ON SS-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS AND THEIR PROPERTIES

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Abstract. A subgroup $H$ of a finite group $G$ is called SS-supplemented in $G$ if there exists a subgroup $K$ of $G$ such that $HK = G$ and $H \cap K$ is $S$-quasinormal in $K$. In this paper, we characterize the finite groups in which every subgroup is SS-supplemented and the influence of SS-supplementation of some subgroups on the structure of finite groups is considered. Some recent results on SS-quasinormal subgroups and $C$-supplemented subgroups are strengthened and enriched.

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1. Introduction. All groups considered in this paper are finite.

A group $G$ is said to be factorized into its subgroups $A$ and $B$ if $G$ is the product of $A$ and $B$. Obviously, the structure of the factorized group $G = AB$ is restricted by its subgroups $A$ and $B$. There has been interest in the past in investigating the structure of the factorized group $G = AB$ by means of the structure of $A$ and $B$. For instance, in 1955, Ito found an impressive and very satisfying theorem. He proved in [17] that $G = AB$ is a metabelian group if $A$ and $B$ are abelian. The most famous theorem of this type was due to Kegel (see [18]) and Wielandt (see [28, 29]) as they stated the solvability of the factorized group $G = AB$ if $A$ and $B$ are both nilpotent. It is also well-known that the group $G = AB$ is nilpotent if $A$ and $B$ are both normal nilpotent subgroups of $G$. However, it is known that the factorized group $G = AB$ is not necessary supersolvable if both $A$ and $B$ are normal supersolvable subgroups of $G$ (see [3]). Thus, the following interesting question arises:

Let $\mathcal{F}$ be a formation (may be a saturated formation). What will be the conditions needed for the subgroups $A$ and $B$ so that the factorized group $G = AB \in \mathcal{F}$ when both $A$ and $B$ are in the formation $\mathcal{F}$?

In answering the above question, Asaad and Shaalan first proved a theorem in 1989 [1] by showing that if $G = HK$ is a factorized group with supersolvable subgroups $H$ and $K$ such that every subgroup of $H$ is permutable with every subgroup of $K$, then $G$ is supersolvable. In 1992, Maier in [23] further proved that the above result can also be obtained by considering the general completeness property of all saturated formations containing the class of supersolvable groups. Along this direction, Ballester-Bolinches and some others have investigated the totally permutable products and the mutually
permutable products of finite groups, and consequently many interesting results have been given (for example, see [4, 6]).

Motivated by the above results, we now call a subgroup $H$ of a group $G$ $SS$-supplemented in $G$ if there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K$ is an $S$-quasinormal subgroup in $K$. In this case, the subgroup $K$ is said to be an $SS$-supplement of $H$ in $G$.

Recall that a subgroup $H$ of a group $G$ is $S$-quasinormal in $G$ if $H$ permutes with every Sylow subgroup of $G$. After the introduction of the above concept by Kegel (see [19]), the structure of a group has been extensively investigated under some additional assumptions on the subgroups of a given group (see [2, 24]). On the other hand, a subgroup $H$ of a group $G$ is called a complemented subgroup of $G$ if there exists another subgroup $K$ of $G$ such that $G = HK$ and $H \cap K = 1$. By using the concept of complemented subgroups, Hall established a fundamental theorem for solvable groups in [14] by proving that a group $G$ is solvable if and only if every Sylow subgroup is complemented. Recently, the authors have also investigated the finite $p$-nilpotent groups with some subgroups c-supplemented in [13]. Research on the complemented subgroups of a given group still continues and many related results have been recently obtained (see [5, 11, 12]).

In this paper, we first describe the relationship between the $SS$-supplemented subgroups and the complemented subgroups or $S$-quasinormal subgroups of a given group $G$. Next, we study the structure of the finite groups whose subgroups are $SS$-supplemented. Some applications of our results are considered so that a number of related results in the literature are extended and generalized.

2. Preliminaries. In this section, we first discuss the properties of $SS$-supplemented subgroups and give some lemmas which will be used in the sequel. For the sake of convenience, we recall that a subgroup $H$ of a group $G$ is $C$-supplemented in $G$ if there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_G$ (see [7]), where $H_G$ is the core of $H$ in $G$. It is obvious that a subgroup $H$ of a group $G$ is $C$-supplemented in $G$ if and only if there exists a subgroup $K_1$ of $G$ such that $G = HK_1$ and $H \cap K_1 = H_G$. Hence, the concept of $C$-supplemented subgroups can be regarded as a generalization of both $C$-normal subgroups and complemented subgroups; therefore, it is worthwhile to investigate the structure of a group by considering its $C$-supplemented subgroups. On the other hand, we recall a new concept (see [21]), which is a generalization of $S$-quasinormality. A subgroup $H$ of a group $G$ is called to be $SS$-quasinormal in $G$ if there is a subgroup $K$ of $G$ such that $G = HK$ and $H$ permutes with every Sylow subgroup of $K$. Many interesting results on $SS$-quasinormality of a group have been recently given by Li and others (for instance, see [21, 22]).

Definition 2.1. A subgroup $H$ of a group $G$ is said to be $SS$-supplemented in $G$ if there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K$ is $S$-quasinormal in $K$. In this case, we say that $K$ is an $SS$-supplement of $H$ in $G$.

It is clear that a $C$-supplemented subgroup of a group $G$ must be $SS$-supplemented in $G$. We now assume that $H$ is a $SS$-quasinormal subgroup of a group $G$. Then, there exists a subgroup $K$ of $G$ such that $G = HK$ and $H$ permutes with every Sylow subgroup of $K$. Let $P$ be a Sylow subgroup of $K$. Then, by $HP = PH$, we deduce that $(H \cap K)P = P(K \cap H)$. This shows that $H$ must be $SS$-supplemented in $G$. On the other hand, a $SS$-quasinormal subgroup of a group may not be $C$-supplemented and a
C-supplemented subgroup of a group may not be SS-quasinormal (see Example 2.2). Furthermore, the following Example 2.3 illustrates that a SS-supplemented subgroup of a group may be neither C-supplemented nor SS-quasinormal. Hence the class of all SS-supplemented subgroups and the class of all SS-quasinormal subgroups in a group contains properly both the class of all C-supplemented subgroups and the class of all SS-quasinormal subgroups in the group.

Example 2.2. Let $G = S_4$ be the symmetric group of degree 4 and let $H = \langle (123) \rangle$. Then, $H$ is C-supplemented in $G$ since $G = HA_4$ and $H \cap A_4 = 1$. However, $H$ is not SS-quasinormal in $G$ because $HP \neq PH$ when $P = \langle (123) \rangle$.

Let $P = \langle x, y : x^{16} = y^4 = 1, x^4 = y^4 \rangle$. Then, $\Phi(P) = \langle x^2, y^2 \rangle = \langle x^2 \rangle \times \langle y^2 \rangle$. It is easy to see that $H = \langle y^2 \rangle$ is S-quasinormal in $P$ and so SS-quasinormal in $P$. However, $H$ is not C-supplemented in $P$.

Example 2.3. Let $G$ be the direct product of $S_4$ and $P$ with $S_4$ and $P$ as in Example 2.2. Now let $H = C_2 \times P_1$, $K = A_4 \times P$, where $C_2 = \langle (34) \rangle$, $P_1 = \langle y^2 \rangle$ and $A_4$ is the alternating group of degree 4. Then, $G = HK$ and $H \cap K$ is S-quasinormal in $K$ since $H \cap K \cong P_1$. Hence, $H$ is SS-supplemented in $G$. However, $H$ is neither C-supplemented nor SS-quasinormal in $G$.

We now give some basic properties of SS-supplemented subgroups.

Lemma 2.4. Let $G$ be a group and $H$ an SS-supplemented subgroup of $G$. Then, the following statements hold:

1. If $M$ is a subgroup of $G$ and $H \leq M$, then $H$ is SS-supplemented in $M$.
2. If $N$ is a normal subgroup of $G$ and $N \leq H$, then $H/N$ is SS-supplemented in $G/N$.
3. Let $\pi$ be a set of primes. If $H$ is a $\pi$-subgroup of $G$ and $N$ is a normal $\pi'$-subgroup of $G$, then $HN/N$ is SS-supplemented in $G/N$.
4. If $L$ is a subgroup of $G$ and $H \leq \Phi(L)$, then $H$ is S-quasinormal in $G$.

Proof. By the hypothesis, there exists $K \leq G$ such that $HK = G$ and $H \cap K$ is S-quasinormal in $K$. Let $K_1 = M \cap K$. Then, $M = HK_1$ and $H \cap K_1 = H \cap K$ is S-quasinormal in $K_1$. This shows that $H$ is SS-supplemented in $M$ and thus (1) is proved.

It follows from $G = HK$ that $H/N \cdot KN/N = G/N$. By using the well-known Dedekind identity, we have $H/N \cap KN/N = N(H \cap K)/N$. For any prime number $p$, it is well known that any Sylow $p$-subgroup of $KN/N$ has the form $K_pN/N$, where $K_p$ is a Sylow $p$-subgroup of $K$. Thus, $(H/N \cap KN/N)(K_pN/N)$ is a subgroup of $G/N$ since $(H \cap K)K_p$ is a subgroup of $G$. This implies that $H/N \cap KN/N$ is S-quasinormal in $KN/N$. Therefore, $H/N$ is SS-supplemented in $G/N$ and (2) is proved.

Since $(|N|, |H|) = 1$, $N \leq K$ and $NH \cap K = N(H \cap K)$. This shows that $NH \cap K$ is S-quasinormal in $K$, and hence, $NH$ is SS-supplemented in $G$. By (2), $HN/N$ is SS-supplemented in $G/N$ and (3) follows.

Since $H \leq \Phi(L)$, $L = H(L \cap K) = L \cap K$. It follows that $K = G$ and $H$ is S-quasinormal in $G$. Thus, (4) holds and the proof is completed.

The following lemmas are known results of S-quasinormal subgroups of a given group $G$.

Lemma 2.5. ([19]) Let $G$ be a group and $H \leq G$. If $H$ is S-quasinormal in $G$, then $H$ is subnormal in $G$.

Lemma 2.6. ([24]) If $H$ is a $p$-subgroup of a group $G$ for some prime $p$, then $H$ is S-quasinormal in $G$ if and only if $O^p(G) \leq N_G(H)$.
Lemma 2.7. ([8]) If $A$ is subnormal in $G$ and $B$ is a minimal normal subgroup of $G$, then $B \leq N_G(A)$.

Recall that a class $\mathcal{F}$ of groups is called a formation if $G \in \mathcal{F}$ and $N \unlhd G$ then $G/N \in \mathcal{F}$, and if $G/N_1 \in \mathcal{F}, i = 1, 2$, then $G/N_1 \cap N_2 \in \mathcal{F}$. In addition, if $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$, then we call $\mathcal{F}$ a saturated formation. A well-known example of saturated formations is the class $\mathcal{U}$ of supersolvable groups.

Concerning saturated formations, we have the following known results.

Lemma 2.8. ([25]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, let $G$ be a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If $H$ is cyclic, then $G \in \mathcal{F}$.

Lemma 2.9. ([26]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $G$ a group with a solvable normal subgroup $H$ such that $G/H \in \mathcal{F}$. If for every maximal subgroup $M$ of $G$, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$.

3. SS-supplemented subgroups of a group. A group $G$ is said to be SS-supplemented if every subgroup of $G$ is SS-supplemented in $G$. In this section, we first investigate the solvability of groups by using SS-supplemented subgroups and then the SS-supplemented group will hence be characterized.

Theorem 3.1. Let $G$ be a group. Then, $G$ is solvable if and only if every Sylow subgroup of $G$ is SS-supplemented in $G$.

Proof. If the given group $G$ is solvable, then every Sylow subgroup of $G$ is complemented and hence $G$ is SS-supplemented.

Conversely, we assume that every Sylow subgroup $P$ of $G$ is SS-supplemented in $G$. Then, by definition, there exists $K \leq G$ such that $PK = G$ and $P \cap K$ is S-quasinormal in $K$. By Lemma 2.5, $P \cap K$ is subnormal in $K$. Note that since $P \cap K$ is a Sylow subgroup of $K$, we can easily see that $P \cap K$ is also a normal Sylow subgroup of $K$. By applying the Schur–Zassenhaus theorem [9, Theorem 6.2.1], we have $K = (P \cap K)K_p'$, where $K_p'$ is a Hall $p'$-subgroup of $K$. Now $G = PK_p = PK_p'$ and $P \cap K_p' = 1$. Hence $P$ is complemented in $G$. The theorem is proved.

By using the same arguments as in Theorem 3.1, we deduce the following corollary.

Corollary 3.2. Let $G$ be a group and $H$ a Hall subgroup of $G$. Then $H$ is complemented in $G$ if and only if $H$ is SS-supplemented in $G$.

If we only assume that all maximal subgroups are SS-supplemented in a group $G$, then $G$ need not be solvable. In fact, $L_2(7), L_2(11)$ and $L_5(2)$ are nonabelian simple groups in which every maximal subgroup is complemented (see [20], main theorem). However, we have the following result.

Theorem 3.3. Let $G$ be a group. Then, $G$ is solvable if and only if every maximal subgroup of $G$ has a subnormal SS-supplement in $G$.

Proof. Let $G$ be a solvable group and $H$ a maximal subgroup of $G$. We now proceed to show that $H$ has a subnormal SS-supplement in $G$. Assume that $H_G \neq 1$. Consider $G/H_G$. By using induction on $|G|$, we know that $H/H_G$ has a subnormal SS-supplement $K/H_G$ in $G/H_G$. Clearly, $K$ is a subnormal SS-supplement of $H$ in $G$. Assume that $H_G = 1$. Let $N$ be a minimal normal subgroup of $G$. Then, $HN = G$ and $H \cap N \leq H_G = 1$. Hence, $N$ is a normal SS-supplement of $H$ in $G$. 

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Conversely, assume that the result is not true so that we can let $G$ be a counterexample of minimal order. Consider a maximal subgroup $H$ of $G$. Then there exists a subnormal subgroup $K$ of $G$ such that $HK = G$ and $H \cap K$ is $S$-quasinormal in $K$. If $G$ is a nonabelian simple group, then $K = G$ since $H \neq G$. By Lemma 2.5, we know that $H$ is subnormal in $G$ and hence $H = 1$. It follows that $G$ is solvable, which is a contradiction. Now, we let $N$ be a minimal normal subgroup of $G$. Then, it is easy to see that the hypothesis is still true for the quotient group $G/N$. By the minimality of $G$, we infer that $G/N$ is solvable. Furthermore, we may assume that $N$ is the unique minimal normal subgroup of $G$ and $N$ is not contained in $\Phi(G)$. Then, in this case, we can let $M$ be a maximal subgroup of $G$ with $M_G = 1$. By our hypothesis, there exists a subnormal subgroup $K$ of $G$ such that $MK = G$ and $M \cap K$ is $S$-quasinormal in $K$. Since $K$ is subnormal in $G$, Lemma 2.5 implies that $M \cap K$ is subnormal in $G$. Assume $M \cap K \neq 1$, then we may take a minimal subnormal subgroup $L$ of $G$ contained in $M \cap K$. Since $L \cap N \leq L$, either $L \cap N = 1$ or $L \cong N$. By Lemma 2.7, $N$ normalizes $L$. If $L \cap N = 1$, it follows that $N_L = N \times L$ and $L \leq C_G(N) = 1$. Suppose $L \leq N$, then $L^G = L^{NM} = L^M \leq M_G = 1$. We also get $L = 1$, a contradiction. Hence $M \cap K = 1$. By using the same arguments, we can similarly prove that all minimal subnormal subgroups of $G$ are contained in $N$. Let $N = N_1 \times \cdots \times N_r$, where each $N_i$ is isomorphic to a fixed nonabelian simple group. Then, it is easy to see that $N_1, \ldots, N_r$ coincide with all minimal subnormal subgroups of $G$. Without loss of generality, we may assume that $N_1 \leq K$. Then, there exists a prime $p$ such that $p$ divides $|G : M| = |K|$. By applying [3, Lemma 3, P.121], we obtain that $N$ is solvable, a contradiction. The proof is now completed.

The following corollary is a direct consequence of Theorem 3.3.

**Corollary 3.4.** ([19]) A group $G$ is solvable if and only if for every maximal subgroup $M$ of $G$, there exists a subnormal subgroup $K$ of $G$ such that $G = MK$ and $M \cap K \leq M_G$.

**Remark.** From the proof it can be noted that Theorem 3.3 is also valid if ‘subnormal’ is replaced by ‘normal’. The same is valid for Corollary 3.4.

If a group $G$ has a solvable maximal subgroup $M$ such that $M$ is $SS$-supplemented in $G$, then $G$ need not be solvable, for instance, $A_5$. However, we have the following result.

**Theorem 3.5.** Let $G$ be a group. Then, $G$ is solvable if and only if $G$ has a solvable maximal subgroup $H$ such that $H$ has a normal $SS$-supplement $K$ in $G$.

**Proof.** If $G$ is solvable, then $G$ has a normal maximal subgroup $H$. It is easy to see that $H$ has a normal $SS$-supplement $K$ in $G$, namely $G$. Conversely, assume that the theorem is not true. Then, we let $G$ be a counterexample of minimum order. If $H_G \neq 1$, then $H/H_G$ is a solvable maximal subgroup of $G/H_G$ and $KH_G/H_G$ is a normal $SS$-supplement of $H/H_G$ in $G/H_G$. The choice of $G$ implies that $G/H_G$ is solvable and therefore $G$ is solvable, a contradiction. Hence, $H_G = 1$. Let $N$ be a minimal normal subgroup of $G$ and $C = C_G(N)$. Then, it follows from [8, A, 17.2] that either $N$ is the unique minimal normal subgroup of $G$ and $C \leq N$ or $G$ has precisely two minimal normal subgroups $N$ and $R$ so that $N \rhd R$ is nonabelian, and hence, $R = C$ and $N \cap H = 1 = R \cap H$. By our hypotheses, we deduce that $H \cap K$ is $S$-quasinormal in $K$ and therefore, by Lemma 2.5, we know that $H \cap K$ is subnormal in $K$ and is hence in $G$. Now, assume that $H \cap K \neq 1$ and let $L$ be a minimal subnormal subgroup of $G$ contained in $H \cap K$. If $L \leq N$, then $L^G = L^{NH} = L^H \leq H_G = 1$, a contradiction.
This shows that \( L \) is not contained in \( N \) and \( L \) is analogously not contained in \( R \). It hence follows that \( N \cap L = 1 = R \cap L \). On the other hand, by Lemma 2.7, we have \( NL = N \times L \) and therefore \( L \leq C \), which contradicts \( C \leq N \) or \( C = R \). Hence, we conclude that \( H \cap K = 1 \). This implies that \( G = [K]H \) and \( K \) is a minimal normal subgroup of \( G \).

Now, we let \( T \) be a minimal normal subgroup of \( H \). Then, \( T \) is clearly an elementary abelian \( p \)-group for some \( p \in \pi(H) \). Since \( C_K(T) \) is normalized by both \( H \) and \( K \), we know that \( C_K(T) \leq G \). If \( C_K(T) = K \), then \( T \leq H_G \), a contradiction. Hence, \( C_K(T) = 1 \). It now follows from [9, Theorem 6.2.2] that \( K \) is a \( p' \)-group. By [9, Theorem 6.2.3], \( K \) contains a unique \( T \)-invariant Sylow \( q \)-subgroup \( Q \) for every prime \( q \in \pi(K) \). For any \( h \in H \), we have \((Q^h)^T = (Q^T)^h = Q^h \), that is, \( Q^h \) is also a \( T \)-invariant Sylow \( q \)-subgroup of \( K \), and thereby \( Q = Q^h \). Consequently, we have \([Q]H = G = [K]H \) and so \( K = Q \) is a \( q \)-group. This implies that \( G \) is a solvable group, a contradiction. Thus, the proof is completed.

We now characterize the \( SS \)-supplemented groups.

**Theorem 3.6.** Let \( G \) be a group. Then, the following statements are pairwise equivalent.

1. \( G \) is an \( SS \)-supplemented group.
2. \( G \) is supersolvable, every Sylow subgroup of \( G/\Phi(G) \) is elementary abelian and every subgroup of \( \Phi(G) \) is \( S \)-quasinormal in \( G \).
3. every subgroup of \( G/\Phi(G) \) is complemented and every subgroup of \( \Phi(G) \) is \( S \)-quasinormal in \( G \).

**Proof.** (1) \( \Rightarrow \) (2). We first prove that \( G \) is supersolvable. By the hypotheses and Theorem 3.1, \( G \) is solvable. Let \( N \) be a minimal normal subgroup of \( G \). Then, \( N \) is an elementary abelian \( p \)-group for some prime \( p \). By Lemma 2.4(2), it is known that \( G/N \) is \( SS \)-supplemented and hence \( G/N \) is supersolvable by induction. It follows that in order to prove that \( G \) is supersolvable, it suffices to prove that \( N = \langle x \rangle \) is cyclic. Let \( P \) be a Sylow \( p \)-subgroup of \( G \) and let \( x \in N \cap Z(P) \) with \( |x| = p \). Then, there exists \( K \leq G \) such that \( \langle x \rangle K = G \) and \( \langle x \rangle \cap K \) is \( S \)-quasinormal in \( K \). Since \( \langle x \rangle \cap K \) is normal by all \( p' \)-elements of \( K \) and centralized by \( P \), it follows that \( \langle x \rangle \cap K \) is a normal subgroup of \( G \). By minimality of \( N \), \( \langle x \rangle \cap K = 1 \) or \( N \leq K \). Assume that \( \langle x \rangle \cap K = 1 \). By order considerations, it follows that \( N = \langle x \rangle \). Assume now that \( N \leq K \). Then \( \langle x \rangle = \langle x \rangle \cap K \leq N \) and so \( N = \langle x \rangle \).

Let \( P \) be a Sylow \( p \)-subgroup of \( G \) and \( H \) is a subgroup of \( \Phi(P) \). Then by Lemma 2.4(4), \( H \) is \( S \)-quasinormal in \( G \). By Lemma 2.6, we deduce that \( \Phi(P) \) is normal in \( G \). Hence, \( \Phi(P) \leq \Phi(G) \) and, therefore every Sylow subgroup of \( G/\Phi(G) \) is elementary abelian. The last argument follows from Lemma 2.4(4).

(2) \( \Rightarrow \) (3). This part follows from [15, Theorem 2].

(3) \( \Rightarrow \) (1). Assume that every subgroup of \( G/\Phi(G) \) is complemented and every subgroup of \( \Phi(G) \) is \( S \)-quasinormal in \( G \). Let \( H \) be a subgroup of \( G \). Then, there exists a subgroup \( K/\Phi(G) \) of \( G/\Phi(G) \) such that \((H\Phi(G))/\Phi(G)(K/\Phi(G)) = G/\Phi(G) \) and \((H\Phi(G))/\Phi(G)) \cap (K/\Phi(G)) = (H \cap K)/\Phi(G)/\Phi(G) = 1 \). It follows that \( HK = G \) and \( H \cap K \leq \Phi(G) \). Hence, \( H \cap K \) is \( S \)-quasinormal in \( G \). By definition, \( H \) is \( SS \)-supplemented in \( G \) and hence \( G \) is an \( SS \)-supplemented group. The proof of theorem is now complete. \( \square \)
4. Applications. In this section, we concentrate on the structure of a group under the assumption that some subgroups of Sylow subgroups are SS-supplemented. Many known results will be generalized. In our first result, the $p$-nilpotency of a group is studied.

**Theorem 4.1.** Let $G$ be a group and let $p$ be the smallest prime divisor of $|G|$. Let $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is SS-supplemented in $G$, then $G$ is $p$-nilpotent.

**Proof.** Assume that the theorem is false and let $G$ be a counterexample of minimal order. Then, it follows from [16, IV, 2.8] that $P$ is not cyclic. Let $P_1$ be a maximal subgroup of $P$. Then, there exists $K \leq G$ such that $P_1K = G$ and $P_1 \cap K$ is $S$-quasinormal in $K$. It follows from Lemma 2.6 and $|P \cap K : P_1 \cap K| \leq p$ that $P_1 \cap K$ is normal in $K$. Applying [16, IV, 2.8] again, $K/P_1 \cap K$ is $p$-nilpotent with normal Hall $p'$-subgroup $H/P_1 \cap K$. Then, by the Schur-Zassenhaus theorem [9, Theorem 6.2.1], we know that $P_1 \cap K$ has a $p$-complement $M$ in $H$. By using the Frattini argument, we deduce that $K = HN_K(M) = (P_1 \cap K)N_K(M)$ and hence $G = P_1N_G(M)$. By the choice of $G$, it implies that $N_G(M) < G$ and $P \cap N_G(M) < P$. Now, choose a maximal subgroup $P_2$ of $P$ such that $P \cap N_G(M) \leq P_2$. By repeating the above argument once again, we can show that there also exists $K_1 \leq G$ such that $P_2K_1 = G$ and $P_2 \cap K_1$ is $S$-quasinormal in $K_1$ and $G = P_2N_G(M_1)$, where $M_1$ is a Hall $p'$-subgroup of $G$. If $p = 2$, then by applying the Gross theorem [10, main theorem], we obtain that $M_1^2 = M$ for some $g \in P$. If $p > 2$, then the odd order theorem implies the same conclusion. Therefore, $G = P_2N_G(M_1) = (P_2N_G(M_1))^g = P_2N_G(M)$. It follows that $P = P_2(P \cap N_G(M)) = P_2$, a contradiction. The proof is completed.

**Theorem 4.2.** Let $F$ be a saturated formation containing the class $\mathcal{U}$ of all supersoluble groups and $H$ a normal subgroup of a group $G$ such that $G/H \in F$. If all maximal subgroups of every non-cyclic Sylow subgroup of $H$ are SS-supplemented in $G$, then $G \in F$.

**Proof.** Let $p$ be the smallest prime divisor of $|H|$ and $P$ a Sylow $p$-subgroup of $H$. If $P$ is cyclic, then by [16, IV, 2.8], $H$ is $p$-nilpotent. If $P$ is non-cyclic, then by Lemma 2.4 (1) and Theorem 4.1, we deduce that $H$ is $p$-nilpotent. By using the same argument and induction, we may conclude that $H$ is a Sylow tower group.

Now, let $q$ be the largest prime dividing $|H|$ and $Q$ a Sylow $q$-subgroup of $H$. Then, $Q$ is normal in $G$. If $Q_1$ is a normal subgroup of $G$ with $1 \neq Q_1 \leq Q$, then, by Lemma 2.4 (2) or (3), $G/Q_1$ satisfies the hypotheses of the theorem and therefore we have $G/Q_1 \in F$, by induction. If $Q_1 \leq \Phi(G)$, then it follows from $G/Q_1 \in F$ that $G \in F$. Hence, in this case, we may assume that $Q$ is not contained in $\Phi(G)$ and $Q$ is a minimal normal subgroup of $G$. If $Q$ is not a cyclic group, then we let $\{N_1, \ldots, N_t\}$ be the set of all maximal subgroups of $Q$. For each $N_i$, by the hypotheses, there exists $K_i \leq G$ such that $N_iK_i = G$ and $N_i \cap K_i$ is $S$-quasinormal in $K_i$. Hence, we have $Q = N_i(Q \cap K_i)$ and $Q \cap K_i \leq G$. By the minimality of $Q$, we deduce that $Q \cap K_i = 1$ or $Q \leq K_i$. If $Q \cap K_i = 1$, then $Q = N_i$, a contradiction. Thus, $Q \leq K_i$ and so $N_i$ is $S$-quasinormal in $G$. Now, Lemma 2.6 implies that $|G : N_G(N_i)| = q^k$ for some nonnegative integer $k$. It hence follows from [16, III, 8.5(d)] that some maximal subgroup of $N$ is normal in $G$, which is a contradiction. This shows that $Q$ is a cyclic group of order $q$. By Lemma 2.8, we conclude that $G \in F$. The proof is completed.

The following corollary follows immediately from Theorem 4.2.
Corollary 4.3. Let $N$ be a normal subgroup of a group $G$ such that $G/N$ is supersolvable. If every maximal subgroup of every Sylow subgroup of $N$ is $c$-supplemented in $G$, then $G$ is supersolvable.

Theorem 4.4. Let $\mathcal{F}$ be a saturated formation containing the formation $\mathcal{U}$ of all supersoluble groups and $H$ a solvable normal subgroup of a group $G$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of every Sylow subgroup of the Fitting subgroup $F(H)$ of $H$ are SS-supplemented in $G$, then $G \in \mathcal{F}$.

Proof. Let $M$ be a maximal subgroup of $G$ not containing $F(H)$. Then, by Lemma 2.9, it suffices to prove that $F(H) \cap M$ is maximal in $F(H)$. To proceed with the proof, let $P$ be a Sylow $p$-subgroup of $F(H)$ not contained in $M$ and let $G_p$ be a Sylow $p$-subgroup of $G$. Then, $PM = G$ and $G_p \cap M < G_p$. Choose a maximal subgroup $G_1$ of $G_p$ such that $G_p \cap M \leq G_1$ and let $P_1 = G_1 \cap P$. Then, $P_1$ is a maximal subgroup of $P$ and $P_1 \cap M = P \cap M$. Now, we suppose that $P \cap \Phi(G) \neq 1$. Then, we can let $N$ be a minimal normal subgroup of $G$ contained in $P \cap \Phi(G)$. In this case, we have $F(H)/N = F(H)/N$ and $G/N$ satisfies the hypotheses. By using induction, we know that $G/N \in \mathcal{F}$ and therefore $G \in \mathcal{F}$. Hence, we may assume that $P \cap \Phi(G) = 1$ and therefore $\Phi(P) = 1$. Thus, $P \cap M \leq G$ and $P \cap M \leq (P_1)_G$. It hence follows that $(P_1)_G \cap M < G$ and so $P \cap M = (P_1)_G$. By the hypotheses, there exists $K_1 \leq G$ such that $P_1K_1 = G$ and $P_1 \cap K_1$ is $S$-quasinormal in $K_1$. If $Q$ is a Sylow $q$-subgroup of $K_1$ with $q \neq p$ then it is clear that $Q$ normalizes $P_1 \cap K_1$. On the other hand, since $PK_1 = G$ and $P$ is abelian, we have that $P \cap K_1$ is normal in $G$. It follows from $G_p = PG_1$ that $P_1 \cap K_1 = G_1 \cap P \cap K_1$ is normalized by $G_p$. Therefore, we have $P_1 \cap K_1 \leq G$ and $P_1 \cap K_1 \leq (P_1)_G$. Let $K = K_1(P_1)_G$. Then, $P_1 \cap K = (P_1)_G$. The maximality of $M$ implies that $(P \cap K)M = M$ or $(P \cap K)M = G$. If $(P \cap K)M = M$, then $P \cap K \leq P \cap M = (P_1)_G$ and therefore $P \cap K = (P_1)_G = P_1 \cap K$. It follows that $P_1 = P$, a contradiction. Hence, $(P \cap K)M = G$. It follows that $P \cap K = P$ by order considerations and so $P \leq K$. This proves that $P_1 = P_1 \cap K = (P_1)_G = P \cap M$. Consequently, $|F(H)/F(H) \cap M| = |P : P \cap M| = p$ and $F(H) \cap M$ is maximal in $F(H)$, as required. □

Corollary 4.5. ([12]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let $H$ be a solvable normal subgroup of a group $G$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of every Sylow subgroup of $F(H)$ are complemented in $G$, then $G \in \mathcal{F}$.

Now we want to delete the solvability of $H$ in the assumption of Theorem 4.4 by replacing $F(H)$ by $F^*(H)$, the generalized Fitting subgroup of $H$.

Theorem 4.6. Let $G$ be a group with a normal subgroup $H$ such that $G/H$ is supersolvable. If every maximal subgroup of every Sylow subgroup of $F^*(H)$ is SS-supplemented in $G$, then $G$ is supersolvable.

Proof. Suppose that the theorem is false and let $G$ be a counterexample of minimal order. Then, every proper normal subgroup of $G$ containing $F^*(H)$ is supersolvable. In fact, let $N$ be a proper normal subgroup of $G$ containing $F^*(H)$. Then, $N/N \cap H \cong NH/H$ is supersolvable. Since $F^*(H) = F^*(F^*(H)) \leq F^*(H \cap N) \leq F^*(H)$, we see $F^*(H \cap N) = F^*(H)$. Hence, every maximal subgroup of every Sylow subgroup of $F^*(H \cap N)$ is SS-supplemented in $G$ and therefore in $N$ by Lemma 2.4(1). So, $N$ with the normal subgroup $N \cap H$ satisfies the hypotheses of the theorem. The choice of $G$ implies that $N$ is supersolvable.
If \( H < G \), then \( H \) is supersolvable. In this case, \( F^*(H) = F(H) \). Theorem 4.4 implies that \( G \) is supersolvable, a contradiction. Thus, \( H = G \). If \( F^*(G) = G \), then \( G \) is supersolvable by Theorem 4.2 for the special case \( F = U \), a contradiction. Thus, \( F^*(G) < G \). By the above proof, \( F^*(G) \) is supersolvable and so \( F^*(G) = F(G) \).

Let \( P \) be a Sylow \( p \)-subgroup of \( F(G) \). Suppose that \( P \cap \Phi(G) \neq 1 \), and let \( N \) be a minimal normal subgroup of \( G \) contained in \( P \cap \Phi(G) \). Then, \( F(G)/N = F(G/N) \) and \( G/N \) satisfies the hypotheses. By the minimality of \( G \), \( G/N \) is supersolvable and so does \( G \). Hence, \( P \cap \Phi(G) = 1 \), and therefore \( \Phi(P) = 1 \) and \( P \) is abelian.

Let \( P_1 \) be a maximal subgroup of \( P \). Then, there exists \( K \leq G \) such that \( P_1 K = G \) and \( P_1 \cap K \) is \( S \)-quasinormal in \( K \). Thus, \( O^k(K) \leq N_G(P_1 \cap K) \) and so \( P_1 \cap K \leq PO^k(K) \). Obviously, \( F(G) \leq PO^k(K) \). Assume that \( PO^k(K) < G \). Then, \( PO^k(K) \) is supersolvable. Since \( PO^k(K) \leq PK = G \) and \( G/PO^k(K) \) is a \( p \)-group, \( G \) is solvable. By Theorem 4.4, \( G \) is supersolvable, a contradiction. Hence \( PO^k(K) = G \) and \( P_1 \cap K \leq G \). Therefore, \( P_1 \) is \( C \)-supplemented in \( G \). Now applying [27, Theorem 1.1], we get \( G \) is supersolvable, the final contradiction. The proof is hence completed.

**Theorem 4.7.** Let \( F \) be a saturated formation containing the class \( U \) of all supersoluble groups and let \( G \) be a group with a normal subgroup \( H \) such that \( G/H \) \( \in \) \( F \). If every maximal subgroup of every Sylow subgroup of \( F^*(H) \) is \( SS \)-supplemented in \( G \), then \( G \in \mathcal{F} \).

**Proof.** By Lemma 2.4(1), every maximal subgroup of every Sylow subgroup of \( F^*(H) \) is \( SS \)-supplemented in \( H \). Thus, \( H \) is supersolvable by Theorem 4.6. In particular, \( H \) is solvable and so \( F^*(H) = F(H) \). Now Theorem 4.4 implies that \( G \in \mathcal{F} \), as desired.

**Theorem 4.8.** Let \( G \) be a group and \( p \) the smallest prime divisor of \( |G| \). If every cyclic subgroup of \( G \) with order \( p \) and order \( 4 \) (if \( p = 2 \)) is \( SS \)-supplemented in \( G \), then \( G \) is \( p \)-nilpotent.

**Proof.** Assume that the theorem is false and let \( G \) be a counterexample of minimal order. Then, by Lemma 2.4(1), \( G \) is a minimal non-\( p \)-nilpotent group (that is, \( G \) is not \( p \)-nilpotent but every proper subgroup of \( G \) is \( p \)-nilpotent). Now by invoking a known result of Itô [16, III, 5.4], we know that \( G \) is a minimal non-nilpotent group. According to a result of Schmidt in [16, III, 5.2], \( G \) has a normal Sylow \( p \)-subgroup \( P \) such that \( G = PQ \) for a Sylow-\( q \)-subgroup \( Q \) \((q \neq p)\).

Let \( P_0 \leq P \) with order \( p \). Then, there exists \( K \leq G \) such that \( P_0 K = G \) and \( P_0 \cap K \) is \( S \)-quasinormal in \( K \). If \( P_0 \cap K = 1 \), then \( K \leq G \) and \( K \) is nilpotent. Thus, \( Q \leq G \), which is a contradiction. If \( P_0 \leq K \), then \( P_0 \) is \( S \)-quasinormal in \( G \) and therefore \( P_0 Q \) is a group. By the choice of \( G \), we have \( P_0 Q \leq G \) and hence \( P_0 Q = P_0 \times Q \). It follows that \( Q \) centralizes \( \Omega_1(P) \). If \( C_G(\Omega_1(P)) < G \), then \( C_G(\Omega_1(P)) \) is nilpotent and so \( Q \leq G \), again a contradiction. This leads to \( C_G(\Omega_1(P)) = G \) and \( \Omega_1(P) \leq Z(G) \). If \( \exp P = p \), then \( G \) is \( p \)-nilpotent, a contradiction. Thus, \( p = 2 \) and \( \exp P = 4 \). Let \( x \in P \) with \( |\langle x \rangle| = 4 \). Then, there exists \( T \leq G \) such that \( \langle x \rangle T = G \) and \( \langle x \rangle \cap T \) is \( S \)-quasinormal in \( T \). If \( |G:T| = 4 \), then \( \langle x^2 \rangle T \leq G \) and hence \( Q \leq G \), again a contradiction. In the case \( |G:T| = 2 \), we also have \( Q \leq G \), the same contradiction. Therefore \( T = G \) and \( \langle x \rangle = S \)-quasinormal in \( G \). By the choice of \( G \), we have \( \langle x \rangle Q < G \) and hence \( \langle x \rangle \) centralizes \( Q \). Thus, again we have \( Q \leq G \), a contradiction. The proof is hence completed.
Finally, we formulate another new theorem which also gives some other conditions for a finite group to be \( p \)-nilpotent.

**Theorem 4.9.** Let \( G \) be a group which is \( A_4 \)-free and let \( p \) be the smallest prime divisor of \(|G|\). If every subgroup of \( G \) having order \( p^2 \) is SS-supplemented in \( G \), then \( G \) is \( p \)-nilpotent.

**Proof.** Assume that the theorem is false and let \( G \) be a counterexample of minimal order. Let \( M \) be a maximal subgroup of \( G \). Assume \(|M|_p \leq p\). Then, by [16, IV, 2.8], \( M \) is \( p \)-nilpotent. If \(|M|_p > p\), then by Lemma 2.4 (1) and the choice of \( G \) we can deduce that \( M \) is \( p \)-nilpotent. Thus, \( G \) is a minimal non-\( p \)-nilpotent group, and consequently, \( G \) has a normal Sylow \( p \)-subgroup \( P \) such that \( G = PQ \), where \( Q \) is a Sylow \( q \)-subgroup of \( G \) with \( q \neq p \).

Let \( H \leq G \) with \(|H| = p^2\). Then, there exists \( K \leq G \) such that \( HK = G \) and \( H \cap K \) is \( S \)-quasinormal in \( K \). Without loss of generality, we may assume that \( Q \leq K \). Suppose \( H \cap K = 1 \), then \( K \) is nilpotent. Let \( K_p \) be a Sylow \( p \)-subgroup of \( K \) and \( P_1 \) is a maximal subgroup of \( P \) containing \( K_p \). Then, \( N_K(K_p) \) contains \( P_1 \) and \( Q \). It follows that \(|G : N_K(K_p)| \leq p\). If \(|G : N_K(K_p)| = p\), then \( N_K(K_p) \trianglelefteq G \). However, it follows that \( Q \) is normal in \( G \), a contradiction. Assume that \( K_p \trianglelefteq G \). We consider the group \( \bar{G} = G/K_p \). Clearly, \( \bar{G} \cong C_{\bar{G}}(\bar{P}) \) is isomorphic to a subgroup of \( \text{Aut}(\bar{P}) \) so that \( q | p^2 - 1 = (p - 1)(p + 1) \). This implies that \( p = 2 \) and \( q = 3 \). Hence, \( \bar{G} = \Phi(\bar{Q}) \) is isomorphic to \( A_4 \), a contradiction.

If \(|H \cap K| = p\), then \( K \trianglelefteq G \). Hence \( Q \trianglelefteq G \), again a contradiction.

Now, we have \( H \leq K \) and thereby \( H \) is \( S \)-quasinormal in \( G \). If \( HQ = G \), then \( P = H \) is not cyclic. Clearly, \( C_G(P) < G \). Now, \( G/C_G(P) \) is isomorphic to a subgroup of \( \text{Aut}(P) \) so that \( p = 2 \) and \( q = 3 \). Hence, \( G/\Phi(O) \) is isomorphic to \( A_4 \), which is a contradiction. Thus, \( HQ < G \) and \( HQ \) is nilpotent. It follows that \( P \) normalizes \( Q \), which is a contradiction. Thus the proof is completed. \( \square \)

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