A = 1\) and \(A\) is PI of \(K\dim 1\) [Lemma 1.4].
(2) Since \( P/P' \) is a nonzero prime ideal of \( A/P' \), \( \partial(A/P') < 3 \) implies \( \partial(A/P) < 2 \). Then the statement follows from (1). \( \square \)

Let \( R \) be a balanced dualizing complex. The canonical dimension is defined by

\[
\text{Cdim } M = - \min \{ i \mid \text{Ext}^i(M, R) \neq 0 \}.
\]

Note that \( \text{cd } A = \text{Cdim } A \). If \( R \) is Auslander, then \( \text{Cdim } \) is a dimension function for ungraded modules [16, 2.10]. To verify \( \text{Cdim } \) is a graded dimension function, we need to check (d0), (d1) and (d4). But (d0) and (d1) follow from Lemma 1.3(2,3) and (d4) is clear for \( \text{Cdim } \). Therefore we have:

**Proposition 3.4.** If \( R \) is Auslander, then \( \text{Cdim } \) is a graded dimension function.

*Proof of Theorem 0.1.* By Proposition 2.4, (1), (2) and (4) are equivalent.

(2) \( \Rightarrow \) (3) is clear and (3) \( \Rightarrow \) (4) is Lemma 3.3(2).

(1) \( \Rightarrow \) (5) is Proposition 3.4.

(5) \( \Rightarrow \) (7) By [16, 4.14], \( \text{Kdim } M \leq \text{Cdim } M \). It remains to show \( \text{Cdim } M \leq \text{Kdim } M \). This is clear when \( \text{Kdim } M = 0 \) or 2. So we only consider the case when \( \text{Kdim } M = 1 \). Since both dimension functions are exact, we may use noetherian induction on \( M \). So we may assume \( M \) is critical faithful over \( A/P \) for some graded prime \( P \). There are two cases. Case 1 is when \( \text{Kdim } A/P = 2 \). Since \( \text{Kdim } A = 2 \), \( P \) is minimal. Hence \( M \) is \( A/P \)-torsion. Thus \( \text{Cdim } M \leq \text{Cdim } A/P - 1 \leq 1 \). Case 2 is when \( \text{Kdim } A/P \leq 1 \). Then by Lemma 3.3(1), \( A/P \) is PI and of \( \text{GKdim } \leq 1 \). Hence \( \text{Cdim } A/P \leq 1 \) and hence \( \text{Cdim } M \leq 1 \).

(7) \( \Rightarrow \) (6) Clear.

(6) \( \Rightarrow \) (1) By the proof of Proposition 2.4, it suffices to show \( \text{Ext}^{-2}(N, R) = 0 \) for all submodules \( N \subseteq \text{Ext}^{-1}(M, R) \) and for all noetherian graded module \( M \). By Lemma 2.3(1) (for \( (p, q) = (-2, -1) \)), \( \text{Cdim } \text{Ext}^{-1}(M, R) < 2 \). Since \( \text{Cdim } \) is exact, \( \text{Cdim } N < 2 \), i.e., \( \text{Ext}^{-2}(N, R) = 0 \). \( \square \)

4. **AS-Gorenstein rings of injective dimension 2.** By [14, 4.14] a noetherian AS-Gorenstein ring \( A \) of injective dimension \( d \) has a balanced dualizing complex \( R = A^\sigma[-d](\sigma e) \) for some graded automorphism \( \sigma \) of \( A \) and some integer \( e \) in the Definition 0.3(1).

**Proposition 4.1.** If \( A \) is AS-Gorenstein of injective dimension 2 and has an artinian quotient ring (with respect to the set of the regular elements of \( A \)), then \( A \) is Auslander-Gorenstein.

Note that Proposition 4.1 is [7, 5.13] without the hypothesis that \( \text{GKdim } A = 2 \). It also follows from Theorem 0.4 that the artinian quotient ring of \( A \) is in fact self-injective.

*Proof.* By Theorem 0.1, it suffices to show that, for any chain of graded primes \( P' \subsetneq P \subset A \), \( \text{Kdim } A/P \leq 1 \). Suppose this is not true, namely, there are graded primes \( P' \subsetneq P \subset A \) such that \( \text{Kdim } A/P > 1 \). By Lemma 2.2(1),

\[
\text{Hom}(A/P, A) \cong \text{Ext}^{-2}(A/P, R(e)) \cong \text{Ext}^{-2}(A/P, R)(e) \neq 0
\]
because $R = A^0[-2](-e)$. Let $Q$ be the artinian quotient ring of $A$. Then $\text{Hom}(A/P, A) \neq 0$ implies that $A/P \otimes_A Q \neq 0$. Next we show that this contradicts to the fact $P$ is not a minimal prime. By [8, 4.1.3(iv)], $A/N(A) \otimes_A Q = Q/N(Q) := Q'$ where $N(-)$ is the prime radical of $-$, and $Q/Q(N)$ is the artinian quotient ring of $A/N(A)$. Since $P$ is not a minimal prime, $P/N(A)$ is an essential (left and right) ideal of $A/N(A)$. By Goldie’s theorem, $Q'/PQ' = 0$. Hence

$$A/P \otimes_A Q = Q/PQ = Q'/PQ' = 0.$$ 

This contradicts to $A/P \otimes_A Q \neq 0$. 

Let $\delta$ be a (graded) dimension function. A nonzero (graded) module $N$ is called $p$-pure with respect to $\delta$ if $\delta M = p$ for all nonzero noetherian (graded) submodule $M \subset N$. In most cases we will take $\delta = \text{Cdim}$. Let $R$ be the balanced dualizing complex over $A$ and let $\delta$ be a graded dimension function. Following Yekutieli’s definition [15, 2.3], $R$ is called residue complex over $A$ (with respect to $\delta$) if (i) each $A$-bimodule $R^q$ is graded injective on both sides and (ii) each $A$-bimodule $R^q$ is pure of $\delta$-dimension $-q$ on both sides. Note that in [15] this is called strong residue complex, and later in [17], this is called residue complex. By [15, 2.6], if $R$ is a residue complex over $A$ with respect to $\delta$, then $R$ is Auslander and $\delta M = \text{Cdim} M$ for all noetherian graded modules $M$.

We are ready to prove Theorem 0.4.

**Proof of Theorem 0.4.** First of all Theorem 0.4(1) is equivalent to Theorem 0.1(1) for AS-Gorenstein rings.

(1) $\Rightarrow$ (2) follows from [16, 6.23.3], (2) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (1) is Proposition 4.1.

(1) $\Rightarrow$ (4) is [16, 6.23.2] and (4) $\Rightarrow$ (1) follows from the implication (2) $\Rightarrow$ (1) of Theorem 0.1.

(5) $\Rightarrow$ (1) is [15, 2.6].

(1) $\Rightarrow$ (5) By Theorem 0.1(6) $\text{Cdim} M = \text{Kdim} M$ for graded noetherian modules $M$. (Note that $\text{Cdim} M$ and $\text{Kdim} M$ could be different for ungraded module $M$ though both dimensions are well-defined.) Hence purity with respect to Cdim is equivalent to purity with respect to Kdim.

Take the minimal injective resolution of the left $A$-module complex $R = A^0[-2](-e)$, say

$$\ldots \rightarrow I^{-2} \rightarrow I^{-1} \rightarrow I^0 \rightarrow 0 \cdots \quad (E4.2)$$

By Lemma 1.7(1), $I^0 = A'$ which is 0-pure. Let $Q$ be the (ungraded) artinian quotient ring of $A$. Since $I^{-2}$ (as ungraded module) is a submodule of $Q$ and since $Q$ is 2-pure as an ungraded module, $I^{-2}$ is 2-pure. Since $Q/A$ is $A$-torsion, so is $I^{-2}/A(-e)$. Thus $\text{Cdim} N \leq 1$ for all noetherian submodule $N \subset I^{-2}/A(-e)$. Since $A$ is AS-Gorenstein, $I^{-2}/A(-e) \subset I^{-1}$ contains no finite dimensional submodules. Therefore $I^{-2}/A(-e)$ is 1-pure. Since the complex (E4.2) is exact at $I^{-1}$ and since $I^0$ is 0-pure, $I^{-1}$ is 1-pure. Similarly, the minimal injective resolution of the right $A$-module complex $R$ has a pure resolution. By [17, 3.8], the Cousin complex $ER$ of $R$ is a residue complex, namely, $ER$ is a balanced dualizing complex in $D_{fg}(\text{Gr}A^e)$ which is a pure minimal injective resolution on both sides.
Remark 4.3. Part (5) is a generalization of a result of Ajitabh [1, 3.12], which proves the existence of residue complexes for AS-regular algebras of global dimension 2.

Corollary 4.4. Let $A$ be AS-Gorenstein of injective dimension 2. If $\partial$ is a graded dimension function such that $\partial A \leq 2$ then

$$\partial M = \text{Kdim } M = \text{Cdim } M$$

for all noetherian graded left and right $A$-modules $M$.

Proof. First of all, by Theorem 0.1, $A$ is Auslander-Gorenstein and $\text{Cdim } M = \text{Kdim } M$. So it remains to show that $\partial M = \text{Kdim } M$.

Let $P$ be a minimal prime of $A$. By Theorem 0.4(4) $\text{Kdim } A/P = 2$, whence $A/(P + xA)$ is not finite dimensional over $k$ for any $x$ of positive degree. Pick $\bar{x} := x + P$ a homogeneous regular element of $A/P$ of positive degree. Then $\partial A/P \geq \partial A/(P + xA) + 1 \geq 2$. Thus $\partial A = \partial(A/P) = 2$ for all minimal prime $P$. Since both $\text{Kdim}$ and $\partial$ are exact we may use noetherian induction and may assume $M$ is a critical module and faithful over $A/P$ for a graded prime $P$. If $P = m$, then $M$ is $k$ and hence both dimensions are 0. If $M$ is infinitely dimensional and either $P$ is not minimal or $P$ is minimal and $M$ is $A/P$-torsion, then both dimensions are 1 (by (d1) and (d3)). The last possibility is when $M$ is a right ideal of $A/P$ when $P$ is minimal. In this case, $\text{Kdim } M = \text{Kdim } A/P$ and $\partial M = \partial A/P$. We have already shown that both dimensions on $A/P$ are equal to 2. That completes our proof. □

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