TWO REMARKS ON $PQ^{\epsilon}$-PROJECTIVITY OF RIEMANNIAN METRICS

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Abstract. We show that $PQ^{\epsilon}$-projectivity of two Riemannian metrics introduced in [15] (P. J. Topalov, Geodesic compatibility and integrability of geodesic flows, J. Math. Phys. 44(2) (2003), 913–929.) implies affine equivalence of the metrics unless $\epsilon \in \{0, -1, -3, -5, -7, \ldots \}$. Moreover, we show that for $\epsilon = 0$, $PQ^{\epsilon}$-projectivity implies projective equivalence.

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1. Introduction.

1.1. $PQ^{\epsilon}$-projectivity of Riemannian metrics. Let $g$, $\bar{g}$ be two Riemannian metrics on an $m$-dimensional manifold $M$. Consider $(1, 1)$-tensors $P, Q$ that satisfy

\[ g(P \cdot, \cdot) = -g(\cdot, P \cdot), \quad g(Q \cdot, \cdot) = -g(\cdot, Q \cdot) \]
\[ \bar{g}(P \cdot, \cdot) = -\bar{g}(\cdot, P \cdot), \quad \bar{g}(Q \cdot, \cdot) = -\bar{g}(\cdot, Q \cdot) \]
\[ PQ = \epsilon Id, \tag{1} \]

where $Id$ is the identity on $TM$ and $\epsilon$ is a real number, $\epsilon \neq 1, m + 1$. The following definition was introduced in [15].

Definition 1. The metrics $g$, $\bar{g}$ are called $PQ^{\epsilon}$-projective if for a certain $1$-form $\Phi$ the Levi-Civita connections $\nabla$ and $\bar{\nabla}$ of $g$ and $\bar{g}$ satisfy

\[ \bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX \tag{2} \]

for all vector fields $X$, $Y$.

Example 1. If the two metrics $g$, $\bar{g}$ are affinely equivalent, i.e. $\nabla = \bar{\nabla}$, then these are $PQ^{\epsilon}$-projective with $P$, $Q$, $\epsilon$ arbitrary and $\Phi \equiv 0$.

Example 2. Suppose that $\Phi(P \cdot) = 0$ or $Q = 0$ and $\epsilon = 0$. It follows that equation (2) becomes

\[ \bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X. \tag{3} \]

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By Levi-Civita [4], equation (3) is equivalent to the condition that \( g \) and \( \bar{g} \) have the same geodesics considered as unparametrised curves, i.e. \( g \) and \( \bar{g} \) are \emph{projectively equivalent}. The theory of projectively equivalent metrics has a very long tradition in differential geometry, see for example [5, 6, 8, 10, 13] and the references therein.

**Example 3.** Suppose that \( P = Q = J \) and \( \epsilon = -1 \). It follows that \( J \) is an almost complex structure, i.e. \( J^2 = -1 \), and by equation (1) the metrics \( g \) and \( \bar{g} \) are required to be Hermitian with respect to \( J \). Equation (2) now reads

\[
\bar{\nabla}_X Y - \nabla_X Y = \Phi(X) Y + \Phi(Y) X - \Phi(JX) JY - \Phi(JY) JX.
\]

(4)

This equation defines the \emph{h-projective equivalence} of the Hermitian metrics \( g \) and \( \bar{g} \), and was introduced for the first time by Otsuki and Tashiro in [12, 14] for the Kaehlerian metrics. The theory of \( h \)-projectively equivalent metrics was introduced as an analog of projective geometry in the Kaehlerian situation and has been studied actively over the years, see for example [1–3, 7, 11] and the references therein.

**Remark 1.** \( PQ' \)-projectivity of the Riemannian metrics is a special case of the so-called \( F \)-planar mappings introduced and investigated in [9], whose defining equation, i.e. equation (1) in [9] clearly generalises equation (2) above.

1.2. Results. The aim of our paper is to give a proof of the following two theorems.

**Theorem 1.** Let Riemannian metrics \( g \) and \( \bar{g} \) be \( PQ' \)-projective. If \( g \) and \( \bar{g} \) are not affinely equivalent, the number \( \epsilon \) is either zero or an odd negative integer, i.e. \( \epsilon \in \{0, -1, -3, -5, -7, ...\} \).

**Theorem 2.** Let Riemannian metrics \( g \) and \( \bar{g} \) be \( PQ' \)-projective. If \( \epsilon = 0 \) then \( g \) and \( \bar{g} \) are projectively equivalent.

1.3. Motivation and open questions. As was shown in [15], \( PQ' \)-projectivity of the metrics \( g, \bar{g} \) allows us to construct a family of commuting integrals for the geodesic flow of \( g \) (see Fact 2 and equation (9)). The existence of these integrals is an interesting phenomenon on its own. Besides, it appeared to be a powerful tool in the study of projectively equivalent and \( h \)-projectively equivalent metrics (Examples 2 and 3), see [3, 5–8]. Moreover, it was shown in [15] that given one pair of \( PQ' \)-projective metrics, one can construct an infinite family of \( PQ' \)-projective metrics. Under some non-degeneracy condition, this gives rise to an infinite family of integrable flows.

From the other side, the theories of projectively equivalent and \( h \)-projectively equivalent metrics appeared to be very useful mathematical theories of deep interest.

The results in our paper suggest to look for other examples in the case when \( \epsilon = -1, -3, -5, ... \). If \( \epsilon = -1 \) but \( P^2 \neq -1 \), a lot of examples can be constructed using the ‘hierarchy construction’ from [15]. It is interesting to ask whether every pair of \( PQ^{-1} \)-projective metrics is in the hierarchy of some \( h \)-projectively equivalent metrics?

Another attractive problem is to find interesting examples for \( \epsilon = -3, -5, ... \). Besides the relation to integrable systems provided by [15], one could find other branches of differential geometry of similar interest as projective or \( h \)-projective geometry.
1.4. PDE for $PQ^\epsilon$-projectivity. Given a pair of Riemannian metrics $g, \bar{g}$ and tensors $P, Q$ satisfying equation (1), we introduce the $(1, 1)$-tensor $A = A(g, \bar{g})$ defined by

$$A = \left( \frac{\det \bar{g}}{\det g} \right)^{\frac{1}{m+1}} \bar{g}^{-1}. \tag{5}$$

Here we view the metrics as vector bundle isomorphisms $g : TM \to T^*M$ and $\bar{g}^{-1} : T^*M \to TM$. We see that $A$ is non-degenerate and self-adjoint with respect to $g$ and $\bar{g}$. Moreover, $A$ commutes with $P$ and $Q$.

**FACT 1.** (Lemma 2 in [15], see also Theorems 5 and 6 in [9]). Two metrics $g$ and $\bar{g}$ are $PQ^\epsilon$-projective if for a certain vector field $\Lambda$, the $(1, 1)$-tensor $A$ defined in (5) is a solution of

$$(\nabla_X A) Y = g(Y, X)\Lambda + g(Y, \Lambda) X + g(Y, QX) P\Lambda + g(Y, P\Lambda) QX \text{ for all } X, Y \in TM. \tag{6}$$

Conversely, if $A$ is a $g$-self-adjoint positive solution of (6), which commutes with $P$ and $Q$, the Riemannian metric

$$\bar{g} = (\det A)^{-\frac{1}{2}} g A^{-1}$$

is $PQ^\epsilon$-projective to $g$.

**REMARK 2.** Taking the trace of the $(1, 1)$-tensors in equation (6) acting on the vector field $Y$, we obtain

$$\Lambda = \frac{1}{2(1 - \epsilon)} \text{grad trace } A. \tag{7}$$

Hence, (6) is a linear first-order PDE on the $(1, 1)$-tensor $A$.

**REMARK 3.** From Fact 1 it follows that the metrics $g, \bar{g}$ are affinely equivalent if and only if $\Lambda \equiv 0$ on the whole $M$.

**REMARK 4.** Relation between the 1-form $\Phi$ in equation (2) and the vector field $\Lambda$ in equation (6) is given by $\Lambda = -A g^{-1} \Phi$ (again $g^{-1} : T^*M \to TM$ is considered as a bundle isomorphism), see [15]. Recall from Example 2 that projective equivalence is a special case of $PQ^\epsilon$-projectivity with $\Phi(P.) = 0$ or $Q = 0$ and $\epsilon = 0$. In view of Fact 1, we now have that $g$ and $\bar{g}$ are projectively equivalent if and only if $A = A(g, \bar{g})$ given by equation (5) (with $\epsilon = 0$), satisfies equation (6) with $P\Lambda = 0$ or $Q = 0$, i.e.

$$g(Y, X)\Lambda + g(Y, \Lambda) X \text{ for all } X, Y \in TM. \tag{8}$$

2. Proof of the results.

2.1. Topalov’s integrals. We first recall the following.

**FACT 2.** (Proposition 3 in [15]). Let $g$ and $\bar{g}$ be $PQ^\epsilon$-projective metrics and let $A$ be defined by (5). We identify $TM$ with $T^*M$ by $g$, and consider the canonical symplectic
structure on $TM \cong T^*M$. Then the functions $F_t : TM \to \mathbb{R}$,

$$F_t(X) = |\det (A - tId)|^{-\frac{1}{2}} g((A - tId)^{-1}X, X), \quad X \in TM$$

(9)

are commuting quadratic integrals for the geodesic flow of $g$.

**Remark 5.** Note that the function $F_t$ in equation (9) is not defined in the points $x \in M$ such that $t \in \text{spec} A|_x$. It will be clear from the proof of Theorem 1 that in the nontrivial case one can extend the functions $F_t$ to these points as well.

**2.2. Proof of Theorem 1.** Suppose that $g$ and $\bar{g}$ are $PQ^\epsilon$-projective Riemannian metrics, and let $A = A(g, \bar{g})$ be the corresponding solution of equation (6) defined by equation (5). Since $A$ is self-adjoint with respect to the positively definite metric $g$, the eigenvalues of $A$ in every point $x \in M$ are real numbers. We denote these by $\mu_1(x) \leq \cdots \leq \mu_m(x)$; depending on the multiplicity, some of the eigenvalues might coincide. The functions $\mu_i$ are continuous on $M$. Denote by $M^0 \subseteq M$ the set of points where the number of different eigenvalues of $A$ is maximal on $M$. Since the functions $\mu_i$ are continuous, $M^0$ is open in $M$. Moreover, it was shown in [15] that $M^0$ is dense in $M$ as well. The implicit function theorem now implies that $\mu_i$ are differentiable functions on $M^0$.

From Remark 3 and equation (7) we immediately obtain that $g$ and $\bar{g}$ are affinely equivalent if and only if all eigenvalues of $A$ are constant. Suppose that $g$ and $\bar{g}$ are not affinely equivalent, that is there is a non-constant eigenvalue $\rho$ of $A$ with multiplicity $k \geq 1$. Let us choose a point $x_0 \in M^0$ such that $d\rho|_{x_0} \neq 0$, define $c := \rho(x_0)$ and consider the hypersurface $H = \{x \in U : \rho(x) = c\}$, where $U \subseteq M^0$ is a geodesically convex neighbourhood of $x_0$. We think that $U$ is sufficiently small such that $\mu(x) \neq c$ for all eigenvalues $\mu$ of $A$ different from $\rho$ and all $x \in U$.

**Lemma 1.** There is a smooth nowhere vanishing $(0, 2)$-tensor $T$ on $U$ such that on $U \setminus H$, $T$ coincides with

$$\text{sgn}(\rho - c)|\det (A - cId)|^{\frac{1}{2}} g((A - cId)^{-1}X, X).$$

(10)

**Proof.** Let us denote by $\rho = \rho_1, \rho_2, \ldots, \rho_r$ different eigenvalues of $A$ on $M^0$ with multiplicities $k = k_1, k_2, \ldots, k_r$, respectively. Since the eigenspace distributions of $A$ are differentiable on $M^0$, we can choose a local frame $\{U_1, \ldots, U_m\}$ on $U$ such that $g$ and $A$ are given by matrices

$$g = \text{diag}(1, \ldots, 1) \quad \text{and} \quad A = \text{diag}(\rho, \ldots, \rho, \underbrace{\rho, \ldots, \rho}_k, \ldots, \underbrace{\rho, \ldots, \rho}_r)$$
Figure 1. Case $\frac{1}{1-\epsilon} - \frac{1}{k} > 0$: We connect the point $y \in U \setminus H$ with the points in $H$ by geodesics. The value of the integral $F_c$ is zero on each of these geodesics.

with respect to this frame. The tensor (10) can now be written as

$$\text{sgn}(\rho - c)\left|\det (A - c\text{Id})\right|^\frac{1}{2} g(A - c\text{Id})^{-1} =$$

$$(\rho - c) \prod_{i=2}^{r} |\rho_i - c|^{\frac{1}{k}} \text{diag}\left(\frac{1}{\rho - c}, \ldots, \frac{1}{\rho - c}, \ldots, \frac{1}{\rho - c}, \ldots, \frac{1}{\rho - c}ight)$$

$$= \prod_{i=2}^{r} |\rho_i - c|^{\frac{1}{k}} \text{diag}\left(1, \ldots, 1, \ldots, \frac{\rho - c}{\rho_c - c}, \ldots, \frac{\rho - c}{\rho_c - c}ight). \quad (11)$$

Since $\rho_i \neq c$ on $U \subseteq M^0$ for $i = 2, \ldots, r$, we see that (11) is a smooth nowhere vanishing $(0, 2)$-tensor on $U$.

**Lemma 2.** The multiplicity of the non-constant eigenvalues of $A$ is equal to $1 - \epsilon$.

*Proof.* Let us consider the integral $F_c : TM \to \mathbb{R}$ defined in equation (9). Using the tensor $T$ from Lemma 1, we can write $F_c$ as

$$F_c(X) = \frac{\text{sgn}(\rho - c)\left|\det (A - c\text{Id})\right|^\frac{1}{2}}{1 - \epsilon} T(X, X), \quad X \in TM. \quad (12)$$

Our goal is to show that $\frac{1}{1-\epsilon} - \frac{1}{k} = 0$.

First suppose that $\frac{1}{1-\epsilon} - \frac{1}{k} > 0$ and let $y \in U \setminus H$. We choose a geodesic $\gamma : [0, 1] \to U$ such that $y = \gamma(0)$ and $\gamma(1) \in H$, see Figure 1. Since $\rho(\gamma(t)) \overset{t \to 1}{\to} c$, we see from equation (12) that $f_c(\gamma(t)) \overset{t \to 1}{\to} 0$. It follows that $F_c(\dot{\gamma}(t)) \overset{t \to 1}{\to} 0$. On the other hand, since $F_c$ is an integral for the geodesic flow of $g$ (see Fact 2), the value $F_c(\dot{\gamma}(t))$ is independent of $t$, and hence $F_c(\dot{\gamma}(0)) = 0$. We have shown that $F_c(\dot{\gamma}(0)) = 0$ for all initial velocities $\dot{\gamma}(0) \in T_{\gamma}M$ of geodesics connecting $y$ with points of $H$. Since $H$ is a hypersurface, it follows that the quadric $\{X \in T_{\gamma}M : F_c(X) = 0\}$ contains an open subset that implies $F_c \equiv 0$ on $T_{\gamma}M$. This is a contradiction to Lemma 1, since $T$ is non-vanishing in $y$. We obtain that $\frac{1}{1-\epsilon} - \frac{1}{k} \leq 0$.

Let us now treat the case when $\frac{1}{1-\epsilon} - \frac{1}{k} < 0$. We choose a vector $X \in T_{x_0}M$ which is not tangent to $H$ and satisfies $T(X, X) \neq 0$. Such a vector exists, since $T_{x_0}M \setminus T_{x_0}H$ is open in $T_{x_0}M$ and $T$ is not identically zero on $T_{x_0}M$ by Lemma 1. Let us consider the geodesic $\gamma$ with $\gamma(0) = x_0$ and $\dot{\gamma}(0) = X$, see Figure 2. Since $X \notin T_{x_0}H$, the geodesic
metrics, it follows from a classical result that was already known to Levi-Civita that for projectively equivalent metrics, it is an arbitrary $X$ that has to leave another eigenvalue of $A$ and $g$ of least one eigenvalue of $A$ corresponding to the eigenvalue $\rho$.

Again this contradicts the fact that the value of $F_c$ must remain constant along $\gamma$ by Fact 2. We have shown that $1 / (1 - \epsilon) - 1 / \kappa = 0$, and finally Lemma 2 is proven.

As a consequence of Lemma 2, if the metrics $g, \bar{g}$ are not affinely equivalent (i.e. at least one eigenvalue of $A$ is non-constant), $\epsilon$ is an integer less or equal to zero. If $\epsilon \neq 0$, the condition $PQ = \epsilon \text{Id}$ in equation (1) implies that $P$ is non-degenerate and by the first condition in equation (1), $g(P, \ldots)$ is a non-degenerate 2-form on each eigenspace of $A$ (note that $A$ and $P$ commute). This implies that for $\epsilon \neq 0$ the eigenspaces of $A$ have even dimension, in particular, $1 - \epsilon \in \{2, 4, 6, 8, \ldots\}$. Theorem 1 is proven.

### 2.3. Proof of Theorem 2.

Let $g, \bar{g}$ be two $PQ^\epsilon$-projective metrics and let $A$ be the corresponding solution of equation (6) defined by equation (5). As it was already stated in the proof of Theorem 1, the eigenspace distributions of $A$ are differentiable in a neighbourhood of almost every point of $M$. First let us prove the following.

**Lemma 3.** Let $X$ be an eigenvector of $A$ corresponding to the eigenvalue $\rho$. If $\mu$ is another eigenvalue of $A$ and $\rho \neq \mu$, then $X(\mu) = 0$. In particular, grad $\mu$ is an eigenvector of $A$ corresponding to the eigenvalue $\mu$.

**Remark 6.** Lemma 3 is known for projectively equivalent (Example 2) and $h$-projectively equivalent (Example 3) metrics. For projectively equivalent metrics, it is a classical result that was already known to Levi-Civita [4]. For $h$-projectively equivalent metrics, it follows from [1, 7].

**Proof.** Let $Y$ be an eigenvector field of $A$ corresponding to the eigenvalue $\mu$. For arbitrary $X \in TM$, we obtain $\nabla_X(AY) = \nabla_X(\mu Y) = X(\mu)Y + \mu \nabla_X Y$ and $\nabla_X(AY) = (\nabla_X A)Y + A\nabla_X Y$. Combining these equations and replacing the expression $(\nabla_X A)Y$ by equation (6) we obtain

$$\begin{align*}
(A - \mu \text{Id})\nabla_X Y &= X(\mu)Y - g(Y, X)\Lambda - g(Y, \Lambda)X - g(Y, QX)\Lambda - g(Y, P\Lambda)QX.
\end{align*}$$

(13)

Now let $X$ be an eigenvector of $A$ corresponding to the eigenvalue $\rho$ and suppose that $\rho \neq \mu$. Since $A$ is $g$-self-adjoint, the eigenspaces of $A$ corresponding to different eigenvalues are orthogonal to each other. Moreover, since $A$ and $Q$ commute, $Q$ leaves
the eigenspaces of $A$ invariant. Using equation (13) we obtain

$$(A - \mu \text{Id})\nabla X Y + g(Y, \Lambda)X + g(Y, P\Lambda)QX = X(\mu)Y.$$  

Since the left-hand side is orthogonal to the $\mu$-eigenspace of $A$, we necessarily have $X(\mu) = 0$. We have shown that $g(\text{grad} \, \mu, X) = X(\mu) = 0$ for any eigenvalue $\mu$ and any eigenvector field $X$ corresponding to an eigenvalue which is different from $\mu$. This forces $\text{grad} \, \mu$ to be contained in the eigenspace of $A$ corresponding to $\mu$. □

Now suppose $\epsilon = 0$. Let us denote the non-constant eigenvalues of $A$ by $\rho_1, \ldots, \rho_l$. Using Lemma 2, the corresponding eigenspaces are 1-dimensional and Lemma 3 implies that these are spanned by the gradients $\text{grad} \, \rho_1, \ldots, \text{grad} \, \rho_l$ respectively. Since $P$ and $A$ commute, $P$ leaves the eigenspaces of $A$ invariant, hence $P\text{grad} \, \rho_i = p_i\text{grad} \, \rho_i$ for some real number $p_i$. Now $P$ is skew with respect to $g$ and we obtain $0 = g(\text{grad} \, \rho_i, P\text{grad} \, \rho_i) = p_i g(\text{grad} \, \rho_i, \text{grad} \, \rho_i)$, which implies that

$$P\text{grad} \, \rho_i = 0.$$  

On the other hand, by equation (7)

$$\Lambda = \frac{1}{2} \text{grad} \, \text{trace} \, A = \frac{1}{2} (\text{grad} \, \rho_1 + \ldots + \text{grad} \, \rho_l).$$  

Combining the last two equations, we obtain $P\Lambda = 0$. It follows from Remark 4 that $g$ and $\bar{g}$ are projectively equivalent and hence Theorem 2 is proved.

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