TOTALLY REAL PSEUDO-UMBILICAL SUBMANIFOLDS OF A QUATERNION SPACE FORM

by HUAFEI SUN†

(Received 12 June, 1996)

1. Introduction. Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature $c$, and let $P(H)$ denote the $4n$-dimensional quaternion projective space of constant quaternion sectional curvature $4$. Let $N$ be an $n$-dimensional Riemannian manifold isometrically immersed in $M(c)$. We call $N$ a totally real submanifold of $M(c)$ if each tangent 2-plane of $N$ is mapped into a totally real plane in $M(c)$. B. Y. Chen and C. S. Houh proved in [1].

**Theorem A.** Let $M$ be an $n$-dimensional compact totally real minimal submanifold of the quaternion projective space $P(H)$. If

$$S \leq \frac{3n(n+1)}{(6n-1)},$$

then $N$ is totally geodesic. Here $S$ is the square of the length of the second fundamental form of $N$.

Let $h$ be the second fundamental form of the immersion, and $\xi$ the mean curvature vector. Let $\langle \cdot, \cdot \rangle$ denote the scalar product of $M(c)$. If there exists a function $\lambda$ on $N$ such that

$$\langle h(X, Y), \xi \rangle = \lambda \langle X, Y \rangle$$

(*)

for any tangent vector $X, Y$ on $N$, then $N$ is called a pseudo-umbilical submanifold of $M(c)$. It is clear that $\lambda \geq 0$. If the mean curvature vector $\xi = 0$ identically, then $N$ is called a minimal submanifold of $M(c)$. Every minimal submanifold of $M(c)$ is itself a pseudo-umbilical submanifold of $M(c)$. In this paper, we consider the case when $N$ is pseudo-umbilical and extend Theorem A. Our main results are

**Theorem 1.** Let $N$ be an $n$-dimensional compact totally real pseudo-umbilical submanifold of $M(c)$. Then

$$\int_N \{6S^2 - [(n + 1)c + 16nH^2]S + 4n^2H^2c + 10n^2H^4\} dN \geq 0,$$

where $H$ and $dN$ denote the mean curvature of $N$ and the volume element of $N$ respectively.

**Theorem 2.** Let $N$ be an $n$-dimensional compact totally real submanifold of $M(c)$. If

$$6S^2 - [(n + 1)c + 16nH^2]S + 4n^2H^2c + 10n^2H^4 - 4nH\Delta H \leq 0,$$

(1.1)

then the second fundamental form of $N$ is parallel. In particular, if the equality of (1.1) holds, then either $N$ is totally geodesic or $N$ is flat.

When $H \equiv 0$, i.e. $N$ is minimal, from Theorem 1 we may get (cf. [4]).

†I would like to thank Prof. K. Ogiue for his advice and encouragement and to thank also the referee for valuable suggestions.

COROLLARY. Let $N$ be an $n$-dimensional compact totally real minimal submanifold of $P(H)$. If

$$S \leq \frac{2}{3}(n + 1),$$

then $N$ is totally geodesic or $S = \frac{3}{3}(n + 1)$.

2. Local formulas. We use the same notation and terminologies as in [1] unless otherwise stated. Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature $c$, and let $N$ be an $n$-dimensional totally real submanifold of $M(c)$. We choose a local field of orthonormal frames,

$$e_1, \ldots, e_n, \quad e_{l(1)} = l e_1, \ldots, e_{l(n)} = l e_n, \quad e_{J(1)} = Je_1, \ldots, e_{J(n)} = Je_n, \quad e_{K(1)} = Ke_1, \ldots, e_{K(n)} = Ke_n,$$

in such a way that when restricted to $N$, $e_1, \ldots, e_n$ are tangent to $N$. Here $I, J, K$ are the almost Hermitan structures and satisfy

$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J, \quad I^2 = J^2 = K^2 = -1.$$

We shall use the following convention on the range of indices:

$$A, B, \ldots = 1, \ldots, n, \quad l(1), \ldots, l(n), \quad J(1), \ldots, J(n), \quad K(1), \ldots, K(n),$$

$$\alpha, \beta, \ldots = l(1), \ldots, l(n), \quad J(1), \ldots, J(n), \quad K(1), \ldots, K(n),$$

$$\phi = I, J, K.$$

Let $\{\omega_A\}$ be the dual frame field. Then the structure equations of $M(c)$ are given by

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{CD} K_{ABCD} \omega_C \wedge \omega_D, \quad (2.1)$$

$$K_{ABCD} = \frac{c}{4} (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} + I_{AC} I_{BD} - I_{AD} I_{BC} + 2 I_{AB} I_{CD} + J_{AC} J_{BD} - J_{AD} J_{BC} + 2 J_{AB} J_{CD} + K_{AC} K_{BD} - K_{AD} K_{BC} + 2 K_{AB} K_{CD}).$$

Restricting these forms to $N$, we get the following structure equations of the immersion:

$$d\omega_\alpha = 0, \quad \omega_{\alpha i} = \sum_j h^{\alpha}_{ij} \omega_j, \quad h^{\alpha}_{ij} = h^{\phi}_{ij}, \quad h^\phi_{ij} = h^\phi_{ik},$$

$$d\omega_{ij} = -\sum_k \omega_{jk} \wedge \omega_{kj} + \frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l,$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h^\alpha_{ij} h^\alpha_{kl} - h^\alpha_{ik} h^\alpha_{jl}), \quad (2.2)$$

$$d\omega_{\alpha \beta} = -\sum_\gamma \omega_{\alpha \gamma} \wedge \omega_{\gamma \beta} + \frac{1}{2} \sum_{ij} R_{\alpha \beta ij} \omega_i \wedge \omega_j,$$

$$R_{\alpha \beta ij} = K_{\alpha \beta ij} + \sum_k (h^\alpha_{ij} h^\beta_{kj} - h^\beta_{ik} h^\alpha_{kj}), \quad (2.3)$$
We call \( h = \sum_{ij\alpha} h_{ij\alpha}^\alpha \omega_i \omega_j e_\alpha \) the second fundamental form of the immersed manifold \( N \). We denote by \( \mathcal{S} = \sum_{ij\alpha} (h_{ij\alpha}^\alpha)^2 \) the square of the length of \( h \). \( \xi = \frac{1}{n} \sum_{\alpha} \text{tr} \: H_{\alpha} e_\alpha \) and \( H = \frac{1}{n} \sqrt{\sum_{\alpha} (\text{tr} \: H_{\alpha})^2} \) denote the mean curvature vector and the mean curvature of \( N \), respectively. Here \( \text{tr} \) is the trace of the matrix \( H_{\alpha} = (h_{ij\alpha}^\alpha) \). Now, let \( e_{k(n)} \) be parallel to \( \xi \). Then we have
\[
\text{tr} \: H_{k(n)} = nH, \quad \text{tr} \: H_{\alpha} = 0, \: \alpha \neq k(n).
\]

We define \( h_{ijk}^\alpha \) and \( h_{ijkl}^\alpha \) by
\[
\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum_l h_{ijl}^\alpha \omega_l - \sum_l h_{ij\alpha l}^\alpha \omega_l + \sum_\beta h_{ij\alpha \beta}^\beta \omega_\beta,
\]
and
\[
\sum_l h_{ijk l}^\alpha \omega_l = dh_{ijk}^\alpha - \sum_l h_{ijk l}^\alpha \omega_l - \sum_l h_{i\alpha k l}^\alpha \omega_l - \sum_l h_{ij\alpha l}^\alpha \omega_l + \sum_\beta h_{ij\alpha \beta l}^\beta \omega_\beta,
\]
respectively. Where
\[
h_{ijk}^\alpha = h_{ikj}^\alpha
\]
and
\[
h_{ijkl}^\alpha - h_{ijk l}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{mij}^\alpha R_{mkjl} - \sum_\beta h_{ij\alpha \beta l}^\beta R_{mkjl}.
\]

The Laplacian \( h_{ij}^\alpha \) of the second fundamental form \( h_{ij}^\alpha \) is defined by \( \Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha \). By a direct calculation we have (cf. \([1, 2, 3]\))
\[
\frac{1}{2} \Delta h_{ij}^\alpha = \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + \sum_{ij\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha
\]
\[
= \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + \sum_{ijk\alpha} h_{ijk}^\alpha h_{kkij}^\alpha + \sum_{ijkl\alpha} h_{ijl}^\alpha h_{ikjk}^\alpha R_{ik} + \sum_{ijkl\alpha} h_{ijl}^\alpha h_{ikjk}^\alpha R_{ik} + \sum_{ijkl\alpha} h_{ijl}^\alpha h_{ikjk}^\alpha R_{ik}. \quad (2.6)
\]

3. Proofs of Theorems. From (*) and (2.4) we get \( \sum_{\alpha} \text{tr} \: H_{\alpha} h_{ij}^\alpha = n\lambda \delta_{ij} \), \( H^2 = \lambda \) and
\[
h_{ij}^{k(n)} = H \delta_{ij}. \quad (3.1)
\]

Using (3.1) we have
\[
\sum_{ijk\alpha} h_{ijk}^\alpha h_{kij}^\alpha = nH \Delta H. \quad (3.2)
\]
Using (2.1)–(2.4) and (3.1), we derive (cf. [1, 2, 3])

\[
\sum_{ijkl} h_{ij}^a h_{kl}^b R_{ijkl} + \sum_{ijkl} h_{ij}^a h_{kl}^b R_{ikjk} + \sum_{ijkl} h_{ij}^a h_{kl}^b R_{jikl} = \frac{c}{4} (n + 1)S - n^2 H^2 c + \sum_{ijkl} h_{ij}^a h_{kl}^b h_{ij}^b h_{kl}^b + \sum_{ijkl} \text{tr} (H_a H_\beta - H_\beta H_a)^2 - \sum_{\alpha\beta} (\text{tr} H_\alpha H_\beta)^2
\]

\[
= \frac{c}{4} (n + 1)S - n^2 H^2 c + \sum_{ijkl} \text{tr} (H_a H_\beta - H_\beta H_a)^2 - \sum_{\alpha\beta} (\text{tr} H_\alpha H_\beta)^2.
\]  

(3.3)

Substituting (3.2) and (3.3) into (2.6), we obtain

\[
\frac{1}{2} \Delta S = \sum_{ijkl} (h_{ij}^a)^2 + nH\Delta H + \frac{c}{4} (n + 1)S - n^2 H^2 c + \sum_{\alpha\beta} \text{tr} (H_a H_\beta - H_\beta H_a)^2 - \sum_{\alpha\beta} (\text{tr} H_\alpha H_\beta)^2.
\]  

(3.4)

In order to prove our Theorems, we need the following Lemmas.

**Lemma 1** [4]. Let $H_i$, $i \geq 2$ be symmetric $n \times n$-matrices, $S_i = \text{tr} H_i^2$, $S = \sum_i S_i$. Then

\[
\sum_{ij} \text{tr} (H_i H_j - H_j H_i)^2 - \sum_{ij} (\text{tr} H_i H_j)^2 \geq -\frac{3}{2} S^2,
\]

and the equality holds if and only if all $H_i = 0$ or there exist two of $H_i$ different from zero. Moreover, if $H_1 \neq 0$, $H_2 \neq 0$, $H_i = 0$, $i \neq 1, 2$, then $S_1 = S_2$ and there exists an orthogonal $(n \times n)$-matrix $T$ such that

\[
TH_i'T = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

where $a = \sqrt{\frac{S_i}{2}}$.

**Lemma 2.**

\[
\sum_{\alpha\beta} \text{tr} (H_a H_\beta - H_\beta H_a)^2 - \sum_{\alpha\beta} (\text{tr} H_a H_\beta)^2 \geq \frac{3}{2} S^2 + 3nH^2 S - \frac{5}{2} n^2 H^4.
\]

In fact, using (2.4), (3.1) and noting that $\alpha$ runs up to $3n > 2$, we have

\[
\sum_{\alpha\beta} \text{tr} (H_a H_\beta - H_\beta H_a)^2 - \sum_{\alpha\beta} (\text{tr} H_a H_\beta)^2 = \sum_{\alpha\beta \neq k(n)} \text{tr} (H_a H_\beta - H_\beta H_a)^2 - \sum_{\alpha\beta \neq k(n)} (\text{tr} H_a H_\beta)^2 - (\text{tr} H_k^2)^2.
\]  

(3.5)
Applying Lemma 1 to (3.5), we get
\[
\sum_{\alpha} \text{tr} (H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha} (\text{tr} H_\alpha H_\beta)^2 \geq -\frac{3}{2} \left( \sum_{\alpha \neq k(n)} \text{tr} H_\alpha^2 \right)^2 - (\text{tr} H_{k(n)}^2)^2
\]
\[
= -\frac{3}{2} (S - \text{tr} H_{k(n)}^2)^2 - (\text{tr} H_{k(n)}^2)^2 = -\frac{3}{2} (S - nH^2)^2 - n^2 H^4 = -\frac{3}{2} S^2 + 3nH^2S - \frac{5}{2} n^2 H^4.
\]
On the other hand, by (3.1) we have
\[
\sum_{ijk\alpha} (h_{ijk}^\alpha)^2 \geq \sum_{i\kappa} (h_{i\kappa}^{k(n)})^2 = n \sum_i (\nabla_i H)^2 = n |\nabla H|^2.
\]
It is obvious that
\[
\frac{1}{2} \Delta H^2 = H \Delta H + |\nabla H|^2.
\]
Therefore, using Lemma 2, (2.6) and (2.7) by (3.4) we get
\[
\frac{1}{2} \Delta S = \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + \sum_{ij\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha
\]
\[
= \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + nH \Delta H + \frac{c}{4} (n + 1) S + nH^2S - n^2 H^2 c
\]
\[
+ \sum_{\alpha\beta} (\text{tr} (H_\alpha H_\beta - H_\beta H_\alpha))^2 - \sum_{\alpha} (\text{tr} H_\alpha H_\beta)^2
\]
\[
\geq \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + nH \Delta H + \frac{c}{4} (n + 1) S + 4nH^2S - n^2 H^2 c - \frac{3}{2} S^2 - \frac{5}{2} n^2 H^4
\]
\[
\geq n |\nabla H|^2 + nH \Delta H + \frac{c}{4} (n + 1) S + 4nH^2S - n^2 H^2 c - \frac{3}{2} S^2 - \frac{5}{2} n^2 H^4
\]
\[
= \frac{1}{2} n H \Delta H^2 + \frac{c}{4} (n + 1) S + 4nH^2S - \frac{3}{2} S^2 - n^2 H^2 c - \frac{5}{2} n^2 H^4.
\]
Since $N$ is compact, we obtain from (3.8)
\[
\int_N \left\{ 6S^2 - [(n + 1)c + 16nH^2]S + 4n^2 H^2 c + 10n^2 H^4 \right\} dN \geq 0.
\]
From the first inequality of (3.8) we know that if $N$ is compact and
\[
6S^2 - [(n + 1)c + 16nH^2]S + 4n^2 H^2 c + 10n^2 H^4 - 4nH \Delta H = 0, \quad (3.9)
\]
then $\sum_{ijk\alpha} (h_{ijk}^\alpha)^2 = 0$, that is, the second fundamental form $h_{ij}^\alpha$ is parallel. In particular, when the equality of (3.9) holds, we see from (3.8) that the equality
\[
\sum_{\alpha \neq k(n)} \text{tr} (H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha \neq k(n)} (\text{tr} H_\alpha H_\beta)^2 = -\frac{3}{2} \left( \sum_{\alpha \neq k(n)} \text{tr} H_\alpha^2 \right)^2
\]
holds. Thus, by Lemma 1 we see that (i) \( H_\alpha = 0 \) (\( \alpha \neq k(n) \)) or (ii) there exist two non-zero \( H_\alpha \). In the case (i), we get \( S = nH^2 \). Hence noting \( H = \text{constant} \) and substituting it into the equality of (3.9), we obtain

\[
(3n - 1)cnH^2 = 0.
\]

This implies \( H = 0 \), so that \( N \) is totally geodesic or \( c = 0 \) so that \( N \) is flat. Now, we will prove that the case (ii) can not occur. Otherwise, using the same method as in [3]), we may see \( n = 2 \). Thus we may assume

\[
H_{(1)} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad H_{(2)} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad H_{K(2)} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad H_\alpha = 0. \tag{3.10}
\]

Here \( a \neq 0 \), \( \alpha \neq I(1), I(2), K(2) \).

Let the codimension of \( N \) be \( p(=3n) \). Put

\[
S_\alpha = \sum_{ij} (h_\alpha^{ij})^2,
\]

\[
p\sigma_1 = \sum_\alpha S_\alpha = S,
\]

\[
p(p - 1)\sigma_2 = 2 \sum_{\alpha < \beta} S_\alpha S_\beta.
\]

It can be easily seen (cf. [3])

\[
p^2(p - 1)(\sigma_1^2 - \sigma_2) = \sum_{\alpha < \beta} (S_\alpha - S_\beta)^2. \tag{3.11}
\]

By a direct calculation using (3.10), we get

\[
p^2(p - 1)\sigma_1^2 = (p - 1)(4a^2 + 2H^2)^2, \tag{3.12}
\]

\[
p^2(p - 1)\sigma_2 = p(8a^4 + 16a^2H^2), \tag{3.13}
\]

and

\[
\sum_{\alpha < \beta} (S_\alpha - S_\beta)^2 = 8(a^2 - H^2)^2. \tag{3.14}
\]

Substituting (3.12)–(3.14) into (3.11), we obtain

\[
(p - 1)(4a^2 + 2H^2)^2 - p(8a^4 + 16a^2H^2) = 8(a^2 - H^2)^2. \tag{3.15}
\]

From (3.15) we get

\[
(p - 3)(2a^4 + H^4) = 0,
\]

namely

\[
(3n - 3)(2a^4 + H^4) = 0,
\]

implying \( n = 1 \), because \( 2a^4 + H^4 \neq 0 \). This is a contradiction, since \( n = 2 \).
REFERENCES


DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
HACHIOJI, TOKYO, 192-03
JAPAN