THE SPECTRAL THEOREM IN BANACH ALGEBRAS

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Introduction. The concept of a hermitian element of a Banach algebra was first introduced by Vidav [21] who proved that, if a Banach algebra $\mathscr{A}$ has " enough " hermitian elements, then $\mathscr{A}$ can be renormed and given an involution to make it a stellar algebra. (Following Bourbaki [5] we shall use the expression " stellar algebra " in place of the term " C^*-algebra ".) This theorem was improved by Berkson [2], Glickfeld [10] and Palmer [17]. The improvements consist of removing hypotheses from Vidav's original theorem and in showing that Vidav's new norm is in fact the original norm of the algebra. Lumer [13] gave a spatial definition of a hermitian operator on a Banach space E and proved it to be equivalent to Vidav's definition when one considers the Banach algebra $\mathscr{L}(E)$ of continuous linear mappings of E into E.

In this paper the theory outlined above will be applied to define a normal element of a Banach algebra and to prove a spectral theorem for such elements. This theorem will then be exploited to prove analogues of well-known theorems for operators in Hilbert spaces.

We shall use the following standard notations. The symbol $\mathbf{N}$ will denote the set $\{0, 1, 2, \dots\}$, $\mathbf{R}$ the set of real numbers, $\mathbf{C}$ the set of complex numbers, $\mathbf{T}^1$ the unit circle in $\mathbf{C}$, and z the identity function of $\mathbf{R}^2$ onto $\mathbf{R}^2$.

The Banach algebras considered here will be assumed to be complex and to have identity element 1 such that $\|1\| = 1$. For an element x of a Banach algebra $\mathscr{A}$, the spectrum of x, denoted by $\operatorname{sp}(x)$, is the set of complex numbers λ such that $\lambda - x$ $(= \lambda 1 - x)$ is not invertible in $\mathscr{A}$. The spectral radius of x is the number

$$\rho(x) = \sup\{|\lambda| : \lambda \in \operatorname{sp}(x)\}.$$

Note that $\rho(x) \leqq \|x\|$.

Let $\mathscr{A}$ be a Banach algebra, and let $x \in \mathscr{A}$. Since the mapping $t \to \|1 + tx\|$ is a convex function of $\mathbf{R}$ into $\mathbf{R}$, one can define

$$\varphi(x) = \lim_{t \to 0+} \frac{\|1 + tx\| - 1}{t}.$$

An element $x \in \mathscr{A}$ is *hermitian* if $\varphi(ix) = \varphi(-ix) = 0$; x is *positive* if x is hermitian and has positive spectrum. Since, for $y \in \mathscr{A}$, $\varphi(y) + \varphi(-y) \geqq 0$, x is hermitian if both $\varphi(ix)$ and $\varphi(-ix)$ are negative and positive if, in addition, $\varphi(-x) \leqq 0$. If $\mathscr{A}$ is a stellar algebra, x is hermitian (in the sense above) if and only if x is self-adjoint $(x = x^*)$. The function Φ of $\mathscr{A}$ into $\mathbf{R}$ defined by the equation

$$\Phi(x) = \sup\{\varphi(\lambda x) : \lambda \in \mathbf{C}, |\lambda| \leqq 1\}$$

is a norm on $\mathscr{A}$ equivalent to the original norm. The above facts are proved in [4].

We shall list here the basic facts that we shall use throughout the paper.

PROPOSITION A. *The element $x \in \mathscr{A}$ is hermitian if and only if $\|e^{itx}\| = 1$ for every $t \in \mathbf{R}$.*

PROPOSITION B [**21**, Hilfsatz 2 (e)]. *If* $x \in \mathscr{A}$ *is hermitian,* $\operatorname{sp}(x) \subset \mathbf{R}$.

PROPOSITION C [**2**, Lemma 3.1]. *Let* H *be the set of hermitian elements of* $\mathscr{A}$. *Then* $H+iH$ *is closed in* $\mathscr{A}$.

Proof. First note that, if $a, b \in H$, then $\varphi(a+ib) \leqq \varphi(a)+\varphi(ib) = \varphi(a) = \varphi(a+ib-ib) \leqq \varphi(a+ib)+\varphi(-ib) = \varphi(a+ib)$; hence $\varphi(a+ib) = \varphi(a)$. Consequently $\Phi(a) \leqq \Phi(a+ib)$.

Now let (x_n) be a sequence in $H+iH$ converging to $x \in \mathscr{A}$. Write $x_n = a_n+ib_n$, $a_n, b_n \in H$. By the above discussion (a_n) is a Cauchy sequence (for Φ and therefore for the original norm). Thus the sequence (a_n) [resp. (b_n)] converges to $a \in \mathscr{A}$ [resp. $b \in \mathscr{A}$]. But H is closed [**21**, Hilfsatz 2(d)]; hence $x = (a+ib) \in H+iH$.

PROPOSITION D [**21**, II]. *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are Banach algebras, and* u *is a norm-decreasing linear mapping of* $\mathscr{B}$ *into* $\mathscr{A}$ *mapping* 1 *onto* 1, *then* u *carries hermitian* [*resp. positive*] *elements of* $\mathscr{B}$ *onto hermitian* [*resp. positive*] *elements of* $\mathscr{A}$.

Proof. For any $y \in \mathscr{B}$, $\varphi(u(y)) \leqq \varphi(y)$. Hence, for $x \in \mathscr{B}$ hermitian, $\varphi(iu(x)) \leqq \varphi(ix) = 0$ and $\varphi(-iu(x)) \leqq \varphi(-ix) = 0$; furthermore, for x positive, $\varphi(-u(x)) \leqq \varphi(-x) \leqq 0$.

PROPOSITION E [**21**, Hilfsatz 2(c)]. *If* $a+ib = a'+ib'$ *where* a, b, a', *and* b' *are hermitian, then* $a = a'$ *and* $b = b'$.

PROPOSITION F [**17**, Theorem]. *If* $\mathscr{A}$ *is* (*algebraically*) *spanned by its hermitian elements* (*i.e.* $\mathscr{A} = H+iH$), *the mapping* $x \to x^*$ *is an involution on* $\mathscr{A}$ *under which* $\mathscr{A}$ *becomes a stellar algebra.* (*If* $x = a+ib$ (a, b *hermitian*) *then* $x^* = a-ib$.)

PROPOSITION G [**21**]. *If* $x \in \mathscr{A}$ *is hermitian and quasi-nilpotent* ($\operatorname{sp}(x) = \{0\}$), *then* $x = 0$.

1. The spectral theorem for normal elements of a Banach algebra. In this section we shall introduce the concept of a normal element of a Banach algebra and prove a spectral theorem for such elements. This theorem depends on the theory of $C^l(\mathbf{R}^2)$-scalar elements (a concept due to Foias (see [**6**]) and Maeda [**16**]; see [**20**] for a complete exposition and for further references).

We shall denote by $C^l(\mathbf{R}^2)$ the Banach algebra of continuous, complex-valued functions defined on $\mathbf{R}^2$ having limits at ∞ and by $\mathscr{K}(\mathbf{R}^2)$ the set of continuous functions with compact support. Note that $C^l(\mathbf{R}^2)$ is a stellar algebra, and that it can be identified with the direct sum of $C_0(\mathbf{R}^2)$ and $\mathbf{C}$ where $C_0(\mathbf{R}^2)$ is the stellar algebra of continuous functions on $\mathbf{R}^2$ vanishing at ∞.

DEFINITION. An element x of a Banach algebra $\mathscr{A}$ is $C^l(\mathbf{R}^2)$-*scalar* if there exists a continuous representation u of $C^l(\mathbf{R}^2)$ into $\mathscr{A}$ mapping 1 onto 1 and such that, for every $f \in \mathscr{K}(\mathbf{R}^2)$ taking the value 1 on $\operatorname{sp}(x)$, $u(f) = 1$ and $u(zf) = x$.

REMARK. It follows from [**6**, Corollary 1.6, p. 98] that, for any $C^l(\mathbf{R}^2)$-scalar element $x \in \mathscr{A}$, there is only one representation u as described in the definition above. It is called the $C^l(\mathbf{R}^2)$-*scalar representation for* x. Furthermore, for any $f, g \in C^l(\mathbf{R}^2)$ which agree on $\operatorname{sp}(x)$, $u(f) = u(g)$ [**6**, Theorem 1.6, p. 60].

If $\mathscr{A}$ and $\mathscr{B}$ are Banach algebras, a linear mapping of $\mathscr{B}$ into $\mathscr{A}$ is *involutive* if it maps hermitian elements of $\mathscr{B}$ into hermitian elements of $\mathscr{A}$.

PROPOSITION 1.1. *Let $\mathscr{A}$ be a Banach algebra, $\mathscr{B}$ a stellar algebra, and u a continuous representation of $\mathscr{B}$ into $\mathscr{A}$ mapping* 1 *onto* 1. *Then u is involutive if and only if u has norm* 1.

Proof. If u has norm 1, use Proposition D to conclude that u is involutive. To prove the converse consider the closure, $\mathscr{A}'$, of the image of u. Since $\mathscr{B}$ is (algebraically) spanned by its hermitian elements, and since u is involutive, $\mathscr{A}'$ is spanned by its hermitian elements (use Proposition C). Thus, by Proposition F, $\mathscr{A}'$ is a stellar algebra. A standard result from the theory of stellar algebras [5, Proposition 1, p. 66] now applies.

DEFINITION. Let $\mathscr{A}$ be a Banach algebra. An element $x \in \mathscr{A}$ is *normal* if there exist commuting elements $a, b \in \mathscr{A}$ such that

(1) $a^m b^n$ is hermitian for every $m, n \in \mathbf{N}$;
(2) $x = a + ib$.

We shall call a the *real part* of x and b the *imaginary part* of x. (Note that, by Proposition E, a and b are unique.)

If E is a Banach space, an operator T on E is *normal* if T is normal as an element of the Banach algebra $\mathscr{L}(E)$.

LEMMA 1.1. *Let X be a compact Hausdorff space, $\mathscr{A}$ a Banach algebra, and $x \in \mathscr{A}$. Suppose that there exists a continuous representation v of $C(X)$ into $\mathscr{A}$ which has* 1 *and x in its image (in particular $v(1) = 1$). Then there exists a $C^1(\mathbf{R}^2)$-scalar representation u for x such that $\|u\| \leqq \|v\|$.*

Proof. Define $u(f) = v(f \circ h)$, where $h \in C(X)$ is such that $x = v(h)$.

THEOREM 1.1 (The spectral theorem in Banach algebras). *An element $x \in \mathscr{A}$ is normal if and only if x is $C^1(\mathbf{R}^2)$-scalar and the $C^1(\mathbf{R}^2)$-scalar representation for x has norm* 1.

Proof. First suppose that x is normal. Let a be the real part of x, b the imaginary part of x, and let $\mathscr{B}$ be the smallest closed subalgebra of $\mathscr{A}$ containing the set $\{a, b, 1\}$. By Proposition F and the Gelfand Isomorphism Theorem [5, Théorème 1, p. 67] there exists an isometric isomorphism v of $C(\mathrm{sp}(x))$ onto $\mathscr{B}$. One now uses Lemma 1.1 to obtain the desired conclusion.

To prove the converse suppose that u is the $C^1(\mathbf{R}^2)$-scalar representation of x (so that $\|u\| = 1$). Let r and s be elements of $C^1(\mathbf{R}^2)$ such that

$$r(z) = \mathscr{R}(z) \quad \text{and} \quad s(z) = \mathscr{I}(z)$$

for every $z \in \mathrm{sp}(x)$, and let

$$a = u(r) \quad \text{and} \quad b = u(s).$$

Clearly $x = a + ib$. For $m, n \in \mathbf{N}$, $a^m b^n = u(r^m s^n)$ is hermitian by Proposition D.

COROLLARY 1. *An element $x \in \mathscr{A}$ is normal if and only if x is $C^l(\mathbf{R}^2)$-scalar and the $C^l(\mathbf{R}^2)$-scalar representation for x is involutive.*

COROLLARY 2. *Let E be a Banach space. A necessary and sufficient condition for an operator $T \in \mathscr{L}(E)$ to be $C^l(\mathbf{R}^2)$-scalar is that there exist a norm on E, equivalent to the original norm, under which T is normal.*

Proof. The sufficiency follows easily from previous results. To prove necessity suppose that T is $C^l(\mathbf{R}^2)$-scalar and let U be the $C^l(\mathbf{R}^2)$-scalar representation for T. By Theorem 1.1 we need only exhibit an equivalent norm on E such that, when E is endowed with the new norm, $\| U \| = 1$. Such a norm is given by

$$x \to \sup \{\| U(f)x \| : \| f \| \leqq 1\}.$$

REMARK. Corollary 2 above is valid in the context of Hilbert spaces [**15, 22**]. However, the proof given above can not be used in this case since the new norm need not be a Hilbert space norm.

COROLLARY 3. *If $x \in \mathscr{A}$ is normal, then $\| x \| = \rho(x)$.*

Proof. Let u be the $C^l(\mathbf{R}^2)$-scalar representation for x, and choose $f \in C^l(\mathbf{R}^2)$ of norm $\rho(x)$ and such that $f(z) = z$ for $z \in \operatorname{sp}(x)$. Then $\| x \| = \| u(f) \| \leqq \| f \| = \rho(x)$.

COROLLARY 4 [**2**, Theorem 2.1]. *If $p_1, \ldots, p_n$ are non-zero disjoint projections (hermitian idempotents) in a Banach algebra $\mathscr{A}$ and $\lambda_1, \ldots, \lambda_n$ are complex numbers, then*

$$\| \sum_{i=1}^{n} \lambda_i p_i \| = \sup \{| \lambda_i | : 1 \leqq i \leqq n\}.$$

In particular, if p is a non-trivial projection, $\| p \| = \| 1-p \| = 1$.

Proof. For $f \in C^l(\mathbf{R}^2)$ define $u(f) \in \mathscr{A}$ by

$$u(f) = \sum_{i=1}^{n} f(\lambda_i)p_i + f(0)(1 - \sum_{i=1}^{n} p_i).$$

Then u is an involutive $C^l(\mathbf{R}^2)$-scalar representation for $x = \sum_{i=1}^{n} \lambda_i p_i$. By Proposition 1.1 and Theorem 1.1, x is normal. By Corollary 3,

$$\| x \| = \rho(x) = \sup \{| \lambda_i | : 1 \leqq i \leqq n\}.$$

REMARK. A necessary and sufficient condition for an idempotent $p \in \mathscr{A}$ to be a projection is that $\| p + \lambda(1-p) \| = 1$ for every $\lambda \in \mathbf{T}^1$. To prove this fact simply use the equality

$$\| e^{itp} \| = \| e^{it}(p + e^{-it}(1-p)) \| = \| p + e^{-it}(1-p) \|$$

together with Proposition A. This equivalence was first noted by Palmer [**18**].

EXAMPLE. If $E = \mathbf{C}^2$ with the norm $(x, y) \to |x|+|y|$ and P is the idempotent in $\mathscr{L}(E)$ defined by

$$P(x, y) = \tfrac{1}{2}(x+y, x+y),$$

$\|P\| = \|I-P\| = 1$. However

$$\|(P+i(I-P))(1, 0)\| = \sqrt{2}.$$

Hence P is not hermitian.

E. Berkson has shown [3] that the above example is valid when $\mathbf{C}^2$ is endowed with the "p-norm" for any p such that $1 \leqq p \leqq \infty$, $p \neq 2$.

For the next two results we shall need two definitions. An element x of a Banach algebra $\mathscr{A}$ is *power hermitian* if x^n is hermitian for every $n \in \mathbf{N}$. The element x is *unitary* if x is normal and invertible and if both x and x^{-1} have norm 1. If $\mathscr{A}$ is a stellar algebra, then every hermitian element is power hermitian. However, there is an example of a hermitian operator on a Banach space which is not power hermitian [**14**].

PROPOSITION 1.2 [**4**]. *If a and b are commuting hermitian elements of a Banach algebra $\mathscr{A}$, and if $x = a+ib$ has real spectrum, then x is hermitian* (*i.e.* $b = 0$). *Consequently, the element $x \in \mathscr{A}$ is power hermitian if and only if x is normal and has real spectrum.*

Proof. The hypotheses of the proposition imply that $\rho(e^{-ia})\rho(e^{b}) = \rho(e^{-ix}) = 1$; hence, by [**5**, Cor. to Prop. 5, p. 26] and the fact that $\rho(e^{ia}) = \rho(e^{-ia}) = 1$, $\rho(e^{b}) = 1$. Similarly, $\rho(e^{-b}) = 1$. By the Spectral Mapping Theorem, b is quasi-nilpotent and therefore 0 by Proposition G.

PROPOSITION 1.3 (see [**19**]). *A normal element x of a Banach algebra $\mathscr{A}$ is unitary if and only if* $\mathrm{sp}(x) \subset \mathbf{T}^1$.

Proof. The fact that a unitary element has a spectrum contained in $\mathbf{T}^1$ follows from the Spectral Mapping Theorem. To prove the converse, let u be the $C^l(\mathbf{R}^2)$-scalar representation for x, and let $f \in C^l(\mathbf{R}^2)$ be the identity on $\mathbf{T}^1$ and have norm 1. From the fact that $f\bar{f} = 1$ on $\mathrm{sp}(x)$, $x^{-1} = u(\bar{f})$; the result follows.

The next proposition, which we shall state here without proof, is analogous to Theorem 1 of [**19**].

PROPOSITION 1.4. *If x is a normal, invertible element of a Banach algebra $\mathscr{A}$, then there exist a positive, power hermitian element $y \in \mathscr{A}$ and a unitary element $z \in \mathscr{A}$ such that $x = yz$. Furthermore, if a is the real part of z and b is the imaginary part of z, then y, a, and b commute and $y^k a^m b^n$ is hermitian for every k, m, $n \in \mathbf{N}$.*

If $x \in \mathscr{A}$ is $B^\infty(\mathbf{R}^2)$-scalar, the assumption of invertibility can be omitted from the hypotheses of Proposition 1.4. (For the definition of $B^\infty(\mathbf{R}^2)$ see §2.)

2. Scalar operators. In this section we shall examine the preceding results in the context of the scalar operators of Dunford [**7, 8**]. We shall begin by reviewing some notation and known theorems, most of which are taken from [**11**].

We shall let $B^\infty(\mathbf{R}^2)$ denote the set of bounded Borel-measurable functions from $\mathbf{R}^2$ into $\mathbf{C}$. With the usual addition, multiplication, involution and norm, $B^\infty(\mathbf{R}^2)$ is a stellar algebra, and $C^l(\mathbf{R}^2)$ is a stellar subalgebra of $B^\infty(\mathbf{R}^2)$. For any subset A of $\mathbf{R}^2$, φ_A will denote the characteristic function of A; A is a Borel set if and only if $\varphi_A \in B^\infty(\mathbf{R}^2)$. If A is a Borel subset of $\mathbf{R}^2$ and if U is a function whose domain is $B^\infty(\mathbf{R}^2)$, we shall use the symbol U_A in place of $U(\varphi_A)$. Note that, if U is a representation of $B^\infty(\mathbf{R}^2)$ into a ring $\mathscr{A}$, U_A is idempotent.

Let E be a Banach space. A continuous representation U of $B^\infty(\mathbf{R}^2)$ into $\mathscr{L}(E)$ is *standard* if, for any bounded sequence (f_n) in $B^\infty(\mathbf{R}^2)$ converging pointwise to 0, the sequence $(U(f_n))$ converges strongly to 0 in $\mathscr{L}(E)$. An operator $T \in \mathscr{L}(E)$ is *scalar* if there exists a standard representation U of $B^\infty(\mathbf{R}^2)$ into $\mathscr{L}(E)$ mapping 1 onto I and such that, for every bounded Borel subset A of $\mathbf{R}^2$, $U(z\varphi_A) = U_A T$. (These operators were called scalar-type by Dunford [**8**].)

If $T \in \mathscr{L}(E)$ is scalar, there is only one standard representation that has the properties listed above. It is called the *spectral representation* for T. Furthermore, $U(zf) = U(f)T$ for every $f \in B^\infty(\mathbf{R}^2)$ with compact support.

The following theorem, which is proved in [**12**], summarizes the relationship between scalar and $C^l(\mathbf{R}^2)$-scalar operators in weakly complete Banach spaces. A proof, based on the theory of spectral measures as developed in [**11**], can be given.

THEOREM 2.1. *On a weakly complete Banach space E, an operator $T \in \mathscr{L}(E)$ is scalar if and only if it is $C^l(\mathbf{R}^2)$-scalar. Furthermore, if U is the spectral representation for T, then the restriction of U to $C^l(\mathbf{R}^2)$* is the *$C^l(\mathbf{R}^2)$-scalar representation for T.*

COROLLARY 1. *If $T \in \mathscr{L}(E)$ is scalar and if U is the spectral representation for T, then the following assertions are equivalent*:

(1) *T is normal.*
(2) *U has norm* 1.
(3) *$U(f)$ is hermitian for every real $f \in B^\infty(\mathbf{R}^2)$.*
(4) *U_A is hermitian for every Borel subset of $\mathbf{R}^2$.*

COROLLARY 2 [**1**, Theorem 4.2; **9**]. *Let E be a weakly complete Banach space. An operator $T \in \mathscr{L}(E)$ is scalar if and only if there is a norm on E, equivalent to the original norm, under which T is normal.*

The following corollary follows easily from Theorem 2.1 and Corollary 1.

COROLLARY 3 (The spectral theorem in Hilbert spaces). *If T is a normal operator on a Hilbert space H, then T is scalar. Furthermore, if U is the spectral representation for T, then U_A is hermitian for every Borel subset A of $\mathbf{R}^2$.*

EXAMPLE. Let $E = C(K)$ where K is an infinite compact subset of $\mathbf{R}^2$. For every $g \in C(\mathbf{R}^2)$ define $U(g) \in \mathscr{L}(E)$ by $U(g)x = gx$ $(x \in E)$, and let $T = U(z)$. Then the restriction of U to $C^l(\mathbf{R}^2)$ is a $C^l(\mathbf{R}^2)$-scalar representation for T. (As a matter of fact U has a norm 1 and therefore T is normal.)

On the other hand, let (t_n) be a discrete convergent sequence in K. Let (Q_n) be a sequence of open subsets of $\mathbf{R}^2$ such that the sequence of closures is disjoint and such that $t_n \in Q_n$ for every $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, let f_n be a unit vector in E with support contained in Q_n. Then (f_n) is a bounded sequence in $B^\infty(\mathbf{R}^2)$ converging pointwise to 0, but the sequence $(U(f_n)1)$ does not converge in E. Consequently, T is not scalar.

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