ON THE OSOFSKY–SMITH THEOREM

SEPTIMIU CRIVEI
Faculty of Mathematics and Computer Science, "Babeș-Bolyai" University,
Str. M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania
e-mail: crivei@math.ubbcluj.ro

CONSTANTIN NĂSTĂSESCU
Faculty of Mathematics and Computer Science, University of Bucharest,
Str. Academiei 14, 010014 Bucharest, Romania
e-mail: cnastase@al.math.unibuc.ro

and BLAS TORRECILLAS
Departamento de Álgebra y Análisis, Universidad de Almería, 04071 Almería, Spain
e-mail: btorreci@ual.es

Abstract. We recall a version of the Osofsky–Smith theorem in the context of a
Grothendieck category and derive several consequences of this result. For example, it
is deduced that every locally finitely generated Grothendieck category with a family
of completely injective finitely generated generators is semi-simple. We also discuss the
torsion-theoretic version of the classical Osofsky theorem which characterizes semi-
simple rings as those rings whose every cyclic module is injective.

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1. Introduction. In the late 1960s, Osofsky showed her classical result which
asserts that a ring is semi-simple if and only if every cyclic module is injective
[8, Theorem], [9, Corollary]. Among the categorical generalizations of the Osofsky
theorem, we mention the version established by Gómez Pardo et al. [5]. They showed
that if $C$ is a locally finitely generated Grothendieck category and $M$ is a finitely
presented object of $C$ which is completely (pure-)injective and has a von Neumann
regular endomorphism ring $S$, then $S$ is a semi-simple ring [5, Theorem 1]. In the early
1990s, Osofsky and Smith established a module counterpart of the original Osofsky
theorem. They proved that if $M$ is a cyclic module with the property that every cyclic
submodule of $M$ is completely extending, then $M$ is a finite direct sum of uniform
modules [10]. As a consequence, if $M$ is a module with every quotient of a cyclic
submodule injective, then $M$ is semi-simple. In the same paper, Osofsky and Smith
noted that their result still holds in a more general categorical setting.

The purpose of this paper is to discuss some categorical version of the Osofsky–
Smith theorem and give several applications. We first consider the setting of a
locally finitely generated Grothendieck category $C$ and deduce that if $C$ has a family
of completely injective finitely generated generators, then $C$ is semi-simple. As an
application, we give a positive partial answer to the following question raised by

*To Professor Patrick F. Smith on the occasion of his 65th birthday.
2. Locally finitely generated Grothendieck categories.

**Definition 2.1.** Let \( C \) be a Grothendieck category. Then an object \( C \) of \( C \) is called \textit{completely injective} if for every object \( M \) of \( C \) and every morphism \( f : C \to M \), \( \text{Im}(f) \) is an injective object.

**Remark.** As an immediate consequence of the existence of an injective hull for every object in \( C \), an object \( C \) of \( C \) is completely injective if and only if for every injective object \( M \) of \( C \) and every morphism \( f : C \to M \), \( \text{Im}(f) \) is an injective object.

We begin with a property that will be needed later.

**Proposition 2.2.** Let \( C \) be a Grothendieck category and \( (U_i)_{i \in I} \) a family of completely injective objects of \( C \). Then every finite direct sum of \( U_i \)'s is completely injective.

**Proof.** Consider a finite direct sum of \( U_i \)'s, say \( U_1 \oplus \cdots \oplus U_n \), and let \( f : U_1 \oplus \cdots \oplus U_n \to M \) be a morphism in \( C \). We show that \( \text{Im}(f) \) is an injective object. We prove it for \( n = 2 \), the general case that follows by induction. Let \( f : U_1 \oplus U_2 \to M \) be a morphism in \( C \). Denote by \( i_1 : U_1 \to U_1 \oplus U_2 \) and \( i_2 : U_2 \to U_1 \oplus U_2 \) the inclusion morphisms. Also, put \( f_1 = f \circ i_1 \) and \( f_2 = f \circ i_2 \). Then it is easy to see that \( \text{Im}(f) = \text{Im}(f_1) + \text{Im}(f_2) \). Let \( X = \text{Im}(f_1) \), \( Y = \text{Im}(f_2) \), and let \( g : U_1 \to X/(X \cap Y) \) be the composition of the natural epimorphisms \( U_1 \to X \) and \( X \to X/(X \cap Y) \). Then \( (X + Y)/Y \cong X/(X \cap Y) \cong \text{Im}(g) \) is an injective object by hypothesis. But \( Y \) is also injective, and so \( \text{Im}(f) = X + Y \) is an injective object.

Recall that a Grothendieck category \( C \) is called \textit{locally finitely generated} if it has a family of finitely generated generators [12].

**Corollary 2.3.** Let \( C \) be a locally finitely generated Grothendieck category with a family of completely injective finitely generated generators. Then every finitely generated object in \( C \) is injective.

**Example 2.4.** The conclusion of Proposition 2.2 does not hold for an infinite family. Indeed, let us consider an infinite family of fields \( (K_i)_{i \in I} \) and let \( R = \prod_{i \in I} K_i \). Then \( R \) is a commutative von Neumann regular ring, that is, a \( V \)-ring, and so every simple \( R \)-module is injective. Now let \( (e_i)_{i \in I} \) be the family of primitive orthogonal idempotents in \( R \). Clearly, each \( S_i = Re_i \) is a simple \( R \)-module, and so injective. Then each \( S_i \) is actually completely injective. Also, we have \( \bigoplus_{i \in I} S_i = \text{Soc}(R) \). Clearly, \( \bigoplus_{i \in I} S_i \) is not injective, because otherwise this would imply that \( R = \text{Soc}(R) \). Now if we take \( M = \bigoplus_{i \in I} S_i \) and \( f \) to be the identity homomorphism, it follows that \( C = M \) is not completely injective.

**Example 2.5.** If \( R \) is a right hereditary ring, then it is clear that the class of completely injective objects in the category \( \text{Mod}-R \) of right \( R \)-modules coincides with the class of injective objects in \( \text{Mod}-R \).

In order to be able to state the Osofsky–Smith theorem, we need the definition of an extending object in a Grothendieck category, which is the same as for modules.
DEFINITION 2.6. Let \( \mathcal{C} \) be a Grothendieck category. An object \( M \) of \( \mathcal{C} \) is called *extending* if every subobject of \( M \) is essential in a direct summand of \( M \). Equivalently, \( M \) is extending if and only if every essentially closed subobject of \( M \) is a direct summand of \( M \).

An object \( M \) of \( \mathcal{C} \) is called *completely extending* if for every object \( M \) of \( \mathcal{C} \) and every morphism \( f : C \to M \), \( \text{Im}(f) \) is an extending object.

Let \( \mathcal{C} \) be a Grothendieck category. For a class \( \mathcal{P} \) of objects of \( \mathcal{C} \), by a \( \mathcal{P} \)-subobject we mean a subobject belonging to \( \mathcal{P} \). Let \( \mathcal{P} \) be a class of finitely generated objects in \( \mathcal{C} \) with the following properties:

\( (P_1) \) \( \mathcal{P} \) is closed under quotients.
\( (P_2) \) If \( X \in \mathcal{P} \) and \( Y \) is a \( \mathcal{P} \)-subobject of a quotient object of \( X \), then there is a \( \mathcal{P} \)-subobject \( Z \) of \( X \) that projects onto \( Y \).

Some examples of such classes \( \mathcal{P} \) in \( \mathcal{C} \) are the following: the class of all finitely generated objects, the class of finitely generated semi-simple objects and any class of finitely generated objects closed under subobjects and quotients.

Now basically the same proof of the basic theorem for modules (see [7] or [10]) works in our categorical context. This has also been noted in the original paper of Osofsky and Smith [10].

**Theorem 2.7.** Let \( \mathcal{C} \) be a Grothendieck category. Let \( \mathcal{P} \) be a class of finitely generated objects in \( \mathcal{C} \) satisfying \( (P_1) \) and \( (P_2) \) and let \( M \in \mathcal{P} \) be such that every \( \mathcal{P} \)-subobject of \( M \) is completely extending. Then \( M \) is a finite direct sum of uniform objects.

The next two corollaries are obtained as [10, Corollaries 1 and 2].

**Corollary 2.8.** Let \( \mathcal{C} \) be a Grothendieck category such that every finitely generated object is extending. Then every finitely generated object is a finite direct sum of uniform objects.

**Corollary 2.9.** Let \( \mathcal{C} \) be a Grothendieck category. Let \( M \) be an object of \( \mathcal{C} \) such that every quotient of every finitely generated subobject of \( M \) is injective. Then \( M \) is semi-simple.

Recall that a Grothendieck category \( \mathcal{C} \) is called *semi-simple* if every object of \( \mathcal{C} \) is semi-simple [12]. Now Corollaries 2.3 and 2.9 yield the Osofsky–Smith theorem in locally finitely generated Grothendieck categories, stated as follows.

**Theorem 2.10.** Let \( \mathcal{C} \) be a locally finitely generated Grothendieck category with a family of completely injective finitely generated generators. Then \( \mathcal{C} \) is semi-simple.

By Corollary 2.3, the property of complete injectivity of the finitely generated generators of a locally finitely generated Grothendieck category passes to each finitely generated object. Now we immediately have the following consequences of Theorem 2.10.

**Corollary 2.11** [8, Theorem]. Let \( R \) be a ring with identity such that every cyclic (finitely generated) module is injective. Then \( R \) is semi-simple.

**Corollary 2.12** [3, Corollary 7.14]. Let \( R \) be a ring with identity, \( M \) a module and \( \sigma[M] \) the category of \( M \)-subgenerated modules. Suppose that every cyclic (finitely generated) module in \( \sigma[M] \) is \( M \)-injective. Then \( M \) is semi-simple.
Corollary 2.13. Let $R$ be a ring with enough idempotents such that every cyclic (finitely generated) module is injective. Then $R$ is semi-simple.

Recall that a Grothendieck category $C$ is called spectral if every object of $C$ is injective. It is well known that $C$ is semi-simple if and only if it is locally finitely generated and spectral [12]. This suggests us to raise the following natural question, whose positive answer would generalize the Osofsky–Smith theorem 2.10.

Question 1. If $C$ is a Grothendieck category with a family of completely injective generators, does it follow that $C$ is spectral?

3. Applications to torsion theories. Throughout this section, $R$ is a ring with identity, all modules are unitary right $R$-modules and $M$ is a module. Also, Mod-$R$ denotes the category of unitary right $R$-modules, $\sigma[M]$ denotes the full subcategory of Mod-$R$ consisting of $M$-subgenerated modules and $\tau=(T,F)$ is a hereditary torsion theory in Mod-$R$. Recall that a submodule $B$ of a module $A$ is called $\tau$-dense (respectively $\tau$-closed) in $A$ if $A/B$ is $\tau$-torsion (respectively $\tau$-torsion free). Also, a module $M$ is called $\tau$-injective if for every module $B$ and every $\tau$-dense submodule $A$ of $B$, every homomorphism $A \to M$ extends to a homomorphism $B \to M$. For further background on torsion theories the reader is referred to [4] or [12].

Now we have the following consequence of the categorical Osofsky–Smith theorem for torsion theories.

Corollary 3.1. Suppose that every cyclic $\tau$-torsion module is $\tau$-injective. Then every $\tau$-torsion module is $\tau$-injective.

Proof. Note that $T$ is generated by the modules of the form $R/I$ for the $\tau$-dense right ideals $I$ of $R$. Each factor of such an $R/I$ is cyclic $\tau$-torsion, and hence, $\tau$-torsion $\tau$-injective by hypothesis, and so injective in $T$. Thus, each such generator $R/I$ is completely injective in $T$. Now by Theorem 2.10, $T$ is semi-simple, and so spectral. Then every $\tau$-torsion module is injective in $T$, that is, every $\tau$-torsion module is $\tau$-injective.

A related question is the following one, which was raised by M. Teply:

Question 2. If every cyclic module is $\tau$-injective, does it follow that every module is $\tau$-injective?

Remark. Note that, by Corollary 3.1, if every cyclic $\tau$-torsion module is $\tau$-injective, then every $\tau$-torsion module is $\tau$-injective, and so every $\tau$-torsion module is semi-simple by [4, Proposition 8.15]. Hence, Question 2 reduces to the case of a specialization of the Dickson torsion theory [2]. Recall that the Dickson torsion theory is the hereditary torsion theory generated by all simple modules. Its torsion class consists of all semi-artinian modules, whereas its torsion-free class consists of all modules with zero socle.

In the following we shall obtain a positive answer in case $\tau$ is of finite type. Recall that a torsion theory is called of finite type if its Gabriel filter contains a cofinal subset of finitely generated left ideals. A module is called $\tau$-finitely generated if it has a finitely generated $\tau$-dense submodule. We need the following lemma.

Lemma 3.2. Suppose that every cyclic module is $\tau$-injective. Then every $\tau$-finitely generated module is $\tau$-injective.
Theorem 3.4. Recall that a module \(\tau\)-injective assures that every module is \(\tau\)-finitely generated module, say \(M = Rx_1 + \cdots + Rx_n\). Use induction on \(n\). For \(n = 1\) it is clear. Suppose that every module generated by \(n - 1\) elements is \(\tau\)-injective. Then \(M/(Rx_1 + \cdots + Rx_{n-1}) \cong Rx_n/((Rx_1 + \cdots + Rx_{n-1}) \cap Rx_n)\) is cyclic, and so \(\tau\)-injective. But \(Rx_1 + \cdots + Rx_{n-1}\) is also \(\tau\)-injective, so that \(M\) is \(\tau\)-injective.

Now let \(M\) be a \(\tau\)-finitely generated module; hence, \(M\) has some \(\tau\)-dense finitely generated submodule \(N\). Then \(N\) is \(\tau\)-injective by the argument given in the previous paragraph. Clearly, \(M/N\) is \(\tau\)-torsion, and hence, \(\tau\)-injective by Corollary 3.1. Thus, it follows that \(M\) is \(\tau\)-injective.

**Theorem 3.3.** Let \(\tau\) be of finite type and suppose that every cyclic module is \(\tau\)-injective. Then every module is \(\tau\)-injective.

**Proof.** Let \(I\) be a \(\tau\)-dense left ideal of \(R\). Then there exists a finitely generated left ideal \(J \subseteq I\) and we have \(I/J\) \(\tau\)-torsion. Then \(J\) is \(\tau\)-injective by Lemma 3.2; hence, it is a direct summand of \(R\), and so a direct summand of \(I\), say \(I = J \oplus J^\prime\). But \(J^\prime \cong I/J\) is \(\tau\)-torsion, and hence, \(\tau\)-injective. It follows that \(I\) is \(\tau\)-injective, and hence, \(I\) is a direct summand of \(R\). Therefore, every module is \(\tau\)-injective by [4, Proposition 8.10].

There are situations when the condition that every cyclic \(\tau\)-torsion module is \(\tau\)-injective assures that every module is \(\tau\)-injective. We present one based on the recent result stating that every Baer module over a commutative domain is projective [6, Theorem 3.4]. Recall that a module \(M\) is called \(\tau\)-projective if \(\text{Ext}^1_R(M, T) = 0\) for every \(\tau\)-torsion module \(T\). If \(R\) is a commutative domain and \(\tau\) is the usual torsion theory in \(\text{Mod-R}\), then a \(\tau\)-projective module is called \textit{Baer}. We need the following easy lemma.

**Lemma 3.4.** Every \(\tau\)-torsion module is \(\tau\)-injective if and only if every \(\tau\)-torsion module is \(\tau\)-projective.

**Corollary 3.5.** Let \(R\) be a commutative domain and \(\tau\) the usual torsion theory in \(\text{Mod-R}\). The following are equivalent:

(i) Every cyclic \(\tau\)-torsion module is injective.
(ii) Every \(\tau\)-torsion module is injective.
(iii) Every \(\tau\)-torsion module is Baer.
(iv) Every module is injective.
(v) \(R\) is a field.

**Proof.** Recall that a module is \(\tau\)-torsion if and only if every non-zero element \(x \in M\) is annihilated by a non-zero ideal. Since \(R/I\) is \(\tau\)-torsion for every non-zero ideal of \(R\), \(\tau\)-injectivity coincides with usual injectivity.

(i)⇒(ii) By Corollary 3.1.
(ii)⇒(iii) By Lemma 3.4.
(iii)⇒(iv) By Lemma 3.4, every \(\tau\)-torsion module is Baer, and so projective by [6, Theorem 3.4]. Then every module is \(\tau\)-injective [4, Proposition 8.10], and so injective.

(iv)⇒(v) In this case \(R\) is semi-simple, and so \(R\) must be a field.
(v)⇒(i) Clear.

In the following, we establish a characterization of semi-simple modules using certain relative injective modules. Let \(\tau\) be a hereditary torsion theory in the category \(\sigma[M]\). Recall that a module \(N \in \sigma[M]\) is called \((M, \tau)\)-injective if \(N\) is injective.
with respect to every exact sequence $0 \to K \to L$ in $\sigma[M]$ with $L/K$ $\tau$-torsion. We consider the following notion which generalizes that of complemented module with respect to a hereditary torsion theory in Mod-$R$ from [11]. A module $N \in \sigma[M]$ is called $(M, \tau)$-complemented if every submodule of $N$ is $\tau$-dense in a direct summand of $N$.

**Theorem 3.6.** The following are equivalent:

(i) $M$ is semi-simple.

(ii) Every module in $\sigma[M]$ is $(M, \tau)$-injective $(M, \tau)$-complemented.

(iii) Every cyclic module in $\sigma[M]$ is $(M, \tau)$-injective $(M, \tau)$-complemented.

(iv) Every cyclic module in $\sigma[M]$ is injective in $\sigma[M]$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $M$ is semi-simple. Then every module in $\sigma[M]$ is injective in $\sigma[M]$ [14, 20.3], and hence, $(M, \tau)$-injective. Also, every module in $\sigma[M]$ is semi-simple in $\sigma[M]$ [14, 20.3], and hence, $(M, \tau)$-complemented.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (iv) Let $C$ be the smallest closed subcategory of $\sigma[M]$ containing the $(M, \tau)$-complemented modules. Then $C = \sigma[N]$ for some module $N \in \sigma[M]$, and a family of finitely generated generators for $C$ consists of the modules $R/I$ with $R/I \in \sigma[N]$. Each such $R/I$ is $(M, \tau)$-complemented, and so an object of $C$. Thus, $C = \sigma[M]$. By an easy adaptation of [13, Lemma 2] in $\sigma[M]$, it follows that $\tau$ is a generalization of the Goldie torsion theory; hence, $(M, \tau)$-injectivity coincides with injectivity.

(iv) $\Rightarrow$ (i) By Corollary 2.12. □

Now we have the following characterization of semi-simple rings.

**Corollary 3.7.** $R$ is semi-simple if and only if every cyclic module is $\tau$-injective $\tau$-complemented.

The classical Osofsky theorem is obtained by taking $\tau = \tau_G$, i.e. the Goldie torsion theory, or $\tau = \chi$, i.e. the torsion theory with all modules torsion. Note that a module is $\tau_G$-injective $\tau_G$-complemented if and only if it is injective. Also, every module is $\chi$-complemented.

In [1] it has been shown that the class of $\tau$-injective $\tau$-complemented modules is strictly contained in the class of quasi-injective modules. Now recall the following result.

**Theorem 3.8** [7, Theorem 6.83]. The following are equivalent:

(i) $R$ is semi-simple.

(ii) Every module is quasi-injective.

(iii) Every finitely generated module is quasi-injective.

The condition that every cyclic module is quasi-injective is, in general, weaker than that in the previous theorem. For instance, $R = \mathbb{Q}[x]/(x^2)$ is self-injective, and every cyclic module is quasi-injective, but $R$ is not semi-simple [7]. Hence, Corollary 3.7 may be seen as a refinement of Theorem 3.8 for cyclic modules.

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