CHARACTERS OF PRIME DEGREE

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(Received 4 June 2008; accepted 26 May 2011)

Abstract. Let $G$ be a finite nilpotent group, $\chi$ and $\psi$ be irreducible complex characters of $G$ with prime degree. Assume that $\chi(1) = p$. Then, either the product $\chi \psi$ is a multiple of an irreducible character or $\chi \psi$ is the linear combination of at least $\frac{p+1}{2}$ distinct irreducible characters.

2010 Mathematics Subject Classification. 20c15

1. Introduction. Let $G$ be a finite group and $\chi, \psi \in \text{Irr}(G)$ be irreducible complex characters of $G$. We can check that the product $\chi \psi$ of $\chi$ and $\psi$, where $\chi(g) = \chi(g)\psi(g)$ for all $g$ in $G$, is a character and so it can be expressed as a linear combination of irreducible characters. Let $\eta(\chi \psi)$ be the number of distinct irreducible constituents of the product $\chi \psi$.

Theorem A. Let $G$ be a finite nilpotent group, $\chi$ and $\psi$ be irreducible complex characters of prime degree. Assume that $\chi(1) = p$. Then, one of the following holds:

(i) $\chi \psi$ is the sum of $p^2$ distinct linear characters.
(ii) $\chi \psi$ is the sum of $p$ distinct linear characters of $G$ and of $p - 1$ distinct irreducible characters of $G$ with degree $p$.
(iii) all the irreducible constituents of $\chi \psi$ are of degree $p$. Also, either $\chi \psi$ is a multiple of an irreducible character, or it has at least $\frac{p+1}{2}$ distinct irreducible constituents and at most $p$ distinct irreducible constituents, i.e.

$$\text{either } \eta(\chi \psi) = 1 \text{ or } \frac{p+1}{2} \leq \eta(\chi \psi) \leq p.$$ 

(iv) $\chi \psi$ is an irreducible character.

It is proved in Theorem A of [1] that given any prime $p$, any $p$-group $P$, any faithful characters $\chi, \psi \in \text{Irr}(P)$, either the product $\chi \psi$ is a multiple of an irreducible, or $\chi \psi$ is the linear combination of at least $\frac{p+1}{2}$ distinct irreducible characters, i.e. either $\eta(\chi \psi) = 1$ or $\eta(\chi \psi) \geq \frac{p+1}{2}$. It is proved in [4] that given any prime $p$ and any integer $n > 0$, there exists a $p$-group $P$ and characters $\varphi, \gamma \in \text{Irr}(P)$ such that $\eta(\varphi \gamma) = n$. Thus, without the hypothesis that the characters in Theorem A of [1] are faithful, the result may not hold. In this note, we are proving that if the characters have ‘small’ degree then the values that $\eta(\chi \psi)$ can take have the same constraint as if they were faithful.
2. Proofs. We are going to use the notation of [5]. In addition, we denote by 
\( \text{Lin}(G) = \{ \chi \in \text{Irr}(G) \mid \chi(1) = 1 \} \) the set of linear characters, and by 
\( \text{Irr}(G/\text{Ker}(\chi) \geq N) = \{ \chi \in \text{Irr}(G) \mid \text{Ker}(\chi) \geq N \} \) the set of irreducible characters of \( G \) that contain in their 
kernel the subgroup \( N \). Also, denote by \( \overline{\chi} \) the complex conjugate of \( \chi \), i.e. 
\( \chi(g) = \overline{\chi(g)} \) for all \( g \) in \( G \).

**Lemma 2.1.** Let \( G \) be a finite group and \( \chi, \psi \in \text{Irr}(G) \). Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \), for some 
\( n > 0 \), be the distinct irreducible constituents of the product \( \chi \psi \) and \( a_1, a_2, \ldots, a_n \) be the 
unique positive integers such that 
\[
\chi \psi = \sum_{i=1}^{n} a_i \alpha_i.
\]

If \( \alpha_1(1) = 1 \), then \( \psi \overline{\alpha_1} = \overline{\chi} \). Hence, the distinct irreducible constituents of the 
character \( \chi \overline{\chi} \) are \( 1_G, \overline{\alpha_1} \alpha_2, \overline{\alpha_1} \alpha_2, \ldots, \overline{\alpha_1} \alpha_n \), and
\[
\chi \overline{\chi} = a_1 1_G + \sum_{i=2}^{n} a_i (\overline{\alpha_1} \alpha_i).
\]

**Proof.** See Lemma 4.1 of [3]. \( \square \)

**Lemma 2.2.** Let \( G \) be a finite \( p \)-group for some prime \( p \) and \( \chi \in \text{Irr}(G) \) be a character 
of degree \( p \). Then, one of the following holds:

(i) \( \chi \overline{\chi} \) is the sum of \( p^2 \) distinct linear characters.

(ii) \( \chi \overline{\chi} \) is the sum of \( p \) distinct linear characters of \( G \) and of \( p - 1 \) distinct irreducible 
characters of \( G \) with degree \( p \).

**Proof.** See Lemma 5.1 of [2]. \( \square \)

**Lemma 2.3.** Let \( G \) be a finite \( p \)-group, for some prime \( p \), and \( \chi, \psi \in \text{Irr}(G) \) be characters 
of degree \( p \). Then, either 
\[
\eta(\chi \psi) = 1 \quad \text{or} \quad \eta(\chi \psi) \geq \frac{p+1}{2}.
\]

**Proof.** Assume that the lemma is false. Let \( G \) and \( \chi, \psi \in \text{Irr}(G) \) be a 
counterexample of the statement, i.e. \( \chi(1) = \psi(1) = p \) and \( 1 < \eta(\chi \psi) < \frac{p+1}{2} \).

Working with the group \( G/(\text{Ker}(\chi) \cap \text{Ker}(\psi)) \), by induction on the order of \( G \), 
we may assume that \( \text{Ker}(\chi) \cap \text{Ker}(\psi) = \{1\} \). Set \( n = \eta(\chi \psi) \). Let \( \theta_i \in \text{Irr}(G) \), for \( i = 1, \ldots, n \), be the distinct irreducible constituents of \( \chi \psi \). Set
\[
\chi \psi = \sum_{i=1}^{n} m_i \theta_i \tag{2.4}
\]

where \( m_i > 0 \) is the multiplicity of \( \theta_i \) in \( \chi \psi \).

If \( \chi \psi \) has a linear constituent, then by Lemmas 2.1 and 2.2 we have that \( \eta(\chi \psi) \geq p \).
If \( \chi \psi \) has an irreducible constituent of degree \( p^2 \), then \( \chi \psi \in \text{Irr}(G) \) and so \( \eta(\chi \psi) = 1 \).
Thus, we may assume that \( \theta_i(1) = p \) for \( i = 1, \ldots, n \).

Since \( G \) is a \( p \)-group, there must exist a subgroup \( H \) and a linear character \( \xi \) of 
\( H \) such that \( \xi^G = \chi \). Then, \( |G : H| = \chi(1) = p \) and thus \( H \) is a normal subgroup. By 
Clifford theory, we have then
\[
\chi_H = \sum_{i=1}^{p} \xi_i \tag{2.5}
\]
for some $\xi_1 = \xi, \ldots, \xi_p$ distinct linear characters of $H$.

Claim 2.6. $H$ is an abelian group.

Proof. Suppose that $\psi_H \in \mathrm{Irr}(H)$. Since $(\xi \psi_H)^G = \chi \psi$ by Exercise 5.3 of [5], and $\xi \psi_H \in \mathrm{Irr}(H)$, it follows that either $\xi \psi_H$ induces irreducibly, and thus $\eta(\chi \psi) = 1$, or $\xi \psi_H$ extends to $G$ and thus $(\xi \psi_H)^G$ is the sum of the $p$ distinct extensions of $\xi \psi_H$, i.e. $\eta(\chi \psi) = p$. Therefore, $\psi_H \not\in \mathrm{Irr}(G)$ and since $H$ is normal in $G$ of index $p$ and $\psi(1) = p$, $\psi$ is induced from some $\tau \in \mathrm{Lin}(H)$.

Since both $\xi$ and $\tau$ are linear characters, we have that $\ker(\xi) \cap \ker(\tau) \geq [H, H]$. Observe that $\mathrm{core}_G(\ker(\xi) \cap \ker(\tau)) = \mathrm{core}_G(\ker(\xi)) \cap \mathrm{core}_G(\ker(\tau)) = \ker(\chi) \cap \ker(\psi)$. Since $H$ is a normal subgroup of $G$, so is $[H, H]$ and thus $\{1\} = \ker(\chi) \cap \ker(\psi) \geq [H, H]$. Therefore, $H$ is abelian. $\Box$

By the previous claim, observe that $\psi$ is also induced by some linear character $\tau$ of $H$ and thus

$$\psi_H = \sum_{i=1}^{p} \tau_i \quad (2.7)$$

for some $\tau_1 = \tau, \ldots, \tau_p$ distinct linear characters of $H$. Observe also that the centre of both $\chi$ and $\psi$ is contained in $H$.

Claim 2.8. $Z(G) = Z(\chi) = Z(\psi)$.

Proof. Suppose that $Z(\chi) \neq Z(\psi)$. Set $U = Z(\chi) \cap Z(\psi)$. Either $U$ is properly contained in $Z(\chi)$, or it is properly contained in $Z(\psi)$. We may assume that $U \subset Z(\psi)$ and thus we may find a subgroup $T \leq Z(\psi)$ such that $T \not\subset U$ is chief factor of $G$.

Since $H$ is abelian, $Z(\psi) < H$ and $\tau^G = \psi$, then $\psi_T = p\tau_T$ and so $(\tau_T)_T = \tau_T$ for $i = 1, \ldots, p$. By [2.5], we have $\chi(\xi^G) = \chi$, $\xi \in \mathrm{Lin}(H)$ and $T \not\subset Z(\chi)$, the stabilizer of $\xi^G$ is $H$. Thus, the stabilizer of $\xi^G_T$ in $G$ is $H$. By Clifford theory, we have that $\xi_{\tau_T} \in \mathrm{Lin}(H)$ induces irreducibly and $\xi_{\tau_T}$ are distinct characters for $i = 1, \ldots, p$. By (2.7), we have that $\chi \psi = (\xi \psi_H^G) = (\xi^G + \cdots + \xi^G) = (\xi^G + \cdots + \xi^G)^G$, and thus $\eta(\chi \psi) = p$.

We conclude that such $T$ cannot exist and so $Z(\chi) = Z(\psi)$.

Given any $z \in Z(\chi)$ and $g \in G$, we have $z^g \equiv z \pmod{\ker(\chi)}$ since $Z(G/\ker(\chi)) = Z(\chi)/\ker(\chi)$. Hence, $[z, g] = z^{-1}z^g$ lies in $\ker(\chi)$. This same $z$ lies in $Z(\psi) = Z(\chi)$. Hence, $[z, g]$ also lies in $\ker(\psi)$. Therefore, $[z, g] \in \ker(\chi) \cap \ker(\psi) = 1$ for every $z \in Z(\chi) = Z(\psi)$ and every $g \in G$. This implies that $Z(\chi) = Z(\psi) = Z(G)$. $\Box$

Set $Z = Z(G)$. Since $Z$ is the centre of $G$, $\xi^G = \chi$ and $\tau^G = \psi$, we have

$$\chi_Z = p\xi_Z \text{ and } \psi_Z = p\tau_Z. \quad (2.9)$$

Because $\chi_Z \psi_Z = p^2 \xi_Z \tau_Z$, (2.4) implies that

$$\psi_Z = p\xi_Z \tau_Z \quad (2.10)$$

for all $i = 1, \ldots, n$.

Let $Y/Z$ be a chief factor of $G$ with $Y \leq H$. Since $Z$ is the centre of $G$ and $Z = Z(\chi)$, the set $\mathrm{Lin}(Y \mid \xi_Z)$ of all extensions of $\xi_Z$ to linear characters is $\{((\xi_1)_Y = \xi_Y, (\xi_2)_Y, \ldots, (\xi_p)_Y\}$ and it is a single $G$-conjugacy class. By Clifford theory, we have
we have that
\[ (2.4) \text{ and (2.10), given any } \]
Since \( H \) is the stabilizer of \( \tau_Y \) in \( G \) and \( \psi(1) = p \), as before we have that the set \( \text{Lin}(Y \mid \tau_z) = \{(\tau_1)_Y = \tau_Y, (\tau_2)_Y, \ldots, (\tau_p)_Y\} \) and
\[ \psi_Y = \sum_{i=1}^{p} (\tau_i)_Y. \]  
(2.12)

**Claim 2.13.** The stabilizer \( G_{\xi_Y \tau_Y} = \{ g \in G \mid (\xi_Y \tau_Y)^g = \xi_Y \tau_Y \} \) of \( \xi_Y \tau_Y \in \text{Lin}(Y) \) in \( G \) is \( H \).

**Proof.** Assume notation (2.4). Since \( H \) is an abelian subgroup of index \( p \) in \( G \), we have that \( G_{\xi_Y \tau_Y} \geq H \) and thus either \( G_{\xi_Y \tau_Y} = H \) or \( G_{\xi_Y \tau_Y} = G \). Suppose \( \xi_Y \tau_Y \) is a \( G \)-invariant character, i.e. \( G_{\xi_Y \tau_Y} = G \). Since \( |Y : Z| = p \) and \( \xi_Y \tau_Y \) is an extension of \( \xi_Z \tau_Z \), it follows then that all the extensions of \( \xi_Z \tau_Z \) to \( Y \) are \( G \)-invariant. Thus, by (2.4) and (2.10), given any \( i \), there exists some extension \( \nu_i \in \text{Lin}(Y) \) of \( \xi_Z \tau_Z \) such that \( (\theta_i)_Y = p \nu_i \). Thus, \( (\chi \psi)_Y = (\sum_{i=1}^{n} m_i \theta_i)_Y = \sum_{i=1}^{n} m_i \nu_i) = \sum_{i=1}^{n} m_i \nu_i \) has at most \( n < \frac{p+1}{2} \) distinct irreducible constituents. On the other hand, by (2.11) and (2.12) we have
\[ (\chi \psi)_Y = \chi \psi_Y = \left( \sum_{i=1}^{p} (\xi_i)_Y \right) \left( \sum_{j=1}^{p} (\tau_j)_Y \right) = \sum_{j=1}^{p} (\xi_Y \tau_Y)_Y, \]
and so \( (\chi \psi)_Y \) has \( p \) distinct irreducible constituents. That is a contradiction and thus \( G_{\xi_Y \tau_Y} = H \).

By Clifford theory and the previous claim, we have that for each \( i = 1, \ldots, n \), there exists a unique character \( \sigma_i \in \text{Lin}(H \mid \xi_Y \tau_Y) \) such that
\[ \theta_i = (\sigma_i)^G. \]  
(2.14)

If \( Y = H \), then \( |G : Z| = |G : H| |H : Z| = p^2 \). Since \( \chi(1) = \psi(1) = p \), by Corollary 2.30 of [5] we have that \( \chi \) and \( \psi \) vanish outside \( Z \). Since \( \theta_i(1) = p \) for all \( i \) and \( |G : Z| = |G : Z(\theta_i)| = p^2 \), it follows that there exists a unique irreducible character lying above \( \xi_Z \tau_Z \) and thus \( \eta(\chi \psi) = 1. \)

**2.15.** Fix a subgroup \( X \leq H \) of \( G \) such that \( X/Y \) is a chief factor of \( G \). Let \( \alpha, \beta \in \text{Lin}(X) \) be the linear characters such that
\[ \alpha = \xi_X \]  
and \( \beta = \tau_X. \)

Since \( \sigma_i \) lies above \( \xi_Y \tau_Y \in \text{Lin}(Y) \) for all \( i \) and \( X/Y \) is a chief factor of a \( p \)-group, there is some \( \delta_i \in \text{Irr}(X \mod Y) \) such that
\[ (\sigma_i)_X = \delta_i \alpha \beta. \]  
(2.16)

**Claim 2.17.** The subgroup \([X, G] \) generates \( Y = [X, G]Z \) modulo \( Z \).
Proof. Working with the group \( \tilde{G} = G/\text{Ker}(\chi) \), using the same argument as in the proof of Claim 3.26 of [1], we have that \([\tilde{X}, \tilde{G}]\) generates \( \tilde{Y} = [\tilde{X}, \tilde{G}]\tilde{Z} \) modulo \( \tilde{Z} \). Since \( \tilde{Z} = \mathbb{Z}(\chi) \), we have that \( \text{Ker}(\chi) \leq \tilde{Z} \). Thus, \( \tilde{Z} = \tilde{Z}/\text{Ker}(\chi) \) and the claim follows. \( \square \)

2.18. Observe that \( G/H \) is cyclic of order \( p \). So, we may choose \( g \in G \) such that the distinct cosets of \( H \) in \( G \) are \( H, Hg, Hg^2, \ldots, Hg^{p-1} \).

Since \( \chi = \xi^G \) and \( \xi_X = \alpha \), it follows from 2.15 that

\[
\chi_X = \alpha + \alpha g + \cdots + \alpha g^{p-1} = \sum_{i=0}^{p-1} \alpha g^i.
\]

Similarly, we have that

\[
\psi_X = \beta + \beta g + \cdots + \beta g^{p-1} = \sum_{j=0}^{p-1} \beta g^j.
\]

Combining the two previous equations, we have that

\[
\chi_X \psi_X = \left( \sum_{j=0}^{p-1} \alpha g^j \right) \left( \sum_{j=0}^{p-1} \beta g^j \right) = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \alpha g^i \beta g^j.
\]

By (2.4) and (2.16), we have that

\[
(\chi \psi)_X = \left( \sum_{i=1}^{n} m_i \delta_i \right) \chi = \sum_{i=1}^{n} m_i \left[ \sum_{j=0}^{p-1} (\delta_i \alpha \beta)^{g^j} \right].
\]

Claim 2.21. Let \( g \in G \) be as in 2.18. For each \( i = 0, 1, \ldots, p-1 \), there exist \( j \in \{0, 1, \ldots, p-1\} \) and \( \delta_g \in \text{Lin}(X \text{mod} Y) \) such that

\[
\alpha \beta g^i = (\alpha \beta)^{g^j} \delta_g.
\]

Also, \(|\{\delta_g \mid i = 0, 1, 2, \ldots, p-1\}| \leq n\).

Proof. See Proof of Claim 3.30 of [1].

Claim 2.23. Let \( g \in G \) be as in 2.18. Then, there exist three distinct integers \( i, j, k \in \{0, 1, 2, \ldots, p-1\} \), and some \( \delta \in \text{Irr}(X \text{mod} Y) \) such that

\[
\alpha \beta g^i = (\alpha \beta)^{g^j} \delta, \quad \alpha \beta g^j = (\alpha \beta)^{g^k} \delta \quad \text{and} \quad \alpha \beta g^k = (\alpha \beta)^{g^i} \delta,
\]

for some \( r, s, t \in \{0, 1, 2, \ldots, p-1\} \).

Proof. See Proof of Claim 3.34 of [1].

Claim 2.24. We can choose the element \( g \) in 2.18 such that one of the following holds:

(i) There exists some \( j = 2, \ldots, p-1 \) such that

\[
\alpha \beta g^j = (\alpha \beta)^{g^r} \quad \text{and} \quad \alpha \beta g^r = (\alpha \beta)^{g^s},
\]

for some \( r, s \in \{0, 1, \ldots, p-1\} \) with \( r \neq 1 \).
Claim 2.30. Suppose that $X$ follows that $Y$ and $Z$.

Observe that Ker($\xi$) follows.

Finally, $zg = zg$.

Let $g$ be as in Claim 2.24. Since $X / Y$ is cyclic of order $p$, we may choose $x \in X$ such that $X = Y < x >$. Since $H$ is abelian, we have $[X, H] = 1$. Suppose that $[x, g^{-1}] \in Z$. Then, $x$ centralizes both $g^{-1}$ and $H$ modulo $Z$. Hence, $xZ = Z(G/Z)$ and so $[x, G] \leq Z$ since $Y / Z$ is a chief section of the $p$-group $G$, we have that $[Y, G] \leq Z$ and so $< x > Y, G \leq [X, G] \leq Z$ which is false by Claim 2.17. Hence $[x, g^{-1}] \in Y \setminus Z$ and so

$$Y = Z < y >$$

is generated over $y = [x, g^{-1}]$. (2.25)

Since $[Y, G] \leq Z$, we have that $z = [y, g^{-1}] \in Z$. If $z = 1$, then $G = H < g >$ centralizes $Y = Z < y >$, since $H$ centralizes $Y < x >$ by 2.15, and $G$ centralizes $Z$. This is impossible because $Z = Z(G) < Y$. Thus,

$$z = [y, g^{-1}]$$

is a non-trivial element of $Z$. (2.26)

By (2.25), we have $y = [x, g^{-1}] = x^{-1}xg^{-1}$. By (2.26), we have $z = [y, g^{-1}] = y^{-1}xg^{-1}$.

Finally, $z^{-1} = z$ since $z \in Z$. Since $X = Z < x, y >$ is abelian, it follows that

$$z^{-j} = z, y s^{-j} = yz^{-j} \text{ and } x s^{-j} = xy^{-j}z^{-j}$$

for any integer $j = 0, 1, \ldots, p - 1$. Because $g^{-p} \in H$ centralizes $X$ by 2.15, we have

$$z^p = 1 \text{ and } y^p z^{(i)} = 1.$$ 

Observe that the statement is true for $p \leq 3$ since then $p^2 + 1 / 2 \leq 2$. Thus, we may assume that $p$ is odd. Hence, $p$ divides $(i)$ and $z^{(i)} = 1$. Therefore,

$$y^p = z^p = 1.$$ 

It follows that $y^j, z^j$ and $z^{2j}$ depend only on the residue of $i$ modulo $p$, for any integer $i \geq 0$.

2.29. Observe that Ker($\xi Z$) $\cap$ Ker($\tau Z$) $\leq$ Ker($\chi$) $\cap$ Ker($\psi$) = 1 implies that $z$ is not in both Ker($\xi Z$) and Ker($\tau Z$). Without loss of generality, we may assume that $\tau Z(z) \neq 1$.

Since $\beta$ is an extension of $\tau Z$, we may assume that $\beta(z) \neq 1$.

Claim 2.30. $\xi Z \tau Z(z)$ is primitive $p$th root of unit.

Proof. Suppose that $(\xi Z \tau Z(z)) = 1$. Then, $(\xi Z \tau Z)([y, g^{-1}]) = 1$ and so $(\xi Z \tau Z)^{(i)}(y) = (\xi Z \tau Z(y))$. Since $H$ is abelian, $|G : H| = p$, $\theta_i$ lies above $\xi Z \tau Z$ for all $i$ and $g \in G \setminus H$, it follows that $Y = < y, Z(G) >$ is contained in $Z(\theta_i)$. This is contradiction with Claim 2.13. Thus, $(\xi Z \tau Z(z)) \neq 1$. Since $z$ is of order $p$ and $\xi Z \tau Z$ is a linear character, the claim follows.

Claim 2.31. Suppose that

$$\alpha \beta^g = (\alpha \beta)^g \delta,$$

(2.32)
and
\[ \alpha \beta^j = (\alpha \beta)^j \delta, \]
for some \( j \in \{0, 1, \ldots, p-1\}, \) \( j \neq 1, \) some \( \delta \in \text{Irr}(X \mod Y) \) and some \( r, s \in \{0, 1, \ldots, p-1\}. \) Then,
\[ \delta(x) = \beta(z)^{h(r-1)}, \]
where \( 2h \equiv 1 \mod p. \)

Proof. By Claim 2.30 and the same argument as in the proof of Claim 3.40 of [I], the statement follows.

Suppose that Claim 2.24 (ii) holds. Then, by Claim 2.31, we have that \( \delta(x) = \beta(z)^{h(r-1)} \) and \( \delta(x) = \beta(z)^{hk(r-1)}. \) Since \( \beta(z) = \tau_z(z) \) is a primitive \( p \)-th root of unit by 2.29, we have that \( hj(r-1) \equiv hk(r-1) \mod p. \) Since \( r \neq 1 \mod p \) and \( 2h \equiv 1 \mod p, \) we have that \( k \equiv j \mod p, \) which is a contradiction. Thus, Claim 2.24 (i) must hold.

We now apply Claim 2.31 with \( \delta = 1. \) Thus, \( 1 = \delta(x) = \beta(z)^{h(r-1)}. \) Therefore, \( hj(r-1) \equiv 0 \mod p. \) Since \( 2h \equiv 1 \mod p, \) either \( j \equiv 0 \mod p \) or \( r-1 \equiv 0 \mod p. \) Neither is possible. That is our final contradiction and Lemma 2.3 is proved.

Proof of Theorem A. Since \( G \) is a nilpotent group, \( G \) is the direct product \( G_1 \times G_2 \) of its Sylow \( p \)-subgroup \( G_1 \) and its Hall \( p^2 \)-subgroup \( G_2. \) We can then write \( \chi = \chi_1 \times \chi_2 \) and \( \psi = \psi_1 \times \psi_2 \) for some characters \( \chi_1, \psi_1 \in \text{Irr}(G_1) \) and some characters \( \chi_2, \psi_2 \in \text{Irr}(G_2). \) Since \( \chi_1(1) = p, \) we have that \( \chi_2(1) = 1 \) and thus \( \chi_2 \psi_2 \in \text{Irr}(G_2). \) If \( \psi(1) \neq p, \) since \( \psi(1) \) is a prime number, we have that \( \psi_1(1) = 1 \) and thus \( \chi_2 \psi_2 \) is an irreducible. Therefore, \( \chi \psi \in \text{Irr}(G) \) and (iv) holds. We may assume then that \( \psi(1) = p \) and thus \( \psi_2(1) = 1. \) Then, \( \chi_2 \psi_2 \) is a linear character and so we may assume that \( G \) is a \( p \)-group.

If \( \chi \psi \) has a linear constituent, by Lemmas 2.1 and 2.2, we have that (i) or (ii) holds. So, we may assume that all the irreducible constituents of \( \chi \psi \) are of degree at least \( p. \) If \( \chi \psi \) has an irreducible constituent of degree \( p^2, \) then \( \chi \psi \in \text{Irr}(G) \) and (iv) holds. We may assume then that all the irreducible constituents of \( \chi \psi \) have degree \( p. \) Since \( \chi \psi(1) = p^2, \) it follows that \( \eta(\chi \psi) \leq p. \) By Lemma 2.3, we have that either \( \eta(\chi \psi) = 1 \) or \( \eta(\chi \psi) \geq \frac{p+1}{2}, \) and so (iii) holds.

Examples. Fix a prime \( p > 2 \)

(i) Let \( E \) be an extraspecial group of order \( p^3 \) and \( \phi \in \text{Irr}(E) \) of degree \( p. \) We can check that the product \( \phi \bar{\phi} \) is the sum of all the linear characters of \( E. \)

(ii) In the proof of Proposition 6.1 of [2], an example is constructed of a \( p \)-group \( G \) and a character \( \chi \in \text{Irr}(G) \) such that \( \chi \bar{\chi} \) is the sum of \( p \) distinct linear characters and of \( p-1 \) distinct irreducible characters of degree \( p. \)

(iii) Given an extraspecial group \( E \) of order \( p^3, \) where \( p > 2, \) and \( \phi \in \text{Irr}(E) \) a character of degree \( p, \) we can check that \( \phi \bar{\phi} \) is a multiple of an irreducible. In Proposition 6.1 of [1], an example is provided of a \( p \)-group \( G \) and a character \( \chi \in \text{Irr}(G) \) such that \( \eta(\chi \chi) = \frac{p+1}{2}. \) In [6], an example is provided of a \( p \)-group \( P \) and two faithful characters \( \delta, \epsilon \in \text{Irr}(\bar{P}) \) of degree \( p \) such that \( \eta(\delta \epsilon) = p-1. \)

Let \( G \) be the wreath product of a cyclic group of order \( p^2 \) with a cyclic group of order \( p. \) Thus, \( G \) has a normal abelian subgroup \( N \) of order \( (p^2)^p \) and index \( p. \) Let \( \lambda \in \text{Lin}(N) \) be a nontrivial character. We can check that \( \chi = \lambda \bar{\lambda} \) and \( \psi = (\lambda \bar{\lambda})^p \) are irreducible characters of degree \( p \) and \( \chi \psi \) is the sum of \( p \) distinct irreducible characters of degree \( p. \)
We wonder if there exists a $p$-group $P$ with characters $\chi, \psi \in \text{Irr}(P)$ of degree $p$ such that $\frac{p+1}{2} < \eta(\chi \psi) < p - 1$.

(iv) Let $Q$ be a $p$-group and $\kappa \in \text{Irr}(Q)$ be a character of degree $p$. Set $P = Q \times Q$, $\chi = \kappa \times 1_G$ and $\psi = 1_G \times \kappa$. Observe that $\chi$, $\psi$ and $\chi \psi$ are irreducible characters of $P$.

ACKNOWLEDGEMENTS. I would like to thank Professor Everett C. Dade for his suggestions. Also, I thank Irene S. Suarez for her encouragement.

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