NORMALLY ORDERED SEMIGROUPS

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Abstract. In this paper we introduce the notion of normally ordered block-group as a natural extension of the notion of normally ordered inverse semigroup considered previously by the author. We prove that the class \( \text{NOS} \) of all normally ordered block-groups forms a pseudovariety of semigroups and, by using the Munn representation of a block-group, we deduce the decompositions in Mal’cev products \( \text{NOS} = \text{EI} \circ \text{POI} \) and \( \text{NOS} \cap \text{A} = \text{N} \circ \text{POI} \), where \( \text{A} \), \( \text{EI} \) and \( \text{N} \) denote the pseudovarieties of all aperiodic semigroups, all semigroups with just one idempotent and all nilpotent semigroups, respectively, and \( \text{POI} \) denotes the pseudovariety of semigroups generated by all semigroups of injective order-preserving partial transformations on a finite chain. These relations are obtained after showing the equalities \( \text{BG} = \text{EI} \circ \text{Ecom} = \text{N} \circ \text{Ecom} \), where \( \text{BG} \) and \( \text{Ecom} \) denote the pseudovarieties of all block-groups and all semigroups with commuting idempotents, respectively.

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Introduction and preliminaries. Let \( X \) be a set. We denote by \( \mathcal{PT}(X) \) the monoid (under composition) of all partial transformations on \( X \), by \( \mathcal{T}(X) \) the submonoid of \( \mathcal{PT}(X) \) of all full transformations on \( X \) and by \( \mathcal{I}(X) \) the symmetric inverse semigroup on \( X \); i.e. the inverse submonoid of \( \mathcal{PT}(X) \) of all injective partial transformations on \( X \). If \( X \) is a finite set with \( n \) elements, we denote \( \mathcal{PT}(X) \), \( \mathcal{T}(X) \) and \( \mathcal{I}(X) \) simply by \( \mathcal{PT}_n \), \( \mathcal{T}_n \) and \( \mathcal{I}_n \), respectively. Now, suppose that \( X \) is a finite chain with \( n \) elements, say \( X = \{1 < 2 < \cdots < n\} \). We say that a transformation \( s \) in \( \mathcal{PT}_n \) is order-preserving if \( x \leq y \) implies that \( xs \leq ys \), for all \( x, y \in \text{Dom}(s) \), and denote by \( \mathcal{PO}_n \) the submonoid of \( \mathcal{PT}_n \) of all partial order-preserving transformations. As usual, \( \mathcal{O}_n \) denotes the monoid \( \mathcal{PO}_n \cap \mathcal{T}_n \) of all full transformations of \( X_n \) that preserve the order and the injective counterpart of \( \mathcal{O}_n \), i.e. the inverse monoid \( \mathcal{PO}_n \cap \mathcal{I}_n \), is denoted by \( \mathcal{POI}_n \).


In the 1987 “Szeged International Semigroup Colloquium” J.-E. Pin asked for an effective description of the pseudovariety (i.e. an algorithm to decide whether or not a finite semigroup belongs to the pseudovariety) of semigroups \( \mathcal{O} \) generated by the semigroups \( \mathcal{O}_n \), with \( n \in \mathbb{N} \). This problem only had essential progresses after 1995. First, Higgins [11] proved that \( \mathcal{O} \) is self-dual and does not contain all \( \mathcal{R} \)-trivial

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semigroups (and so $\mathcal{O}$ is properly contained in $\mathcal{A}$, the pseudovariety of all finite aperiodic semigroups, i.e. $\mathcal{J}$-trivial semigroups), although every finite band belongs to $\mathcal{O}$. Next, Vernitskii and Volkov [18] generalized Higgins's result by showing that every finite semigroup whose idempotents form an ideal is in $\mathcal{O}$, and in [5] the author proved that the pseudovariety of semigroups $\mathcal{POI}$ generated by the semigroups $\mathcal{POI}_n$, with $n \in \mathbb{N}$, is a (proper) subpseudovariety of $\mathcal{O}$. On the other hand, Almeida and Volkov [2] showed that the interval $[\mathcal{O}, \mathcal{A}]$ of the lattice of all pseudovarieties of semigroups has the cardinality of the continuum and Repnitski˘ı and Volkov [16] proved that $\mathcal{O}$ is not finitely based. In fact, Repnitski˘ı and Volkov proved in [16] that any pseudovariety of semigroups $\mathcal{V}$ such that $\mathcal{POI} \subseteq \mathcal{V} \subseteq \mathcal{O} \vee \mathcal{R} \lor \mathcal{L}$, where $\mathcal{R}$ and $\mathcal{L}$ are the pseudovarieties of semigroups of all $\mathcal{R}$-trivial semigroups and of all $\mathcal{L}$-trivial semigroups, respectively, is not finitely based. Another contribution to the resolution of Pin's problem was given by the author [7], who showed that $\mathcal{O}$ contains all semidirect products of a chain (considered as a semilattice) by a semigroup of injective order-preserving partial transformations on a finite chain. Finally, notice that the problem of embeddability into $\mathcal{O}_n$ was solved by Fremlin and Higgins [9]. Nevertheless, Pin's question is still unanswered.

The inverse counterpart of Pin's problem can be formulated by asking for an effective description of the pseudovariety of inverse semigroups $\mathcal{PCS}$ generated by $\{\mathcal{POI}_n \mid n \in \mathbb{N}\}$. In [3], Cowan and Reilly proved that $\mathcal{PCS}$ is properly contained in $\mathcal{A}$ and also that the interval $[\mathcal{PCS}, \mathcal{A}]$ of the lattice of all pseudovarieties of inverse semigroups has the cardinality of the continuum. From Cowan and Reilly's results it can be deduced that a finite inverse semigroup with $n$ elements belongs to $\mathcal{PCS}$ if and only if it can be embedded into the semigroup $\mathcal{POI}_n$. This is in fact an effective description of $\mathcal{PCS}$. On the other hand, in [6] the author introduced the class $\mathcal{NO}$ of all normally ordered inverse semigroups. This notion is deeply related to the Munn representation of an inverse semigroup $S$, an idempotent-separating homomorphism that may be defined by

$$\phi : S \to \mathcal{I}(E)$$
$$s \mapsto \phi_s : Es^{-1} \to Es^{-1}s$$
$$e \mapsto s^{-1}e,$$

with $E$ the semilattice of all idempotents of $S$. Notice that, the kernel of $\phi$ is $\mu$, the maximum idempotent-separating congruence on $S$. Therefore, $\phi$ is an injective homomorphism if and only if $S$ is a fundamental semigroup, see [12] or [13], for more details. Observe that by a fundamental semigroup we mean any semigroup without non-trivial idempotent-separating congruences. Now, a finite inverse semigroup $S$ is said to be normally ordered if there exists a linear order $\sqsubseteq$ in the semilattice $E$ of the idempotents of $S$ preserved by all partial injective mappings $\phi_e$ (i.e. for $e,f \in Es^{-1}$, $e \sqsubseteq f$ implies $e\phi_s \sqsubseteq f\phi_s$), $s \in S$. It was proved in [6] that $\mathcal{NO}$ is a pseudovariety of inverse semigroups and also that the class of all fundamental normally ordered inverse semigroups consists of all aperiodic normally ordered inverse semigroups. Moreover, the author showed that $\mathcal{PCS} = \mathcal{NO} \cap \mathcal{A}$, giving in this way a Cowan and Reilly alternative (effective) description of $\mathcal{PCS}$. In fact, this also led the author [6] to the following refinement of Cowan and Reilly's description of $\mathcal{PCS}$: a finite inverse semigroup with $n$ idempotents belongs to $\mathcal{PCS}$ if and only if it can be embedded into $\mathcal{POI}_n$. Another refinement (in fact, the best possible) will be given in this paper. Notice that in [6] it was also proved that $\mathcal{NO} = \mathcal{PCS} \lor \mathcal{G}$ (the join of $\mathcal{PCS}$ and $\mathcal{G}$, the pseudovariety of all groups).
The work presented in this paper was strongly motivated by the author's attempt to obtain an effective description for the pseudovariety of semigroups \( \text{POI} \), generalizing the ideas of [6]. Notice that \( \text{POI} \) is a subpseudovariety of \( \text{Ecom} \), the pseudovariety of all idempotent commuting semigroups, whence in order to accomplish this aim, a Munn type representation for, at least, idempotent commuting semigroups is required. Such a representation was constructed by the author [8] for a wider class of semigroups: namely \( \text{BG} \), the class of all block-groups. Recall that a block-group is a finite semigroup whose elements have at most one inverse. Clearly, a finite semigroup is a block-group if and only if each \( \mathcal{L} \)-class and each \( \mathcal{R} \)-class contains at most one idempotent. Observe that \( \text{BG} \) is a pseudovariety of semigroups, which plays a main role in the following celebrated result: \( \Diamond \mathcal{G} = \mathcal{PG} = \mathcal{J} \ast \mathcal{G} = \mathcal{J} \otimes \mathcal{G} = \text{BG} = \text{EJ} \), where \( \mathcal{J} \) denotes the pseudovariety of all \( \mathcal{J} \)-trivial semigroups, \( \mathcal{PG} \) and \( \Diamond \mathcal{G} \) denote the pseudovarieties generated by all power monoids of groups and by all Schützenberger products of groups, respectively, and, finally, \( \text{EJ} \) denotes the pseudovariety of all semigroups whose idempotents generate a \( \mathcal{J} \)-trivial semigroup. See [15] for precise definitions and for a complete story of these equalities.

Next, we recall our extension of the Munn representation for block-groups. Let \( S \) be a semigroup. We denote by \( E(S) \) the set of all idempotents of \( S \) and by \( \text{Reg}(S) \) the set of all regular elements of \( S \). Recall the definition of the quasi-orders \( \leq_{\mathcal{R}} \) and \( \leq_{\mathcal{L}} \) associated to the Green relations \( \mathcal{R} \) and \( \mathcal{L} \), respectively: for all \( s, t \in S \), \( s \leq_{\mathcal{R}} t \) if and only if \( sS^1 \subseteq tS^1 \) and \( s \leq_{\mathcal{L}} t \) if and only if \( S^1s \subseteq S^1t \), where \( S^1 \) denotes the monoid obtained from \( S \) through the adjoining of an identity if \( S \) has none and denotes \( S \) otherwise. To each element \( s \in S \), we associate the following two subsets of \( E(S) \): \( \mathcal{R}(s) = \{ e \in E(S) \mid e \leq_{\mathcal{R}} s \} \) and \( \mathcal{L}(s) = \{ e \in E(S) \mid e \leq_{\mathcal{L}} s \} \). Notice that, if \( e \in E(S) \) and \( s, t \in S \) are such that \( e = st \), then \( es \) and \( te \) are mutually inverse elements of \( S \). Now, let \( S \) be a block-group and let \( s^{-1} \) denote the unique inverse of a regular element \( s \in S \). Then, given \( s \in S \), the maps \( \delta_s : \mathcal{R}(s) \rightarrow \mathcal{L}(s) \), \( e \mapsto (es)^{-1}(es) \), and \( \delta_s : \mathcal{L}(s) \rightarrow \mathcal{R}(s) \), \( e \mapsto (se)(se)^{-1} \), are mutually inverse bijections that preserve \( \mathcal{D} \)-classes. Moreover, being \( E = E(S) \), the mapping

\[
\delta : S \rightarrow \mathcal{I}(E) \\
S \mapsto \delta_s : \mathcal{R}(s) \rightarrow \mathcal{L}(s) \\
e \mapsto (es)^{-1}(es)
\]

is an idempotent-separating homomorphism, which we call the \textit{Munn representation} of \( S \). Notice that \( \delta \) coincides with the (usual) Munn representation of an inverse semigroup \( S \). Furthermore, as for inverse semigroups, the kernel of the Munn representation of a block-group is the maximum idempotent-separating congruence of \( S \); (see [8] for details). Now, we can extend, naturally, the concept of “normally ordered” from inverse semigroups to block-groups. We say that a block-group is \textit{normally ordered} if there exists a \textit{normal order} in \( S \); i.e. a linear order \( \subseteq \) in \( E(S) \) preserved by all partial injective mappings \( \delta_s \) \( (s \in S) \), of the Munn representation of \( S \). We denote by \( \text{NOS} \) the class of all normally ordered block-groups.

The remainder of this paper is organized as follows. In Section 1 we study the class \( \text{NOS} \); in particular, we show that \( \text{NOS} \) is a (decidable) pseudovariety of semigroups. Also in this section we present a refinement of the descriptions of \( \text{PCS} \) mentioned above. In the final section, by using the Munn representation of a block-group, we show the following decompositions in Mal’cev products of the pseudovariety of block-groups: \( \text{BG} = \text{EI} \otimes \text{Ecom} = \text{N} \otimes \text{Ecom} \), where \( \text{EI} \) and \( \text{N} \) denote the pseudovarieties.
of all semigroups with just one idempotent and all nilpotent semigroups, respectively. Furthermore, in Section 2, we deduce also the equality NO = EI ⊕ POI and NO ∩ A = N ⊕ POI.

We assume some knowledge of semigroups, namely that of Green’s relations, regular elements and inverse semigroups. Possible references are [12, 13]. For general background on pseudovarieties, pseudoidentities and finite semigroups, we refer the reader to Almeida’s book [1]. All semigroups considered in this paper are finite.

1. Normally ordered block-groups. In this section we study the class NO of all normally ordered block-groups. In particular, we show that NO is a pseudovariety of semigroups. Notice that, an inverse semigroup belongs to the class NO if and only if it belongs to the pseudovariety of inverse semigroups NO.

We begin by recalling the following lemma, the proof of which can be found in [17].

**Lemma 1.1.** Let φ : S → T be an onto homomorphism of semigroups and let J’ be a J’-class of T. Then J’φ⁻¹ = J₁ ∪ ⋯ ∪ Jₖ, for some J’-classes J₁, ⋯, Jₖ of S, and if Jᵢ (1 ≤ i ≤ k) is ≤₃-minimal among J₁, ⋯, Jₖ, then Jᵢφ = J’. Furthermore, if J’ is regular, then the index i is uniquely determined (i.e. Jᵢ is ≤₃-minimum among J₁, ⋯, Jₖ), and Jᵢ is itself regular.

Next, recall that, given two elements a and b of an arbitrary semigroup S, it is well known that ab ∈ Rₜ ∩ Lₜ if and only if La ∩ Rₜ contains an idempotent. Moreover, if S is finite and a ⋆ b, then ab ∈ Rₐ ∩ Lₐ if and only if ab ⋆ a (see [14]).

The next two lemmas help us to show that NO is closed under homomorphic images.

**Lemma 1.2.** Let S and T be two block-groups and let φ : S → T be an onto homomorphism. Let J’ be a regular J’-class of T and J the J’-class of S ≤ₐ-minimum among the J’-classes Q of S such that Qφ ⊆ J’. Then φ induces a bijection from J ∩ E(S) onto J’ ∩ E(T).

**Proof.** First, notice that J is regular and Jφ = J’. Let e’ ∈ J’ ∩ E(T) and let x ∈ J be such that xeφ = e’. Take e = xω. Then eφ = e’ and Jₑφ ⊆ J’. By the minimality of J, we have J ≤ₜ Jₑ. On the other hand Jₑ ≤ₜ Jₑφ = J and so Jₑ = J. Hence e ∈ J ∩ E(S). Thus J’ ∩ E(T) ⊆ (J ∩ E(S))φ and, since the other inclusion is clear, it follows that (J ∩ E(S))φ = J’ ∩ E(T). In order to prove that φ is injective in J ∩ E(S), let e, f ∈ J ∩ E(S) be such that eφ = fφ = e’. Then (ef)φφ = e’, and so, again by the minimality of J, we have J ≤ₜ Jₑf ≤ₜ Jₑφ = J. Hence ef ∈ J. As e, f ∈ J, then ef ∈ Rₑ ∩ Lₑ, whence Lₑ ∩ Rₑ contains an idempotent g. Now, since each R’-class and each L-class of S contains at most one idempotent, we conclude that e = g = f, as required. □

Let S and T be two block-groups and let φ : S → T be an onto homomorphism. Denote by Eₚ(S) the subset of E(S) of all idempotents e such that the J’-class Jₑ is ≤ₜ-minimum among the J’-classes Q of S such that Qφ ⊆ Jₑφ. Therefore, by the previous lemma, the restriction φ|ₑₚ(S) : Eₚ(S) → E(T) is a bijection from Eₚ(S) onto E(T). Furthermore, given s ∈ S and e ∈ R(s), as e ⋆ (es)⁻¹(es), we have e ∈ Eₚ(S) if and only if (es)⁻¹(es) ∈ Eₚ(S).
Next, observe that, since any homomorphism maps an inverse of a regular element into an inverse of its image, in particular given a homomorphism $\varphi : S \rightarrow T$ between block-groups, we have $(s^{-1})\varphi = (s\varphi)^{-1}$, for any regular element $s \in S$.

**Lemma 1.3.** Let $S$ and $T$ be two block-groups and let $\varphi : S \rightarrow T$ be an onto homomorphism. Let $s \in S$, $t = s\varphi$, $a \in \mathcal{R}(t)$ and $e \in E_\varphi(S) \cap a\varphi^{-1}$. Then $e \in \mathcal{R}(s)$.

**Proof.** Since $a \in \mathcal{R}(t)$, then $at$ is regular and $a = t(at)^{-1} = (at)(at)^{-1}$. Moreover, $at \in J_a$ and $(es)\varphi = at$. Then, by the minimality of $J_e$, we have $J_e \leq_\beta J_{es}$, whence $J_e = J_{es}$. In particular, $es$ is regular and so $(es)^{-1}\varphi = (((es)\varphi)^{-1} = (at)^{-1}$. Then, we have $eq = a = t(at)^{-1} = s(q(es))^{-1}\varphi = (s(es)^{-1})\varphi$ and so $eq = (s(es)^{-1})\varphi$. Thus, again by the minimality of $J_e$, it follows that $J_e \leq_\beta J_{(s(es)^{-1})\varphi}$ and, on the other hand, $J_{(s(es)^{-1})\varphi} \leq_\beta J_{s(es)^{-1}} = J_{s(es)^{-1}(es)(s)^{-1}} \leq_\beta J_e$. Then $J_e = J_{(s(es)^{-1})\varphi}$ and hence $e = (s(es)^{-1})\varphi$. Therefore $e \in \mathcal{R}(s)$, as required. \hfill $\square$

Now, we can prove:

**Proposition 1.4.** Any homomorphic image of a normally ordered block-group is a normally ordered block-group.

**Proof.** Let $T$ be a semigroup, let $S$ be a normally ordered block-group and let $\varphi : S \rightarrow T$ be an onto homomorphism. Denote by $\subseteq$ the normal order of $S$. As $\varphi$ is a bijection from $E_\varphi(S)$ onto $E(T)$, we may define a linear order $\subseteq$ in $E(T)$ by $e\varphi \subseteq f\varphi$ if and only if $e \subseteq f$, for all $e, f \in E_\varphi(S)$.

Now, let $t \in T$ and consider $a, b \in \mathcal{R}(t)$ such that $a \subseteq b$. We aim to show that $(at)^{-1}(at) \subseteq (bt)^{-1}(bt)$. Take $e, f \in E_\varphi(S)$ such that $a = eq$ and $b = f\varphi$. Then $e \subseteq f$, by definition. Let $s \in tw^{-1}$. By Lemma 1.3, it follows that $e, f \in \mathcal{R}(s)$ and, as $\subseteq$ is a normal order of $S$, we have $(es)^{-1}(es) \subseteq (fs)^{-1}(fs)$. Since also $(es)^{-1}(es), (fs)^{-1}(fs) \in E_\varphi(S)$, then $(at)^{-1}(at) = (es)^{-1}(eq) = (es)^{-1}(es)\varphi \subseteq (fs)^{-1}(fs)\varphi = (fs)^{-1}(f)\varphi = (bt)^{-1}(bt)$, as required. \hfill $\square$

Let $S$ be a normally ordered block-group and let $T$ be a subsemigroup of $S$. Then, it is clear that the order induced on $E(T)$ by the normal order of $S$ is a normal order in $T$. Hence $T$ is also a normally ordered block-group.

On the other hand, consider a normally ordered block-groups $S_1, S_2, \ldots, S_n$. For $i \in \{1, 2, \ldots, n\}$, denote by $\subseteq_i$ the normal order of $S_i$. Take $S = S_1 \times S_2 \times \cdots \times S_n$. Since $E(S) = E(S_1) \times E(S_2) \times \cdots \times E(S_n)$, we may consider in $E(S)$ the lexicographic order $\leq_{\text{lex}}$ induced by the orders $\subseteq_1, \subseteq_2, \ldots, \subseteq_n$; i.e. given $e = (e_1, e_2, \ldots, e_n), f = (f_1, f_2, \ldots, f_n) \in E(S)$, we have $e \leq_{\text{lex}} f$ if and only if $e = f$ or, for some $p \in \{1, 2, \ldots, n\}$, $e_i = f_i$, with $1 \leq i \leq p - 1$, and $e_p \supseteq f_p$. It is routine to show that $\leq_{\text{lex}}$ is a normal order in $S$, whence the direct product of $S_1, S_2, \ldots, S_n$ is also a normally ordered block-group.

The previous two observations together with Proposition 1.4 allow us to conclude the following result.

**Theorem 1.5.** The class NOS is a pseudovariety of semigroups.

Observe that, as $POI_n \subseteq NO$ by [6], for all $n \in \mathbb{N}$, we have the next result.

**Corollary 1.6.** $POI \subseteq NOS \cap Ecom \cap A$.

As for inverse semigroups [6], we have the following result.
Proposition 1.7. Let \( S \) and \( T \) be two block-groups and let \( \varphi : S \rightarrow T \) be an onto idempotent-separating homomorphism. Then, \( S \in \text{NOS} \) if and only \( T \in \text{NOS} \).

Proof. By Proposition 1.4, it remains to prove that \( T \in \text{NOS} \) implies \( S \in \text{NOS} \). Suppose that \( T \in \text{NOS} \) and let \( \sqsubseteq \) be the normal order of \( T \). Define a relation \( \sqsubseteq \) in \( E(S) \) by \( e \sqsubseteq f \) if and only if \( e \varphi f \varphi \), for all \( e, f \in E(S) \). As \( \varphi \) separates idempotents, then \( \varphi \) induces a bijection from \( E(S) \) onto \( E(T) \) and hence \( \sqsubseteq \) is a linear order of \( E(S) \). Moreover, \( \sqsubseteq \) is a normal order in \( S \). Indeed, take \( s \in S \) and \( e, f \in \mathcal{R}(s) \) such that \( e \sqsubseteq f \). Then \( e \varphi f, f \varphi \in \mathcal{R}(s \varphi) \) and, by definition, \( e \varphi f \varphi \). Hence, \( (e \varphi s) \varphi^{-1}(e \varphi s \varphi) \subseteq (f \varphi s \varphi) \varphi^{-1}(f \varphi s) \), i.e., \( ((e) \varphi^{-1}(es)) \varphi \subseteq ((f) \varphi^{-1}(fs)) \varphi \), since \( es \) and \( fs \) are regular elements of \( S \). Thus, we have \( (es)^{-1}(es) \subseteq (fs)^{-1}(fs) \), as required. \( \square \)

As the kernel of the Munn representation of a block-group \( S \) is the (maximum) idempotent-separating congruence \( \mu \) of \( S \), we have, by Proposition 1.7, \( S \in \text{NOS} \) if and only if \( S/\mu \in \text{NOS} \). On the other hand, if \( S \in \text{NOS} \), then \( S/\mu \) is, up to an isomorphism, a subsemigroup of \( \mathcal{T}(E(S)) \) whose elements preserve the normal order of \( S \) (a linear order in \( E(S) \)). Therefore, we have the following

Corollary 1.8. Let \( S \) be a block-group and let \( \mu \) be the maximum idempotent-separating congruence of \( S \). Then, \( S \in \text{NOS} \) if and only if \( S/\mu \in \text{POI} \).

Also, we have the following result.

Corollary 1.9. Every fundamental normally ordered block-group belongs to \( \text{POI} \).

Notice that any aperiodic inverse semigroup is fundamental. Moreover, a normally ordered inverse semigroup is aperiodic and if only if it is fundamental by [6]. Unfortunately, in general, an aperiodic normally ordered block-group may not be fundamental; for instance, this is the case of a non-trivial zero semigroup. Nevertheless, it seems reasonable to make the following guess.

Conjecture 1.10. \( \text{POI} = \text{NOS} \cap \text{Ecom} \cap \text{A} \).

Observe that, if \( S \in \text{NOS} \cap \text{Ecom} \cap \text{A} \), then \( \text{Reg}(S) \) is a normally ordered aperiodic inverse semigroup; i.e. \( \text{Reg}(S) \in \text{NO} \cap \text{A} = \text{PCS} \), whence \( \text{Reg}(S) \in \text{POI} \).

We finish this section by presenting a refinement of the author’s description [6] (and of Cowan and Reilly’s description [3]) of the pseudovariety of inverse semigroups \( \text{PCS} \).

First, recall the following refinement of the Munn representation of a block-group \( S \) presented by the author in [8]: the mapping

\[
\vartheta : S \rightarrow \mathcal{I}(\mathcal{J}(E(S)))
\]

\[
s \mapsto \vartheta_s : \mathcal{J}(\mathcal{R}(s)) \rightarrow \mathcal{J}(\mathcal{L}(s))
\]

\[
e \mapsto (es)^{-1}(es),
\]

is an idempotent-separating homomorphism, where \( \mathcal{J}(X) \) denotes the set of all join irreducible idempotents belonging to \( X \), for any subset \( X \) of \( E(S) \).

Theorem 1.11. A finite inverse semigroup \( S \) with \( n \) join irreducible idempotents belongs to \( \text{PCS} \) if and only if \( S \) is isomorphic to a subsemigroup of \( \text{POI}_n \).

Proof. If \( S \) is isomorphic to a subsemigroup of \( \text{POI}_n \), then it is clear that \( S \in \text{PCS} \). Conversely, if \( S \in \text{PCS} \), then the author showed in [6] that there exists a linear order...
\[ \subseteq \text{ in } E(S) \text{ preserved by the mappings } \phi_s(= \delta_s), \ s \in S, \text{ of the Munn representation of } S. \text{ Thus, for all } s \in S, \text{ the mapping } \delta_s \text{ is an injective order-preserving partial transformation on the subchain } \Im \tau (E(S)) \text{ of } (E(S), \subseteq). \text{ Since } \Im \tau (E(S)) \text{ has } n \text{ elements, we may consider } \POI_n \text{ built over this chain and look at } \delta_s \text{ as an element of } \POI_n, \text{ for all } s \in S. \text{ On the other hand, as } S \text{ is aperiodic, then } S \text{ is fundamental, whence the homomorphism } \vartheta : S \to \POI_n, \ s \mapsto \delta_s, \text{ is injective, and the result follows.} \]

Observe that Easdown showed in \[4\] that the least non-negative integer \( n \) such that a fundamental inverse semigroup \( S \) embeds in \( \PT_n \) is the number of join irreducible idempotents of \( S \), whence Theorem 1.11 gives us the best possible refinement of the prior descriptions of \( \text{PCS}. \)

**2. Mal’cev decompositions.** Given a pseudovariety of semigroups \( \mathcal{V} \), a semigroup \( S \) is called a \( \mathcal{V} \)-extension of a semigroup \( T \) if there exists an onto homomorphism \( \varphi : S \to T \) such that, for every idempotent \( e \) of \( T \), the subsemigroup \( e \varphi^{-1} \) of \( S \) belongs to \( \mathcal{V} \). Let \( \mathcal{W} \) be another pseudovariety of semigroups. The Mal’cev product \( \mathcal{V} \bowtie \mathcal{W} \) is the pseudovariety of semigroups generated by all \( \mathcal{V} \)-extensions of elements of \( \mathcal{W} \). One can define alternatively the Mal’cev product by using “relational morphisms”. Recall that a relational morphism \( \tau : S \to T \) from a semigroup \( S \) into a semigroup \( T \) is a function \( \tau \) from \( S \) into the power set \( \mathcal{P}(T) \) of \( T \) such that \( a \tau \neq \emptyset \), for \( a \in S \), and \( a \tau b \subseteq (ab) \tau \), for \( a, b \in S \). Observe that, for each idempotent \( e \) of \( T \), the set \( e \tau^{-1} \) is either empty or a subsemigroup of \( S \). Then, a semigroup \( S \) belongs to \( \mathcal{V} \bowtie \mathcal{W} \) if and only if there exists a relational morphism \( \tau \) from \( S \) into a member \( T \) of \( \mathcal{W} \) such that, for each idempotent \( e \) of \( T \), if \( e \tau^{-1} \) is nonempty then \( e \tau^{-1} \in \mathcal{V} \). (See \[14, 10\].)

Now, recall that the pseudovarieties \( \mathcal{BG}, \mathcal{Ecom}, \mathcal{EI} \) and \( \mathcal{N} \) can be defined by just one pseudoidentity: we have \( \mathcal{Ecom} = \{ x^\omega y^\omega = y^\omega x^\omega \} \), \( \mathcal{BG} = \{ (x^\omega y^\omega)\omega = (\omega x^\alpha y^\alpha)\omega \} \), \( \mathcal{EI} = \{ x^\omega = y^\omega \} \) and \( \mathcal{N} = \{ x^\omega = 0 \} \). Notice also that \( \mathcal{EI} \) is equal to the join \( \mathcal{G} \vee \mathcal{N} \). See \[1\].

Let \( S \in \mathcal{BG} \) and \( E = E(S) \). Since the Munn representation \( \delta : S \to \Im (E) \) of \( S \) is an idempotent-separating homomorphism and \( \Im (E) \in \mathcal{Ecom} \), we immediately have \( S \in \mathcal{EI} \bowtie \mathcal{Ecom} \). Hence \( \mathcal{BG} \subseteq \mathcal{EI} \bowtie \mathcal{Ecom} \). Next, by recalling that \( \mathcal{BG} = \mathcal{J} \bowtie \mathcal{G} \), we can consider a relational morphism \( \xi \) from \( S \) into some group \( G \) such that \( 1 \xi^{-1} \in \mathcal{J} \). Define a function \( \tau \) from \( S \) into \( \mathcal{P}(\Im (E) \times G) \) by \( \st a = ((\delta_s, g) \in \Im (E) \times G \mid g \in s \xi) \), for all \( s \in S \). It is easy to show that \( \tau \) is a relational morphism and, given an idempotent \( e \) of \( \Im \delta \), \( (e, 1) \tau^{-1} = e \delta^{-1} \cap 1 \xi^{-1} \in \mathcal{EI} \cap \mathcal{J} \). Since \( \Im (E) \times G \) is an idempotent commuting semigroup and \( \mathcal{EI} \cap \mathcal{J} = \mathcal{N} \). (In fact, we also have \( \mathcal{EI} \cap \mathcal{N} = \mathcal{N} \); recall that \( \mathcal{J} = \{ (xy)^\omega = (y(xy)^\omega), x^\alpha y^{\alpha+1} = x^\alpha \} \) and \( \mathcal{A} = \{ x^\omega x^{\alpha+1} = x^\alpha \} \). We deduce that \( S \in \mathcal{N} \bowtie \mathcal{EI} \) and so we also have \( \mathcal{BG} \subseteq \mathcal{N} \bowtie \mathcal{EI} \bowtie \mathcal{Ecom} \).

On the other hand, let \( S \) be an \( \mathcal{EI} \)-extension of an idempotent commuting semigroup \( T \) and let \( \varphi : S \to T \) be an onto homomorphism such that, for every idempotent \( e \) of \( T \), \( e \varphi^{-1} \in \mathcal{EI} \) (i.e. \( S \) is an arbitrary generator of \( \mathcal{EI} \bowtie \mathcal{Ecom} \)). Take \( x, y \in S \). Then \( x^\omega, y^\omega, \varphi \in E(T) \), whence \( e = (x^\omega y^\omega)\varphi = x^\omega \varphi y^\omega \varphi = y^\omega \varphi x^\alpha \varphi = (y^\omega x^\alpha)\varphi \) is an idempotent of \( T \). Therefore \( (x^\alpha y^\omega)\varphi, (y^\omega x^\alpha)\varphi \in e \varphi^{-1} \) and, since \( e \varphi^{-1} \in \mathcal{EI} \), we have \( (x^\alpha y^\omega)\varphi = (y^\omega x^\alpha)\varphi \). Thus \( S \in \mathcal{BG} \) and so \( \mathcal{EI} \bowtie \mathcal{Ecom} \subseteq \mathcal{BG} \).

As \( \mathcal{N} \subseteq \mathcal{EI} \), then \( \mathcal{N} \bowtie \mathcal{Ecom} \subseteq \mathcal{EI} \bowtie \mathcal{Ecom} \) and so we have proved the following result.

**Theorem 2.1.** \( \mathcal{BG} = \mathcal{EI} \bowtie \mathcal{Ecom} = \mathcal{N} \bowtie \mathcal{EI} \bowtie \mathcal{Ecom}. \) \( \square \)
This result allows us to conclude that block-groups form the largest class of finite semigroups for which one can consider a Munn type representation; i.e. an idempotent-separating representation by partial injective transformations.

Now, let $S$ be a normally ordered block-group and let $\delta : S \rightarrow \mathcal{I}(E(S))$ be the Munn representation of $S$. As already observed, the semigroup $S\delta$ is a subsemigroup of $\mathcal{I}(E(S))$ whose elements preserve the normal order of $S$, which is a linear order in $E(S)$, so $S\delta \in \text{POI}$. Since $\delta$ separates idempotents, it follows that $S \in \text{EI} \underset{\oplus}{\text{POI}}$. Hence, $\text{NOS} \subseteq \text{EI} \underset{\oplus}{\text{POI}}$. On the other hand, let $S$ be an $\text{EI}$-extension of a semigroup $T \in \text{POI}$ and let $\varphi : S \rightarrow T$ be an onto homomorphism such that, for every idempotent $e$ of $T$, $e\varphi^{-1} \in \text{EI}$ (i.e. $S$ is an arbitrary generator of $\text{EI} \underset{\oplus}{\text{POI}}$). Then, $\varphi$ separates idempotents, $T \in \text{POI} \subseteq \text{NOS}$ and $S \in \text{EI} \underset{\oplus}{\text{POI}} \subseteq \text{EI} \underset{\oplus}{\text{Ecom}} = \text{BG}$, whence $S \in \text{NOS}$, by Proposition 1.7. Therefore, $\text{EI} \underset{\oplus}{\text{POI}} \subseteq \text{NOS}$ and so we have proved the following result.

**Theorem 2.2.** $\text{NOS} = \text{EI} \underset{\oplus}{\text{POI}}$.

Next, observe that any aperiodic extension of an aperiodic semigroup is an aperiodic semigroup. In fact, let $T$ be an aperiodic semigroup and let $\varphi : S \rightarrow T$ be an onto homomorphism such that, for every idempotent $e$ of $T$, $e\varphi^{-1} \in \text{A}$. Take $x \in S$ and let $e = (x^\omega)\varphi$. Then, as $T \in \text{A}$, we have $e = (x^\omega)\varphi = (x\varphi)^\omega = (x\varphi)^{\omega+1} = (x^{\omega+1})\varphi$, whence $x^{\omega+1} \in e\varphi^{-1}$. Then $(x^{\omega+1})^{\omega+1} = (x^{\omega+1})^\omega$, since $e\varphi^{-1} \in \text{A}$, and so $x^\omega = (x^{\omega+1})^\omega = (x^{\omega+1})^{\omega+1} = x^{\omega+1}$, by definition. Thus $S \in \text{A}$, as required.

Now, as $N = \text{EI} \cap \text{A}$, we have $N \underset{\oplus}{\text{POI}} \subseteq \text{A} \cap (\text{EI} \underset{\oplus}{\text{POI}}) = \text{A} \cap \text{NOS}$, by the above observation and Theorem 2.2. On the other hand, let $S \in \text{NOS} \cap \text{A}$. Considering again the Munn representation $\delta : S \rightarrow \mathcal{I}(E(S))$ of $S$, we have, as above, $S\delta \in \text{POI}$ and $e\varphi^{-1} \in \text{EI}$, for all $e \in E(T)$. Since $S$ is aperiodic, we have also $e\varphi^{-1} \in \text{A}$, for all $e \in E(T)$, and so $S \in (\text{EI} \cap \text{A}) \underset{\oplus}{\text{POI}} = N \underset{\oplus}{\text{POI}}$. Thus, we have proved the following result.

**Theorem 2.3.** $\text{NOS} \cap \text{A} = N \underset{\oplus}{\text{POI}}$.

**References**


