ON NONABELIAN TENSOR ANALOGUES OF 2-ENGEL CONDITIONS

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Abstract. Tensor analogues of right 2-Engel elements in groups were introduced by D. P. Biddle and L.-C. Kappe. We investigate the properties of right 2-Engel tensor elements and introduce the concept of $2\otimes$-Engel margin. With the help of these results we describe the structure of $2\otimes$-Engel groups. In particular, we prove a tensor version of Levi’s theorem for 2-Engel groups and determine tensor squares of two-generator $2\otimes$-Engel $p$-groups.

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1. Introduction. For any group $G$, the nonabelian tensor square $G \otimes G$ is a group generated by the symbols $g \otimes h$, subject to the relations

$$gg' \otimes h = (g^g \otimes h^g)(g' \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h')(g^h \otimes h'^h),$$

where $g, g', h, h' \in G$ and $g^h = h^{-1}gh$. The more general concept of nonabelian tensor product of groups acting on each other in certain compatible way was introduced by R. Brown and J.-L. Loday in [5], following the ideas of R. K. Dennis [6]. This construction has its origins in algebraic K-theory as well as in homotopy theory, yet it has become interesting from a purely group-theoretical point of view since the paper of R. Brown, D. L. Johnson and E. F. Robertson [4]. Since then, many authors have been concerned with explicit computations of nonabelian tensor squares; see the paper of L.-C. Kappe [9] for a comprehensive survey of these results.

The main topic of [3] is consideration of tensor analogues of the center and centralizers in groups. More precisely, for a given group $G$ the subgroup $Z^\otimes(G)$ consisting of all $a \in G$ with $a \otimes x = 1_\otimes$ for every $x \in G$ is called the tensor center. This concept was introduced by G. J. Ellis [7]. Moreover, for a group $G$ and a non-empty subset $X$, the subgroup $C^\otimes_G(X) = \{a \in G : a \otimes x = 1_\otimes \text{ for all } x \in X\}$ is said to be the tensor annihilator of $X$ in $G$. Also, tensor analogues of right $n$-Engel elements have been defined. Recall that the set of right $n$-Engel elements of a group $G$ is defined by $R_n(G) = \{a \in G : [a, n] = 1 \text{ for all } x \in G\}$. Here $[a, n]$ stands for the commutator $[\cdots [[a, x], x], \cdots]$ with $n$ copies of $x$. It is well-known that $R_1(G) = Z(G)$ and that $R_2(G)$ is a subgroup of $G$ [13]. In contrast with this, it was shown that for $n \geq 3$ the set $R_n(G)$ is not necessarily a subgroup [14]. The set of right $n_{\otimes}$-Engel elements of a group $G$ is

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then defined as
\[ R_n^\otimes(G) = \{ a \in G : [a, x^{-1}] \otimes x = 1_\otimes \text{ for all } x \in G \}. \]

One of the results of [3] shows that \( R_2^\otimes(G) \) is always a characteristic subgroup of \( G \) containing \( Z(G) \) and contained in \( R_3(G) \). It is also shown by an example that these inclusions may be proper.

The purpose of this paper is to further investigate tensor analogues of 2-Engel structure in groups. In the first part of the paper we determine some further information about \( R_2^\otimes(G) \) and provide some new characterizations of this subgroup. In particular, we define the tensor analogue of 2-Engel margin and show that there is a striking resemblance between the results about 2-Engel margin and the results about its tensor analogue. We use these results to obtain the structure of 2\(^{-}\)-Engel groups. Here

It is proved in that paper that a group \( G \) has a finite covering by 2-Engel subgroups if and only if \( \gamma_3(G) \) divides 3. Therefore it is hardly surprising that the following result is obtained: if \( G \) is a 2\(^{-}\)-Engel group, then \( G \otimes G \) is abelian, \( \gamma_3(G) \leq Z^\otimes(G) \) and \( ([x, y] \otimes z)^3 = 1_\otimes \) for every \( x, y, z \in G \). As a consequence, we obtain several characterizations of 2\(^{-}\)-Engel groups, once again indicating the strong correspondence between 2-Engel groups and 2\(^{-}\)-Engel groups.

Let \( \mathcal{G} \) be a group-theoretic property. A group \( G \) is said to have a finite covering by \( \mathcal{G} \)-subgroups if \( G \) equals, as a set, to the union of finite family of \( \mathcal{G} \)-subgroups. The finite coverings of groups by their 2-Engel subgroups were studied by L.-C. Kappe [10]. It is proved in that paper that a group \( G \) has a finite covering by 2-Engel subgroups if and only if \( |G : R_3(G)| < \infty \). The situation is similar in the context of 2\(^{-}\)-Engel groups. We prove that a group \( G \) can be covered by a finite family of 2\(^{-}\)-Engel subgroups if and only if \( |G : R_3^\otimes(G)| < \infty \). Another result of [10] in this direction is that \( G \) has a finite covering by 2-Engel normal subgroups if and only if \( G \) is 3-Engel and \( |G : R_2(G)| < \infty \). It is to be expected that there is a tensor analogue of this result, but we leave it for future consideration. It is not difficult to see that if \( G \) has a finite covering by 2\(^{-}\)-Engel normal subgroups, then \( G \) is 3\(^{-}\)-Engel and \( |G : R_2^\otimes(G)| < \infty \). For the reverse conclusion one would probably need the characterization of 3\(^{-}\)-Engel groups by their normal closures analogous to [12].

Since every 2\(^{-}\)-Engel group has an abelian tensor square, there is a good chance to compute tensor squares of 2\(^{-}\)-Engel groups explicitly. We reduce these computations to consideration of tensor squares of groups of class \( \leq 2 \). With the help of this we compute tensor squares of two-generator 2\(^{-}\)-Engel \( p \)-groups, using the results of [1] and [11]. It is worth mentioning that there is a minor error in the classification of two-generator \( p \)-groups of class 2 given by [1], so we give the correct result here. We also compute the kernel of the commutator map \( \kappa : G \otimes G \to G' \) given by \( g \otimes h \mapsto [g, h] \) for any nonabelian two-generator 2\(^{-}\)-Engel \( p \)-group \( G \). The group \( \ker \kappa \) is of interest as it is isomorphic to the third homotopy group of the space \( SK(G, 1) \) [5]. In addition, we compute the Schur multiplier of \( G \).

2. Preliminary results. In this section we summarize without proofs some basic results regarding computations in tensor squares and the results concerning 2-Engel groups which will be used throughout the paper without any further reference. The first lemma gives the right action version of [5, Proposition 3].
LEMMA 1 ([5]). Let $g, g', h, h' \in G$. The following relations hold in $G \otimes G$:
(a) $(g^{-1} \otimes h)^g = (g \otimes h)^{-1} = (g \otimes h^{-1})^g$.
(b) $(g' \otimes h)^g \otimes h = (g' \otimes h')^g \otimes h$.
(c) $[g, h] \otimes g' = (g \otimes h)^{-1} (g \otimes h)^{g'}$.
(d) $g' \otimes [g, h] = (g \otimes h)^{-g} (g \otimes h)$.
(e) $[g, h] \otimes [g', h'] = (g \otimes h, g' \otimes h')]$.

Note here that $G$ acts on $G \otimes G$ by $(g \otimes h)^g = g^g \otimes h^g$. The next result is crucial in studying the analogy between commutators and tensors.

PROPOSITION 1 ([4]). For a given group $G$ there exists a homomorphism $\kappa : G \otimes G \to G'$ such that $\kappa : g \otimes h \mapsto [g, h]$. Moreover, $\ker \kappa \leq Z(G \otimes G)$ and $G$ acts trivially on $\ker \kappa$.

An element $a$ of a group $G$ is called a right 2-Engel element of $G$ if $[a, x, x] = 1$ for each $x \in G$. In a similar fashion, an element $a$ is said to be a left 2-Engel element of $G$ if $[x, a, a] = 1$ for each $x \in G$. The sets of right 2-Engel elements and left 2-Engel elements of $G$ are denoted by $R_2(G)$ and $L_2(G)$, respectively. For the properties of right 2-Engel elements we refer to [15, Theorem 7.13] and [16, Lemma 2.2, Theorem 2.3].

We list here some of them, especially those which turn out to have tensor analogues.

PROPOSITION 2 ([15], [16]). Let $G$ be a group, $a \in R_2(G)$ and $x, y, z \in G$.
(a) $a$ is also a left 2-Engel element and $a^G$ is abelian.
(b) $[a, x]^2 = [a^x, x]$ for all $r, s \in \mathbb{Z}$.
(c) $[a, x, y] = [a, y, x]^{-1}$.
(d) $[a, [x, y]] = [a, x, y]^2$.
(e) $a^2 \in Z_3(G)$.
(f) $[a, [x, y], z] = 1$.

Here $a^G$ denotes the normal closure of $a$ in $G$. This result is the main ingredient of the proof of Levi’s theorem [15, pp. 45–46] that every 2-Engel group $G$ is nilpotent of class $\leq 3$ and the exponent of $\gamma_3(G)$ divides 3. We also list some characterizations of 2-Engel groups which will serve as a model for $2_\otimes$-Engel groups.

PROPOSITION 3 ([15]). For a group $G$ the following assertions are equivalent:
(a) $G$ is a 2-Engel group.
(b) $C_G(x)$ is a normal subgroup of $G$ for every $x \in G$.
(c) $[x, [y, z]] = [x, y, z]^2$ for any $x, y, z \in G$.
(d) $[x, y, z]^{-1} = [x, y, z]$ for any $x, y, z \in G$.
(e) $x^G$ is abelian for every $x \in G$.

3. Right $2_\otimes$-Engel elements of groups. The main object of this section is the study of tensor analogues of right (left) 2-Engel elements of a given group. More precisely, for an arbitrary group $G$ we define the sets of right (left) $2_\otimes$-Engel elements of $G$ by $R_2^\otimes(G) = \{a \in G : [a, x] \otimes x = 1_\otimes$ for all $x \in G\}$ and $L_2^\otimes(G) = \{a \in G : [x, a] \otimes a = 1_\otimes$ for all $x \in G\}$, respectively. At the beginning we formulate some elementary properties of these two sets.

LEMMA 2. Let $G$ be any group.
(a) $R_2^\otimes(G) \subseteq R_2(G), L_2^\otimes(G) \subseteq L_2(G)$.
(b) Every right $2_\otimes$-Engel element of $G$ also belongs to $L_2^\otimes(G)$.
(c) $L_2^\otimes(G) = \{a \in G : a^x \otimes a^y = a \otimes a$ for all $x, y \in G\}$.

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The first equation yields

\[ b = (a \otimes x)(a \otimes x)^{-1}(x) \]

The following theorem is already proved in [3].

**Theorem 1 ([3]).** For any group \( G \), the set of all right \( 2_{\otimes} \)-Engel elements of \( G \) is a characteristic subgroup of \( G \).

The computations with tensors involving right \( 2_{\otimes} \)-Engel elements are facilitated by the following result which has roots in corresponding rules for computation with 2-Engel elements [15, Theorem 7.13]. Before formulating the result, note that

\[ Z_n^\otimes (G) = \{ a \in G : [a, x_1, \ldots, x_{n-1}] \otimes x_n = 1_\otimes \text{ for all } x_1, \ldots, x_n \in G \} \]

is a characteristic subgroup of \( G \) contained in the \( n \)-th center \( Z_n(G) \). This subgroup is called the \( n \)-th tensor center of \( G \) [3].

**Proposition 4.** Let \( G \) be a group, \( x, y, z \in G \) and \( a \in R_2^\otimes (G) \).

(a) \([a, x] \otimes y = ([a, y] \otimes x)^{-1}\).

(b) \([a, x] \in C_G^\otimes (x^G)\).

(c) \([a, x]^n \otimes y = ([a, x] \otimes y)^n \) for any \( n \in \mathbb{Z} \).

(d) \( a \otimes x^a = (a \otimes x)^a \) for any \( n \in \mathbb{Z} \).

(e) \([a, x] \otimes [y, z] = 1_\otimes\).

(f) \([x, y] \otimes a = ([x, a] \otimes y)^2 \) and \( a \otimes [x, y] = ([a, x] \otimes y)^2 \).

(g) \( a^x \in Z_3^\otimes (G) \).

**Proof.** The identities (a) and (b) are already proved in [3, Lemma 5.1 and Lemma 5.2]. To prove (c), it suffices to assume that \( n > 0 \). Now observe that \([a, x]^n \otimes y = ([a, x] \otimes y)([a, x]^{n-1} \otimes y)\); hence (c) follows by an induction on \( n \).

Before we proceed, note first that (a) implies that the elements of the form \( b \otimes z \), where \( b \in a^G \) and \( z \in G \), commute with each other. Expanding \( a \otimes xy \) and \( xy \otimes a \) using the tensor product rules, we have

\[ a \otimes xy = (a \otimes x)(a \otimes y)([a, x] \otimes y) \]  

and

\[ xy \otimes a = (x \otimes a)(y \otimes a)([x, a] \otimes y). \]

The first equation yields

\[ a \otimes [x, y] = a \otimes (yx)^{-1}(xy) = (a \otimes xy)(a \otimes yx)^{-1}([a, (yx)^{-1}] \otimes xy) \]
by [3, Lemma 5.1]. Since \( xy \) is a conjugate of \(yx\), we have \([a, (yx)^{-1}] \otimes xy = 1_\otimes\) by (b), hence \( a \otimes [x, y] = ([a, x] \otimes y)^2 \). Similarly we prove \( a \otimes [x, y] = ([a, x] \otimes y)^2 \). It is also clear that the equation (1) also implies (d).

It remains to prove that \([a, x] \otimes [y, z] = 1_\otimes\) and \( a^2 \in Z^2_\otimes(G) \). Expanding the identity \([a, x] \otimes yz = ([a, yz] \otimes x)^{-1}\), we obtain that \(([a, x] \otimes z)([a, x] \otimes y)^2 = ([a, z] \otimes x)^{-[a, y]z}(a, y) \otimes y^{-1}, x^{-1})x^{-z} = 1_\otimes \). Since \([a, z, x] \otimes [a^2, y^2] = 1_\otimes\), it follows that 

\[ [a, y]^2 \text{ acts trivially on } [a, z] \otimes x. \]

Thus we obtain, after cancellation and relabelling, 

\[ 1_\otimes = [a, y] \otimes [x, z] = ([a, x, z] \otimes y)^{-1} = ([a, x, z]^2 \otimes y)^{-1}, \] hence \([a^2, x, y] \otimes z = 1_\otimes\). □

The immediate consequence of Proposition 4 is the following characterization of \( R^2_\otimes(G) \).

**Corollary 1.** For any group \( G \) we have \( R^2_\otimes(G) = \{a \in G : [a, x] \in C^2_\otimes(G^\otimes) \text{ for all } x \in G\} \).

It is known that \( a \in R_2(G) \) implies that \( a^G \) is abelian. The following corollary gives the corresponding result for right 2-\( _\otimes \)-Engel elements.

**Corollary 2.** Let \( a \in R^2_\otimes(G) \). Then the normal closure \((a \otimes x)^{G\otimes} \) is an abelian group for any \( x \in G\).

**Proof.** Let \( a \in R^2_\otimes(G) \) and \( \tau \in G \otimes G \). As usual, denote with \( \kappa \) the commutator map \( G \otimes G \rightarrow G' \). Then we have \([([a \otimes x], (a \otimes x)^\kappa)] = [a \otimes x, (a \otimes x)^{\kappa(\tau)}] = [a, x] \otimes [a^\kappa(\tau), x^\kappa(\tau)] = 1_\otimes \) by Proposition 4. It follows by conjugation that every two elements of \((a \otimes x)^{G\otimes} \) commute, as required. □

Let \( \phi(x_1, \ldots, x_n) \) be any word in the variables \( x_1, \ldots, x_n \). For a group \( G \) the associated marginal subgroup \( \phi^*(G) \) (also called the \( \phi \)-margin of \( G \)) consists of all \( a \in G \) such that \( \phi(g_1, \ldots, a g_i, \ldots, g_n) = \phi(g_1, \ldots, g_i, \ldots, g_n) \) for every \( g_i \in G \) and \( 1 \leq i \leq n \). It is clear that \( \phi^*(G) \) is always a characteristic subgroup of \( G \).

Margins were first introduced by P. Hall [8]. In particular, marginal subgroups for the 2-Engel word \( \phi(x, y) = [x, y, y] \) were studied by T. K. Teague [16].

Let \( E_1(G) = \{a \in G : [a, x, y] = [x, y, y] \text{ for all } x, y \in G\} = R_2(G) \) and \( E_2(G) = \{a \in G : [a, ay, ay] = [x, y, y] \text{ for all } x, y \in G\} \). Then the 2-Engel margin of \( G \) is \( E(G) = E_1(G) \cap E_2(G) \). Now, the tensor analogues of these subgroups can be defined as

\[
E^\otimes_1(G) = \{a \in G : [ax, y] \otimes y = [x, y] \otimes y \text{ for all } x, y \in G\},
\]

\[
E^\otimes_2(G) = \{a \in G : [ay, a] \otimes ay = [x, y] \otimes y \text{ for all } x, y, a \in G\},
\]

and let \( E^\otimes(G) = E^\otimes_1(G) \cap E^\otimes_2(G) \). It is not difficult to see that these sets are characteristic subgroups of \( G \). Using Proposition 4, we also conclude that \( E^\otimes_1(G) = R^2_\otimes(G) \).

In [16, Theorem 2.4] it is proved that \( E(G) = \{a \in G : [x, a, y][x, y, a] = 1 \) for all \( x, y \in G\) \). The following result is therefore hardly surprising.

**Theorem 2.** For any group \( G \) we have

\[
E^\otimes(G) = \{a \in G : ([x, a] \otimes y)([x, y] \otimes a) = 1_\otimes \text{ for all } x, y \in G\}.
\]

**Proof.** Let \( S = \{a \in G : ([x, a] \otimes y)([x, y] \otimes a) = 1_\otimes \text{ for all } x, y \in G\} \), let \( a \in S \) and \( x, y \in G \). It is clear that \( a \in R^2_\otimes(G) = E^\otimes_1(G) \). Using Proposition 4, we have that 

\[
[x, ay] \otimes ay = [x, y][x, a] \otimes ay = ([x, y][x, a] \otimes y)[x, y][x, a] \otimes ay = ([x, y] \otimes y)([x, a] \otimes y)\bigl([x, y] \otimes a \bigr) \otimes ay = ([x, y] \otimes y)([x, a] \otimes a)^{(x, a)} \otimes a)^{(x, a)}.\]

Observe that \((a \otimes a)^{-1} = (a \otimes a)^{-1} (a \otimes a) = 1_\otimes \) by Lemma 2;
hence we only have to prove that \([x, a]^y\) acts trivially on \([x, y] \otimes y\). To see this, we first note that \([y, [x, a]]^y = [y, [x, a]]y\), hence \(([x, y] \otimes y)^{[x, a]^y} = [x, y] \otimes [y, [x, a]]y\). As \([x, a] \in R^2_2(G)\), we get \(([x, a], y) \otimes [x, y] = ([x, a], [x, y]) \otimes y = 1_\otimes\) by Proposition 4, thus the inclusion \(S \subseteq E(G)\) is proved. Conversely, every \(a \in E(G)\) also belongs to \(R^2_2(G)\). Reversing the above arguments, we obtain \(a \in S\), as required.

Let us mention an important consequence of this theorem.

**Corollary 3.** Let \(G\) be a group, \(x, y \in G\) and \(a \in E(G)\). Then \(([a, x] \otimes y)^3 = [a^3, x] \otimes y = 1_\otimes\).

**Proof.** For \(a \in E(G)\) we get \(1_\otimes = ([x, y] \otimes a)([x, a] \otimes y) = ([x, a] \otimes y)^3\) by Proposition 4, hence also \([a^3, x] \otimes y = 1_\otimes\).

It is proved in [16] that \(Z_2(G) \leq E(G) \leq Z_3(G)\) for any group \(G\). Similar arguments show the following.

**Proposition 5.** For any group \(G\) we have \(Z^2_2(G) \leq E(G) \leq Z^3_2(G)\).

**Proof.** It is clear that \(Z^2_2(G) \leq E(G)\). Now, if \(a \in E(G)\), then \(a^3 \in Z^2_2(G) \leq Z^3_2(G)\). On the other hand, we have \(a^2 \in Z^3_2(G)\) by Proposition 4, hence \(a \in Z^3_2(G)\).

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**4.** \(2_\otimes\)-Engel groups. A group \(G\) is said to be \(2_\otimes\)-Engel when \([x, y] \otimes y = 1_\otimes\) for any \(x, y \in G\). It is worth noting that \(G\) is \(2_\otimes\)-Engel precisely when \(R^2_2(G) = G\), which is equivalent to \(L^2_2(G) = G\) and is also equivalent to \(E(G) = G\). Using the commutator map argument, it becomes clear that every \(2_\otimes\)-Engel group is also \(2\)-Engel. The structure of \(2_\otimes\)-Engel groups is described in the next result which corresponds to the well-known Levi’s theorem about \(2\)-Engel groups [15, pp. 45–46]:

**Theorem 3.** Let \(G\) be a \(2_\otimes\)-Engel group. Then we have:

(a) \(G \otimes G\) is abelian group;
(b) \(g_3(G) \leq Z^3_2(G)\);
(c) \(([x, y] \otimes z)^3 = 1_\otimes\) for any \(x, y, z \in G\).

**Proof.** It follows directly from Proposition 4 that \(G \otimes G\) is abelian. From the same proposition we obtain \(([x, y, z] \otimes v)^3 = [x, y, z]^3 \otimes v = [x, y, z] \otimes v = ([x, v] \otimes [y, z])^{-1} = 1_\otimes\). Furthermore, since \(E(G) = G\), we get (b) and (c) by Corollary 3.

In contrast with this result, there exists a \(2\)-Engel group \(G\) such that \(cl (G \otimes G) = 2\) \([2]\). The following is a tensor analogue of Proposition 3.

**Corollary 4.** The following statements for a group \(G\) are equivalent.

(a) \(G\) is \(2_\otimes\)-Engel.
(b) \([x, y] \otimes z = ([x, z] \otimes y)^{-1}\) for any \(x, y, z \in G\).
(c) \(x \otimes [y, z] = ([x, y] \otimes z)^2\) for any \(x, y, z \in G\).
(d) \(x^y \otimes x^z = x \otimes x \) for any \(x, y, z \in G\).

Additionally, if \(G\) is a \(2_\otimes\)-Engel group, then \(C^2_2(G) \leq C_G\) for any \(g \in G\).

**Proof.** By Proposition 4, (a), (b) and (c) are equivalent. The equivalence between (a) and (d) is established in Lemma 2, (c). Now let \(G\) be a \(2_\otimes\)-Engel group, let \(g, y \in G\) and let \(x \in C^2_2(G) \leq C_G(g)\). Then we have \(x^y \otimes g = x[y, y] \otimes g = [x, y] \otimes g = ([x, g] \otimes y)^{-1} = 1_\otimes\), thus \(x^y \in C^2_2(G)\). This proves the corollary.
It is evident that the condition “\( C^2_G(g) \triangleleft G \) for any \( g \in G \)” may fail to imply that \( G \) is 2\( _\oplus \)-Engel, as \( C^2_G(g) \) does not necessarily contain \( g \).

Turning our attention to finite coverings by 2\( _\oplus \)-Engel subgroups, we mention here a related result of L.-C. Kappe [10] which states that a group \( G \) has a finite covering by 2-Engel subgroups if and only if \( |G : R_2(G)| < \infty \). Our proof of the tensor analogue follows the lines of Kappe’s proof.

**Theorem 4.** A group \( G \) has a finite covering by 2\( _\oplus \)-Engel subgroups if and only if \( |G : R_2^\oplus(G)| < \infty \).

**Proof.** Suppose that \( G = \bigcup_{i=1}^n H_i \), where \( H_i \) are 2\( _\oplus \)-Engel subgroups of \( G \). The standard reduction step, due to B. H. Neumann (see [10]), shows that we may assume that \( |G : H_i| < \infty \) for every \( i \). Hence the subgroup \( D = \bigcap_{i=1}^n H_i \) has a finite index in \( G \). It is clear that \( D \leq R_2^\oplus(G) \); hence \( |G : R_2^\oplus(G)| < \infty \).

Assume now \( |\bar{G} : \bar{R}_2^\oplus(G)| < \infty \). Let \( \{g_1, \ldots, g_n\} \) be a transversal of \( R_2^\oplus(G) \) in \( G \) and let \( H_i = \langle g_i \rangle R_2^\oplus(G) \). We have \( G = \bigcup_{i=1}^n H_i \), hence it suffices to prove that each \( H_i \) is 2\( _\oplus \)-Engel. Let \( y = g'a \) and \( x = g'b \) be arbitrary elements of \( R_2^\oplus(G) \), where \( i, j \in \mathbb{Z} \) and \( a, b \in R_2^\oplus(G) \). Since \( R_2^\oplus(G) = E^\oplus_2(G) \), we obtain, using Proposition 4, \( [x, y] \otimes y = [g^i, g^j] \otimes g^i a = [g^i, a] \otimes g^i a = ([g^i, a] \otimes a)([g^i, a] \otimes g^i a) = ([g, a] \otimes g)^i j = 1_\ominus \), as required. \( \square \)

**Remark.** Suppose that a group \( G \) has a finite covering by 2\( _\oplus \)-Engel normal subgroups \( N_1, \ldots, N_n \). Again we may assume that \( |G : N_i| < \infty \) and by Theorem 4 we also have \( |G : R_2^\oplus(G)| < \infty \). Since for every \( x \in G \) we have \( x^G \leq N_i \) for some \( i \), we conclude that every normal closure of an element of \( G \) is 2\( _\oplus \)-Engel. In particular, we have \( 1_\ominus = [x^{-\gamma}, x] \otimes x = ([y, x, x] \otimes x)^\gamma \), hence \( G \) is 3\( _\oplus \)-Engel. In view of [10] it is likely that a 3\( _\oplus \)-Engel group \( G \) with \( |G : R_2^\oplus(G)| < \infty \) has a finite normal covering by 2\( _\oplus \)-Engel subgroups, but we have not been able to (dis)prove this, since there are no known tensor analogues of results regarding 3-Engel groups [12].

**5. Tensor squares of 2\( _\oplus \)-Engel groups.** We have proved in the previous section that 2\( _\oplus \)-Engel groups have abelian tensor squares. Moreover, if \( G \) is a 2\( _\oplus \)-Engel group, then \( \gamma_2(G) \leq Z^\oplus(G) \) by Theorem 3. Using a result of G. J. Ellis [7], we see that \( G \otimes G \cong G/\gamma_3(G) \otimes G/\gamma_3(G) \), hence the calculations of tensor squares reduce to the calculations of tensor squares of class 2 groups (of course, the situation becomes even better when \( G \) is abelian).

Let \( G \) be a nonabelian two-generator 2\( _\oplus \)-Engel \( p \)-group. The group \( G/\gamma_3(G) \) is a two-generator 2\( _\oplus \)-Engel \( p \)-group of class 2. From [1] and [11] we obtain the complete classification of two-generator \( p \)-groups of class 2, hence we only have to check which of these groups are 2\( _\oplus \)-Engel. The following lemma provides a useful criterion for this task.

**Lemma 3.** Let \( G \) be a two-generator group of class two. Then \( G \) is 2\( _\oplus \)-Engel if and only if \( G \otimes G \cong G^{ab} \otimes G^{ab} \).

**Proof.** Let \( G = \langle a, b \rangle \) be a group of class two and let \( x, y \in G \). Then \( x = a^{i_1} b^{j_1} [a, b]^{k_1} \) and \( y = a^{i_2} b^{j_2} [a, b]^{k_2} \) for some \( i_1, j_1, j_2, k_1, k_2 \in \mathbb{Z} \). By means of linear expansion we obtain \( [x, y] = [a, b]^{i_1-i_2} \), hence \( [x, y] \otimes y = (a \otimes [a, b])^{i_1-j_1+j_2} (b \otimes [a, b])^{-i_1+j_1-j_2} \). Therefore \( G \) is 2\( _\oplus \)-Engel if and only if \( a \otimes [a, b] = b \otimes [a, b] = 1_\ominus \), which is equivalent to
The recipe for computing tensor squares of two-generator 2-Engel p-groups therefore consists of looking for those two-generator p-groups G of class two which satisfy the condition $G \otimes G \cong G^{ab} \otimes G^{ab}$. Note also that if $G^{ab} \cong \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$, then $G^{ab} \otimes G^{ab}$ is isomorphic to the direct product of all $\mathbb{Z}_{\gcd(a_i, a_j)}$, where $i, j = 1, \ldots, r$.

First assume $p$ is odd. Then we have the following cases [1].

(Case 1.) $G \cong \langle (c) \rangle \times \langle b \rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$ and $\alpha \geq \beta \geq \gamma \geq 1$. Here we have $G \otimes G \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^\gamma}$, hence $G \otimes G \not\cong G^{ab} \otimes G^{ab}$.

(Case 2.) $G \cong \langle a \rangle \times \langle b \rangle$, where $[a, b] = a^{p^\alpha - p^{2\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$ and $\beta \geq \gamma \geq 1$, $\alpha \geq 2\gamma$; by a closer inspection of the proof of [1, Theorem 2.4] it becomes clear that the extra condition $\alpha \geq \beta$ given there is irrelevant. By [1, Theorem 4.2] we have $G \otimes G \cong \langle (a \otimes a) \rangle \times \langle (b \otimes b) \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle (b \otimes a) \rangle$, where $|a \otimes a| = p^{\alpha p^\gamma}$, $|b \otimes b| = p^{\beta p^\gamma}$, $|(b \otimes a)(a \otimes b)| = p^{\min(\alpha, \gamma) p^\beta}$, $|b \otimes a| = n$, where $n = \gcd(p^{\alpha p^\gamma}, \sum_{k=0}^{p^{\beta p^\gamma}} p^{\gamma p^\beta} + 1)$). Applying [1, Lemma 4.1], we immediately obtain $n = p^{\min(\alpha, \beta)}$, hence $G \otimes G$ is isomorphic to $\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^{\alpha + \beta}} \times \mathbb{Z}_{p^{\alpha + \beta - \gamma}}$. Since $G^{ab} \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta}$ we get $G^{ab} \otimes G^{ab} \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta}$. This yields that $G$ is 2-Engel if and only if $\min(\alpha - \gamma, \beta) = \min(\alpha, \beta)$ which is equivalent to $\alpha \geq \beta + \gamma$.

(Case 3.) $G \cong \langle (c) \rangle \times \langle a \rangle$, where $[a, b] = a^{p^\alpha - p^{2\gamma}} c$, $[c, b] = a^{-p^{2\alpha - \gamma}} c^{-p^{\gamma - \gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$, $|c| = p^\rho$, $\alpha \geq \beta \geq \gamma > \sigma \geq 1$ and $\alpha + \sigma \geq 2\gamma$. Let $\delta = \min(\alpha - \gamma, \beta)$ and $\tau = \min(\alpha - \gamma, \sigma)$. Then we have $G \otimes G \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^\gamma}$, hence it is not isomorphic to $G^{ab} \otimes G^{ab}$.

For $p = 2$ the situation is more complicated [11].

(Case 4.) $G \cong \langle (c) \rangle \times \langle a \rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|c| = 2^\gamma$ and $\alpha \geq \beta \geq \gamma \geq 1$. Here we have

$$G \otimes G \cong \begin{cases} 
\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^\gamma}, & : \beta > \gamma, \\
\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^{\alpha + 1}} \times \mathbb{Z}_{2^{\beta - 1}} \times \mathbb{Z}_{2^{\min(\alpha, \beta) - \gamma}} & : \beta = \gamma.
\end{cases}$$

It follows from here that $G \otimes G \not\cong G^{ab} \otimes G^{ab}$.

(Case 5.) $G \cong \langle a \rangle \times \langle b \rangle$, where $[a, b] = a^{2^{\alpha - \gamma}}$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|[a, b]| = 2^\gamma$ and $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, $\beta \geq \gamma$ and $\alpha + \beta > 3$. In this particular case, $G \otimes G$ is isomorphic to $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^{\alpha + 1}} \times \mathbb{Z}_{2^{\min(\alpha, \beta) - \gamma}} \times \mathbb{Z}_{2^{\min(\alpha, \beta)}}$. It is straightforward to verify that $G \otimes G \not\cong G^{ab} \otimes G^{ab}$.

(Case 6.) $G \cong \langle (c) \rangle \times \langle a \rangle$, where $[a, b] = a^{2^{\alpha - \gamma}} c$, $[c, b] = a^{-2^{2\alpha - \gamma}} c^{-2^{\gamma - \gamma}}$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|[a, b]| = 2^\gamma$, $|c| = 2^\rho$ with $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\alpha + \sigma \geq 2\gamma$ and $\beta \geq \gamma$. Let $\rho = \min(\alpha - \gamma + \sigma, \beta)$. Then we have

$$G \otimes G \cong \begin{cases} 
\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^{\beta + 1}} \times \mathbb{Z}_{2^{\gamma - 1}}, & : \alpha = \gamma + 1, \beta = \gamma, \\
\mathbb{Z}_{2^{\alpha + 1}} \times \mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^{\min(\alpha, \beta) - \gamma}} \times \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\beta} & : \alpha \geq \gamma + 2 or \beta \geq \gamma + 1.
\end{cases}$$

It is clear that $G \otimes G$ is not isomorphic to $G^{ab} \otimes G^{ab}$.

We summarize our conclusions in the following theorem.

**Theorem 5.** Let $G$ be a nonabelian two-generator 2-Engel p-group. Then $p \neq 2$ and $G/\gamma_3(G) \cong \langle a \rangle \times \langle b \rangle$, where $[a, b] = a^{p^\alpha - p^\gamma}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$ with $\alpha \geq \beta)$.\]
$\beta \geq \gamma \geq 1, \alpha \geq 2\gamma$ and $\alpha \geq \beta + \gamma$. We have $G \otimes G \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle b \otimes a \rangle \cong \mathbb{Z}_p^3 \times \mathbb{Z}_{p^{\alpha - \gamma}}$.

Our considerations also show the following.

**Corollary 5.** Every $2_{\otimes}$-Engel 2-group is abelian.

More generally, if $G$ is a $2_{\otimes}$-Engel group without elements of order 3, then $G' \leq Z_{(G)}$ by Theorem 3. This, together with the result of Ellis [7], implies $G \otimes G \cong G^{ab} \otimes G^{ab}$.

Let $G$ be a group. From a topological point of view, the third homotopy group $\pi_3 SK(G, 1)$ of the suspension of $K(G, 1)$ is of some interest. A combinatorial description of $\pi_3 SK(G, 1)$ has been given by J. Wu [17]. Observing the formula $\pi_3 SK(G, 1) \cong \ker \kappa$ [5], one can use a different approach when $G \otimes G$ is explicitly computed. Applying Theorem 5, we describe $\pi_3 SK(G, 1)$ for any nonabelian two-generator $2_{\otimes}$-Engel $p$-group $G$. We also determine the Schur multiplier $H_2(G)$ of $G$.

**Corollary 6.** Let $G$ be a nonabelian two-generator $2_{\otimes}$-Engel $p$-group, let $\kappa : G \otimes G \to G'$ be the commutator map and let $a, b, \alpha, \beta, \gamma$ be as in Theorem 5. Then $\pi_3 SK(G, 1) \cong \ker \kappa \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle (b \otimes a)\gamma \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_{p^{\alpha - \gamma}} \times \mathbb{Z}_{p^{\beta - \gamma}}$ and $H_2(G) \cong \mathbb{Z}_{p^{\beta - \gamma}}$.

**Proof.** As $\kappa(a \otimes a) = \kappa(b \otimes b) = \kappa((b \otimes a)(a \otimes b)) = \kappa((b \otimes a)\gamma) = 1$, Theorem 5 gives $\ker \kappa \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle (b \otimes a)\gamma \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_{p^{\alpha - \gamma}} \times \mathbb{Z}_{p^{\beta - \gamma}}$, as required. To compute the Schur multiplier of $G$, note for instance that the exactness of rows and columns in commutative diagram (1) in [4] implies $H_2(G) \cong \ker \kappa/\Delta(G)$, where $\Delta(G) = \langle x \otimes x : x \in G \rangle$. Now, every $x \in \langle a, b \rangle$ can be written in the form $x = a^m b^n [a, b]^k$, where $m, n, k \in \mathbb{Z}$. Expanding $x \otimes x$ linearly, we obtain $x \otimes x = (a \otimes a)\gamma^m (b \otimes b)^n (b \otimes a)(a \otimes b)^{nm}$. This yields $\Delta(G) \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_{p^{\beta - \gamma}}$, hence the result. □

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**References**


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