In the thirty years since it was proved that 0, 1 and 144 were the only perfect squares in the Fibonacci sequence [1, 9], several generalisations have been proved, but many problems remain. Thus it has been shown that 0, 1 and 8 are the only Fibonacci cubes [6], but there seems to be no method available to prove the conjecture that 0, 1, 8 and 144 are the only perfect powers.

In a different direction, generalising the sequence to $P_n(a)$ defined by

$$P_0(a) = 0,$$

$$P_1(a) = 1,$$

$$P_{n+2}(a) = aP_{n+1}(a) + P_n(a)$$

or to $p_n(a)$ defined by

$$p_0(a) = 0,$$

$$p_1(a) = 1,$$

$$p_{n+2}(a) = ap_{n+1}(a) - p_n(a),$$

it has been shown that the problem of determining the squares in these sequences can be handled easily when $a$ is odd, but only in exceptional cases when $a$ is even [2, 3, 4]. In the case of the first of these with $a = 2$, we obtain the Pell sequence, 0, 1, 2, 5, \ldots, 169, \ldots, to which we shall refer below simply as $P_n$. It has been shown by Ljunggren [5] that its only squares are 0, 1 and 169. However, the method of that paper was long and extremely complicated, involving relative units in a biquadratic field, and Mordell asked over 30 years ago [7] whether a simpler proof might not be available. There has indeed been another proof recently [8] which is quite different in conception, depending as it does on purely analytical ideas. Although that proof is a considerable achievement, whether it can be regarded as more simple is a matter of opinion, as it still seems to require tools and a mass of detail disproportionate to the apparent difficulty of the problem. Maybe what Mordell had in mind was a proof akin to that for Fibonacci squares, both short and technically elementary.

Despite this challenge, no such proof has appeared; it may therefore perhaps be of interest to present the following very simple proof of the fact that there are no other powers in the sequence, a result far exceeding the present state of knowledge of the corresponding problem for the Fibonacci sequence.

**Theorem.** The only solutions of $P_n = x^k$ with $k > 2$ are given by $n = 0, 1$.

**Lemma.** The Diophantine equation $y^2 - 2z^k = -1$ with $k > 2$ has only the solutions $y = z = 1$ and $y = 239, z = 13, k = 4$.

**Proof of lemma.** For $k = 4$ or a multiple of 4, the result is Ljunggren's. For other values, $k$ must have an odd prime factor, and so without loss of generality may be taken to be odd, say $k = 2K + 1$. For any solution both $y$ and $z$ must be odd, and factorising in $Q[i]$ gives $(y + i)(y - i) = (1 + i)(1 - i)^{2K+1}$. Since $(1 + i)$ and $(1 - i)$ are associates we find that $y + i = (1 + i)(a + ib)^{2K+1}$ and $z = a^2 + b^2$ for some suitable rational integers $a$ and $b$, since any units, i.e. powers of $i$, can be absorbed into the $a + ib$. Thus we find $2i = (1 + i)(a + ib)^{2K+1} - (1 - i)(a - ib)^{2K+1}$ and so

$$1 + i = (a + ib)^{2K+1} + i(a - ib)^{2K+1} = (a + ib)^{2K+1} + (-1)^K(i(a + b))^{2K+1}.$$

Thus, if $K$ is even, $(1 + i)$ is divisible by $(a + ib) + (ia + b) = (1 + i)(a + b)$ whence

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\[ a + b = \pm 1, \text{ and similarly, if } K \text{ is odd, } a - b = \pm 1. \text{ In either case we obtain } z = a^2 + b^2 = 2a^2 \pm 2a + 1, \text{ and so } 2z = c^2 + 1, \text{ where } c = |2a \pm 1| \geq 1. \]

Our equation can now be rewritten in the form \( y^2 - (c^2 + 1)(z^K)^2 = -1 \), and since the general solution of the Pell equation \( u^2 - (c^2 + 1)v^2 = -1 \) is given by

\[ u + v\sqrt{c^2 + 1} = (c + \sqrt{c^2 + 1})^{2m+1}, \]

we find that

\[ z^K = \left( \frac{1}{2}(c^2 + 1) \right)^K = \sum_{r=0}^{m} \binom{2m + 1}{2r + 1} c^{2m - 2r}(c^2 + 1)^r. \] (1)

Now suppose that \( p \) is any prime dividing \( \frac{1}{2}(c^2 + 1) \). Then \( p \mid (2m + 1) \). Then from (1) we see that \( p \mid (2m + 1) \) and so if \( p^m \mid (2m + 1) \), we see that the first term on the right hand side of (1) is divisible by \( p^n \) precisely, whereas all the other terms are divisible by higher powers. Thus \( \lambda K = \mu \), and since this holds for every prime factor of \( \frac{1}{2}(c^2 + 1) \), it follows that \( \left( \frac{1}{2}(c^2 + 1) \right)^K \) divides \( 2m + 1 \) and so \( 2m + 1 \equiv \left( \frac{1}{2}(c^2 + 1) \right)^K \). On the other hand from (1) we see that \( \left( \frac{1}{2}(c^2 + 1) \right)^K > 2m + 1 \) unless \( m = 0 \) and \( c = 1 \). Thus \( z = 1. \)

**Proof of theorem.** For \( n \) odd, the result follows from the lemma and the identity \( Q_n^2 - 2P_n^2 = (-1)^n \) where the sequence \( Q_n \) satisfies the same recurrence relation as \( P_n \) but with initial conditions \( Q_0 = Q_1 = 1 \). For \( n \) even, \( n \neq 0 \), let \( n = 2^h m \), where \( m \) is odd. Then it is found without difficulty that \( h \geq 2 \) and that \( P_n = 2^h P_m Q_m Q_{2m} X \), where the five factors on the right are pairwise coprime. It thus follows that if \( P_n \) is to be a perfect \( k \)th power, then each factor on the right must also be one. But by the lemma \( P_m \) can be a perfect \( k \)th power only if \( m = 1 \), and then \( Q_{2m} = 3 \) fails to be one, which concludes the proof.

**REFERENCES**


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