ON CERTAIN APPLICATIONS OF THE KHUKHRO–MAKARENKO THEOREM

AHMET ARIKAN
Gazi Üniversitesi, Gazi Eğitim Fakültesi, Matematik Eğitimi Anabilim Dalı 06500 Teknikokullar, Ankara, Turkey
e-mail: arikan@gazi.edu.tr

HOWARD SMITH
Department of Mathematics, Bucknell University, Lewisburg, PA 17837, USA
e-mail: howsmith@bucknell.edu

and NADIR TRABELSI
Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics,
University Ferhat Abbas of Setif, Algeria
e-mail: nadir_trabelsi@yahoo.fr

(Received 23 December 2011; revised 7 May 2012; accepted 9 May 2012; first published online 2 August 2012)

Abstract. Some recent results of Khukhro and Makarenko (see especially Lemma 2.1) establish that, for certain group-theoretic properties \( Y \), the existence of a \( Y \)-subgroup \( H \) of finite index in a group \( G \) ensures that there is a characteristic \( Y \)-subgroup \( C \) of finite index in \( G \). In the present paper we shall use these results to obtain generalisations of some well-known results on groups \( G \), in which all proper subgroups satisfy certain conditions, in several cases the condition in question being either ‘almost in the variety \( \mathcal{X}_\omega \)’ for some outer commutator word \( \omega \) (see for example Theorem 2.4) or ‘\( \mathcal{X}_\omega \)-by-Chernikov’ (see for example Theorem 2.5). We shall also obtain a generalisation of a result on barely transitive \( p \)-groups (see Theorem 2.3). Recall that a group

1. Introduction. Let \( F \) be a free group of countable rank with basis \( \{x_1, x_2, \ldots\} \). Then an outer commutator word of weight 1 is \( x_1 \), and an outer commutator word \( \omega \) of weight \( t > 1 \) is a word of the form

\[
\omega(x_1, \ldots, x_t) = [u(x_1, \ldots, x_r), v(x_{r+1}, \ldots, x_t)],
\]

where \( u, v \) are outer commutator words of weight \( r, t - r \) respectively. Let \( \omega \) be an outer commutator word of weight \( t \). We denote by \( \mathcal{X}_\omega \) the class of groups \( G \) satisfying \( \omega(g_1, \ldots, g_t) = 1 \) for all \( g_1, \ldots, g_t \in G \), i.e. \( \omega(G) = 1 \).

Some recent results of Khukhro and Makarenko (see especially Lemma 2.1) establish that, for certain group-theoretic properties \( \mathcal{Y} \), the existence of an \( \mathcal{Y} \)-subgroup \( H \) of finite index in a group \( G \) ensures that there is a characteristic \( \mathcal{Y} \)-subgroup \( C \) of finite index in \( G \). In the present paper we shall use these results to obtain generalisations of some well-known results on groups \( G \), in which all proper subgroups satisfy certain conditions, in several cases the condition in question being either ‘almost in the variety \( \mathcal{X}_\omega \)’ for some outer commutator word \( \omega \) (see for example Theorem 2.4) or ‘\( \mathcal{X}_\omega \)-by-Chernikov’ (see for example Theorem 2.5). We shall also obtain a generalisation of a result on barely transitive \( p \)-groups (see Theorem 2.3). Recall that a group
of permutations $G$ of an infinite set $\Omega$ is called a barely transitive group if $G$ acts transitively on $\Omega$ and every orbit of every proper subgroup is finite. Equivalently, $G$ is barely transitive if $G$ has a subgroup $H$ such that $|G:H|$ is infinite, $\bigcap_{g \in G} H^g = 1$ and $|K:K \cap H|$ is finite for every proper subgroup $K$ of $G$, where the subgroup $H$ is called a point stabiliser. Finally, in Section 4 of the paper, we obtain some partial generalisations of the Khukhro–Makarenko results.

We shall use the following notation for the given classes of groups.

\begin{itemize}
  \item $\mathfrak{A}$: Abelian groups,
  \item $\mathfrak{N}$: Nilpotent groups,
  \item $\mathfrak{S}$: Soluble groups,
  \item $\mathfrak{S}_d$: Soluble groups of derived length at most $d$,
  \item $\mathfrak{C}$: Chernikov groups,
  \item $\mathfrak{F}$: Groups of finite (Prüfer) rank,
  \item $\mathfrak{D}$: Divisible (radicable) groups,
  \item $\mathfrak{T}$: Periodic groups,
  \item $\mathfrak{L}$: $(\mathfrak{F} \cap \mathfrak{D} \cap \mathfrak{A})$-groups.
\end{itemize}

We also denote the class of all $\mathfrak{X}$-by-$\mathfrak{Y}$-groups by $\mathfrak{X}\mathfrak{Y}$, and $\mathfrak{X}\mathfrak{X}$-groups by $\mathfrak{X}^2$.

### 2. $\mathfrak{X}_\omega\mathfrak{C}$-groups

We will use the following very useful result, referred to here as the Khukhro–Makarenko theorem.

**Lemma 2.1** ([9, Theorem 1], [11, Theorem 1] or [13]). If a group $G$ has a subgroup $H$ of finite index $n$ satisfying the identity $\chi(H) = 1$, where $\chi$ is an outer commutator word of weight $w$, then $G$ has also a characteristic subgroup $C$ of finite $(n, w)$-bounded index satisfying the same identity $\chi(C) = 1$.

Before we give an application of Lemma 2.1, we prove the following lemma.

**Lemma 2.2.** Let $G$ be a group and let $\omega$ be an outer commutator word of weight $t \geq 2$; then $G^{(t-1)} \leq \omega(G)$. In particular,

1. if $\omega(G) = 1$, then $G$ is in $\mathfrak{S}_{t-1}$, i.e. $\mathfrak{X}_\omega \leq \mathfrak{S}_{t-1}$,
2. if $G$ is a perfect group, then $\omega(G) = G$.

**Proof.** We proceed by induction on $t$. If $t = 2$, then $G^{(t-1)} = G^{(1)} = G' = \omega(G)$. Now assume that $t \geq 3$; then there exist outer commutator words $\sigma$, $\tau$ of weight $1 \leq t_1, t_2 < t$, respectively, such that $t = t_1 + t_2$ and $\omega = [\sigma, \tau]$, and then $\omega(G) = [\sigma(G), \tau(G)]$. By induction hypothesis, we have $G^{(t_1-1)} \leq \sigma(G)$ and $G^{(t_2-1)} \leq \tau(G)$. Put $m = \max\{t_1, t_2\}$, then

$$G^{(m)} = [G^{(m-1)}, G^{(m-1)}] \leq [G^{(t_1-1)}, G^{(t_2-1)}] \leq [\sigma(G), \tau(G)] = \omega(G).$$

Clearly $t_1 + t_2 \geq m + 1$ and thus $t - 1 \geq m$. So $G^{(t-1)} \leq G^{(m)} \leq \omega(G)$ and the induction is complete.

1. If $\omega(G) = 1$, then $G^{(t-1)} = 1$. So $G$ is in $\mathfrak{S}_{t-1}$.
2. Assume that $G$ is a perfect group. Since $G^{(t-1)} \leq \omega(G)$, we have $G^{(t-1)} = G$, and hence $G = \omega(G)$, as desired.

As an application of the Khukhro–Makarenko theorem we present the following result.
ON CERTAIN APPLICATIONS OF THE KHUKHRO–MAKARENKO THEOREM 277

Theorem 2.3. Let $G$ be a locally finite barely transitive $p$-group with a point stabiliser $H$ and let $\omega$ be an outer commutator word of weight $t$. If $H \in \mathcal{X}_\omega$, then $G' \neq G$ and $G' \in \mathcal{X}_\omega$.

Proof. Let $N$ be a proper normal subgroup of $G$; then $N \cap H \in \mathcal{X}_\omega$. Since $|N : N \cap H|$ is finite, by Lemma 2.1, $N$ has a characteristic subgroup $K \in \mathcal{X}_\omega$ such that $N/K \in \mathcal{F}$. It is well known that $G$ has no proper subgroup of finite index, so $N/K \leq Z(G/K)$. It follows that $N' \leq K$ and that $N' \in \mathcal{X}_\omega$. Since there exists a chain $\{N_i : i \in I\}$ of proper normal subgroups of $G$ such that $G = \bigcup_{i \in I} N_i$, it follows that

$$G' = \bigcup_{i \in I} N'_i.$$ 

Consequently, we have $G' \in \mathcal{X}_\omega$ and $G \neq G'$ by Lemma 2.2(ii). \hfill \Box

Theorem 2.3 generalises [1] and [2, Theorem 2], and by using Lemma 2.2(i) we can obtain the same results as those in [1] and [2, Theorem 2]. The structure of imperfect locally finite barely transitive groups is described in [7].

Let $v(x_1, \ldots, x_s)$ and $u(x_1, \ldots, x_t)$ be two words. Then the composite of $v$ and $u$, $v \circ u$ is defined as follows:

$$v \circ u = v(u(x_1, \ldots, x_t), \ldots, u(x_{(s-1)t+1}, \ldots, x_{st})).$$

If $v$ is an outer commutator word and $u$ is a word, then it is well known that $v \circ u(G) = v(u(G))$ for any group $G$ (see for example [16, Lemma 2.5]).

We will use this definition to describe the structure of certain groups.

Let $\mathcal{A}$ be a class of groups. Recall that a group $G$ is called a minimal non-$\mathcal{A}$-group if every proper subgroup of $G$ is a $\mathcal{A}$-group, but $G$ itself is not. The minimal non-$\mathcal{A}$-groups are denoted by $MN\mathcal{A}$.

Now define the word $\theta$ as $\theta(x, y) = [x, y]$, which will be used in the sequel.

Theorem 2.4. Let $G$ be an $MN\mathcal{X}_\omega\mathcal{F}$-group, where $\omega$ is an outer commutator word of weight $t > 1$. If $G$ has no infinite simple images, then the following properties hold.

(i) $G$ has no proper subgroup of finite index and no simple images.

(ii) $N' \in \mathcal{X}_\omega$ for every proper normal subgroup $N$ of $G$.

(iii) $G$ is not perfect, $G \in \mathcal{X}_\omega(\mathcal{L} \cap \mathcal{E})$ and $G' \in \mathcal{X}_\omega$. In particular, $G \in \mathcal{S}_t$.

(iv) $(\omega \circ \theta)(G) = 1$, i.e. $G \in \mathcal{X}_{\omega\theta\omega}$.

Proof. We first assume that $G$ has a proper subgroup $K$ of finite index. Since $K \in \mathcal{X}_\omega\mathcal{F}$, $K$ has a normal subgroup $L \in \mathcal{X}_\omega$ such that $K/L \in \mathcal{F}$. Hence, $\text{core}_G L \in \mathcal{X}_\omega$ and has finite index in $G$, i.e. $G \in \mathcal{X}_\omega\mathcal{F}$. But this is a contradiction. So $G$ has no proper subgroup of finite index and it has no simple images. Thus (i) holds.

Now let $N$ be a proper normal subgroup of $G$. Since $N \in \mathcal{X}_\omega\mathcal{F}$, $N$ has a characteristic subgroup $S \in \mathcal{X}_\omega$ of finite index in $N$ by Lemma 2.1. Put $\overline{G} := G/S$, then $\overline{G} = C_{\overline{G}}(N)$, since $G/S$ has no proper subgroup of finite index and so we have $[G, N] \leq S$. Since $\mathcal{X}_\omega$ is subgroup-closed, $N' \in \mathcal{X}_\omega$, and thus (ii) holds.

Now assume that $G$ is perfect. Since $G$ has no simple images, it is a union of a chain of proper normal subgroups. If $N$ is a proper normal subgroup of $G$, then $N' \in \mathcal{X}_\omega$ by (ii) and so $G = G'$ is a union of $\mathcal{X}_\omega$-groups. So $G \in \mathcal{X}_\omega$, a contradiction.

Thus, $G$ is not perfect and $G/G'$ has a proper subgroup $R/G'$ such that $G/R \in \mathcal{L} \cap \mathcal{E}$. Now by Lemma 2.1, $R$ has a characteristic subgroup $W \in \mathcal{X}_\omega$ such that $G/W \in \mathcal{E}$. Since $G/W$ has no proper subgroup of finite index, we have $G/W \in \mathcal{L} \cap \mathcal{E}$. 

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 04 Oct 2019 at 00:19:38, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089512000493
Consequently, \( G \in \mathcal{X}_\omega(\mathcal{L} \cap \mathcal{C}) \). In particular, \( G' \leq W \) and hence \( G' \in \mathcal{X}_\omega \). In particular, \( G \in \mathcal{S}_r \) by Lemma 2.2(i). So (iii) holds.

Finally, since \( G' \in \mathcal{X}_\omega \), we have \((\omega \circ \theta)(G) = \omega(G') = 1\), and (iv) holds. \( \square \)

The following is the \( \mathcal{X}_\omega \mathcal{C} \) version of Theorem 2.4.

**Theorem 2.5.** Let \( G \) be an \( MN \mathcal{X}_\omega \mathcal{C} \)-group. If \( G \) has no infinite simple images, then the following are satisfied.

(i) \( G \) has no proper subgroup of finite index and no simple images.

(ii) \( N' \in \mathcal{X}_{\omega t} \) for every proper normal subgroup \( N \) of \( G \), i.e. \( N \in \mathcal{X}_{\omega t} \).

(iii) \( G \) is not perfect and \( G \in \mathcal{X}_{\omega t}(\mathcal{L} \cap \mathcal{C}) \). In particular, \( G' \in \mathcal{X}_{\omega t} \) and \( G \in \mathcal{S}_{r+1} \).

**Proof.** By a similar argument to that used in the proof of Theorem 2.4, \( G \) has no proper subgroup of finite index. So (i) holds.

Now let \( N \) be a proper normal subgroup of \( G \), then it has a normal subgroup \( S \in \mathcal{X}_\omega \) such that \( N/S \in \mathcal{C} \). So \( N/S \) has a normal subgroup \( R/S \in \mathcal{L} \cap \mathcal{C} \) such that \( N/R \in \mathcal{C} \). Since \( R/S \) is in \( \mathcal{A} \), \( R \in \mathcal{X}_{\omega t} \). By Lemma 2.1 \( N \) has a characteristic subgroup \( M \in \mathcal{X}_{\omega t} \) such that \( N/M \in \mathcal{C} \) and hence \( N' \leq M \), i.e. \( N' \in \mathcal{X}_{\omega t} \). So (ii) holds.

Suppose next that \( G \) has a non-trivial \( \mathcal{C} \)-image \( G/N \). Then \( N \) has a normal subgroup \( S \in \mathcal{X}_\omega \) such that \( N/S \in \mathcal{C} \) and \( N/S \) has a normal subgroup \( M/S \in \mathcal{L} \cap \mathcal{C} \) such that \( N/M \in \mathcal{C} \). So \( N \in \mathcal{X}_{\omega t} \). By Lemma 2.1 \( N \) has a characteristic subgroup \( T \in \mathcal{X}_{\omega t} \) with \( N/T \in \mathcal{C} \). This implies that \( G/T \in \mathcal{C} \) and hence \( G/T \in \mathcal{L} \cap \mathcal{C} \) by (i) and \( G \in \mathcal{X}_{\omega t}(\mathcal{L} \cap \mathcal{C}) \) in this case.

Now if \( G \) is perfect, then as in the proof of Theorem 2.4, \( G \) is a union of proper normal subgroups and so we have \( G \in \mathcal{X}_{\omega t} \), and hence \( \omega(G) = 1 \), a contradiction. So \( G \) is not perfect and \( G/G' \) has a proper normal subgroup \( R/G' \) such that \( G/R \in \mathcal{L} \cap \mathcal{C} \). By the previous argument \( G \in \mathcal{X}_{\omega t}(\mathcal{L} \cap \mathcal{C}) \) and so \( G' \in \mathcal{X}_{\omega t} \) and \( G \in \mathcal{S}_{r+1} \) by Lemma 2.2(i). Thus, (iii) holds. \( \square \)

### 3. Applications to \( MN \mathcal{S}_n \mathcal{C} \) and \( MN \mathcal{S}_n \mathcal{F} \)-groups.

Since a group is in \( \mathcal{S} \mathcal{C} \) if and only if it is in \( \mathcal{S} \mathcal{F} \), we see that a group is in \( MN \mathcal{S} \mathcal{C} \) if and only if it is in \( MN \mathcal{S} \mathcal{F} \).

We know that the celebrated example of Heineken and Mohamed (see [15, Theorem 6.2.1]) is an \( MN \mathcal{S} \mathcal{F} \)-group which is in \( \mathcal{A} \mathcal{C} \). So an \( MN \mathcal{S}_n \mathcal{F} \)-group (for a positive integer \( n \)) is not in general an \( MN \mathcal{S}_n \mathcal{C} \)-group.

The locally graded groups with all proper subgroups in \( \mathcal{S} \mathcal{F} \) are classified by [6, Theorem C], as follows.

**Theorem 3.1.** Let \( G \) be a locally graded group with all proper subgroups in \( \mathcal{S} \mathcal{F} \). Then either

(i) \( G \) is locally soluble, or

(ii) \( G \in \mathcal{S} \mathcal{F} \), or

(iii) \( G \) is \( \mathcal{S} \)-by-\( PSL(2, \mathcal{F}) \), or

(iv) \( G \) is \( \mathcal{S} \)-by-\( S(\mathcal{F}) \).

where \( F \) is an infinite locally finite field with no infinite proper subfields.

By the remark above, we see that Theorem 3.1 also gives a classification of the locally graded groups with all proper subgroups in \( \mathcal{C} \).

If \( G \) is a countable locally graded simple group with all subgroups in \( \mathcal{S} \mathcal{F} \) (or in \( \mathcal{S} \mathcal{C} \)), then a super-inert subgroup \( R \) (see [6] for the definition) of \( G \) either has non-trivial Hirsch–Plokin radical or is in \( \mathcal{F} \), hence \( G \) is locally finite [6, Theorem 2]. So by [12]
ON CERTAIN APPLICATIONS OF THE KHUKHRO–MAKARENKO THEOREM

279

G is isomorphic either to $\text{PSL}(2, \mathbb{F})$ or to $S\mathbb{F}$ for some infinite locally finite field $\mathbb{F}$ containing no infinite proper subfield.

THEOREM 3.2. There are infinite locally finite simple $MN\mathfrak{S}_2\mathfrak{F}$ and $MN\mathfrak{S}_3\mathfrak{F}$-groups.

Proof. Let $G := \text{PSL}(2, \mathbb{F})$ or $G := S\mathbb{F}$ for some infinite locally finite field $\mathbb{F}$ containing no infinite proper subfield. In the first case every proper subgroup is either in $\mathfrak{A}^2$ or in $\mathfrak{F}$ and so in $\mathfrak{S}_2\mathfrak{F}$ by [4, Example 3]. Clearly $G \notin \mathfrak{S}_2\mathfrak{S}$. So $G \in MN\mathfrak{S}_2\mathfrak{F}$.

In the second case every proper subgroup of $G$ is in $\mathfrak{S}$ or is $\mathfrak{A}$-by-locally cyclic (i.e. in $\mathfrak{S}_3\mathfrak{S}$) by the proof of [5, Lemma 2]. Consequently, $G$ is in $MN\mathfrak{S}_3\mathfrak{F}$. □

Let $G$ be a group, $H$ a subgroup of $G$; then the isolator $IG(H)$ of $H$ in $G$ is defined as

$$IG(H) = \{x \in G \mid \text{there is a non-zero integer } n \text{ such that } x^n \in H\}.$$  

We prove the following general lemma.

LEMMA 3.3. Let $\omega$ be an outer commutator word of weight $t$, and let $H$ be a subgroup of the locally nilpotent torsion-free group $G$. Then

$$\omega(IG(H)) \leq IG(\omega(H)).$$

Proof. First let $U$ and $V$ be subgroups of $G$. Then with the notation of [14, Section 2.3] we have $IG(U) \sim U$ and $IG(V) \sim V$. It follows that $[IG(U), IG(V)] \sim [U, V]$ by [14, 2.3.5]. So we see that

$$[IG(U), IG(V)] \leq IG([U, V]).$$

Now we proceed by induction on $t$. If $t = 1$, then the result is immediate. If $t > 1$, then $\omega = [\varphi, \delta]$ for some outer commutator words $\varphi$ and $\delta$ of weights $1 \leq t_1 < t$, $1 \leq t_2 < t$ such that $t_1 + t_2 = t$. By induction hypothesis and the above remark, we have

$$\omega(IG(H)) = [\varphi(IG(H)), \delta(IG(H))] \leq [IG(\varphi(H)), IG(\delta(H))] \leq IG([\varphi(H), \delta(H)]) = IG(\omega(H)),$$

and the proof is complete. □

THEOREM 3.4. Let $G$ be a locally nilpotent torsion-free group.

(i) If all proper subgroups of $G$ are in $\mathfrak{X}_\omega \mathfrak{S}$, then $G \in \mathfrak{X}_\omega$.

(ii) If all proper subgroups of $G$ are in $\mathfrak{X}_\omega \mathfrak{A}$, then $G \in \mathfrak{X}_\omega (\mathfrak{A} \cap \mathfrak{M})$. In particular, $G$ is in $\mathfrak{S}$.

Proof. (i) Let $K$ be a proper subgroup of $G$. Then $K$ has a normal subgroup $N \in \mathfrak{X}_\omega$ such that $K/N \in \mathfrak{S}$, and so $IK(N) = K$. By Lemma 3.3

$$\omega(K) = \omega(IK(N)) \leq IK(\omega(N)) = IK(1) = 1.$$

This means that every proper subgroup $K$ of $G$ is in $\mathfrak{X}_\omega$.

If $G$ is not finitely generated, then every finitely generated subgroup of $G$ is in $\mathfrak{X}_\omega$, and thus $G \in \mathfrak{X}_\omega$. Otherwise, $G$ is finitely generated, and by [18, 5.2.21] it has a normal
subgroup \( H \in \mathcal{X}_\omega \) of finite index, since it is nilpotent. Hence, \( I_G(H) = G \) and as above \( \omega(G) = 1 \). So \( G \in \mathcal{X}_\omega \).

(ii) Assume for a contradiction that \( G \) is not in \( \mathcal{X}_\omega \alpha \), and first suppose that \( G \) has a proper normal subgroup of \( N \) such that \( N/M \in \alpha \). Then \( N \) has a normal subgroup \( M \) such that \( M \in \mathcal{X}_\omega \) and \( N/M \in \alpha \). By [10, Theorem 3], we may assume that \( M \) is characteristic in \( N \) so that \( M \) is normal in \( G \). Clearly \( G/M \) is in \( \alpha \), so \( G \in \mathcal{X}_\omega \alpha \), a contradiction. Therefore, \( G \) has no proper images which are in \( \alpha \) and hence it is perfect.

Let \( H \) be any proper normal subgroup of \( G \) and let \( K \) be a characteristic subgroup of \( H \) such that \( K \in \mathcal{X}_\omega \) and \( H/K \in \alpha \). If \( T/K \) denotes the torsion subgroup of \( H/K \), then \( T \in \mathcal{X}_\omega \mathcal{X} \) and hence \( T \in \mathcal{X}_\omega \) by (i). Since torsion-free locally nilpotent \( \alpha \)-groups are in \( \mathcal{X} \) [17, Theorem 6.36], we have \( H/T \in \alpha \). So \( H/T \) has a finite characteristic series whose factors are torsion-free \( \mathfrak{A} \cap \alpha \)-groups. If \( U \) is such a factor, then \( G/C_G(U) \) is nilpotent by [17, Part 2, Lemma 6.37] and hence \( G = C_G(U) \) since \( G \) is perfect. We deduce that \( H/T \) is contained in the hypercentre of \( G/T \), which equals the centre, as \( G \) is perfect. Thus, \( H/T \leq Z(G/T) \) and so \( H' \leq T \). We deduce that \( H' \in \mathcal{X}_\omega \). As before, since there exists a chain \( \{ N_i : i \in I \} \) of proper normal subgroups of \( G \) such that \( G = \bigcup_{i \in I} N_i \), it follows that \( G = G' = \bigcup_{i \in I} N_i' \). Consequently, we have \( G \in \mathcal{X}_\omega \), a contradiction. Therefore, \( G \in \mathcal{X}_\omega \alpha \).

Let \( N \) be a normal subgroup of \( G \) such that \( N \in \mathcal{X}_\omega \) and \( G/N \in \alpha \). If \( T/N \) denotes the torsion subgroup of \( G/N \), then again by (i) \( T \in \mathcal{X}_\omega \), and since \( G/T \) is a locally nilpotent torsion-free \( \alpha \)-group, it is in \( \alpha \). Therefore, \( G \in \mathcal{X}_\omega (\alpha \cap \alpha) \). By Lemma 2.2(i), we deduce that \( G \) is in \( \mathfrak{S} \), as claimed. \( \square \)

Let us define the outer commutator word \( \phi_j \) for every \( j \geq 0 \) as follows:
\[
\phi_0(x) = x \quad \text{and for } i \geq 1
\]
\[
\phi_i(x_1, \ldots, x_2^i) = [\phi_{i-1}(x_1, \ldots, x_2^{i-1}), \phi_{i-1}(x_2^{i-1}+1, \ldots, x_2^i)].
\]
Then \( G \) is in \( \mathfrak{S} \) if and only if there is a positive integer \( n \) such that \( \phi_n(G) = 1 \).

**Theorem 3.5.** Let \( G \) be a group without infinite simple images. Then the following are satisfied.

(i) If every proper subgroup of \( G \) is in \( \mathfrak{S}_n \mathfrak{S} \) for some fixed positive integer \( n \), then either \( G \in \mathfrak{S}_n \mathfrak{S}\) or \( G \in \mathfrak{S}_n (\mathfrak{L} \cap \mathfrak{C}) \). So if \( G \) is an \( M \mathfrak{S}_n \mathfrak{S}\)-group, then \( G \in \mathfrak{S}_n \mathfrak{C} \cap \mathfrak{S}_n+1 \).

(ii) If every proper subgroup of \( G \) is in \( \mathfrak{S}_n \mathfrak{C} \) for some fixed positive integer \( n \), then \( G \in \mathfrak{S}_n \mathfrak{L} \) or \( G \in \mathfrak{S}_{n+1} (\mathfrak{L} \cap \mathfrak{C}) \). So if \( G \) is an \( M \mathfrak{S}_n \mathfrak{C}\)-group, then \( G \in \mathfrak{S}_{n+1} \mathfrak{C} \cap \mathfrak{S}_{n+2} \).

**Proof.** (i) Take \( \omega = \phi_n \). If \( G \notin \mathfrak{S}_n \mathfrak{S} \), then \( G \) is an \( M \mathfrak{S}_n \mathfrak{S}\)-group. By Theorem 2.4 (iii) \( G \in \mathfrak{X}_{\phi_n} (\mathfrak{L} \cap \mathfrak{C}) \).

(ii) Again take \( \omega = \phi_n \) so that \( \omega \circ \theta = \phi_{n+1} \). If \( G \notin \mathfrak{S}_n \mathfrak{C} \), then by Theorem 2.5 (iii) \( G \in \mathfrak{X}_{\phi_{n+1}} (\mathfrak{L} \cap \mathfrak{C}) \), and the proof is complete. \( \square \)

The following lemma will be generalised in Section 4 (see Lemma 4.1), but since the ‘soluble’ version of the lemma is useful here, we shall prove it.

**Lemma 3.6 (c.f. [3, Proposition 1]).** Let \( G \) be in \( \mathfrak{L} \cap \mathfrak{M} \), \( A \in \mathfrak{G}_n (n \geq 1) \) a normal subgroup of \( G \) such that \( G/A \in \mathfrak{L} \). Then also \( G \in \mathfrak{G}_n \).

**Proof.** We proceed by induction on \( n \). If \( n = 1 \), then \( A \) is in \( \mathfrak{A} \) and hence \( A \leq C_G(A) \). So \( T := G/C_G(A) \in \mathfrak{L} \) is isomorphic to a subgroup of \( \text{Aut } A \). By [3, Lemma 1], \( A \leq Z(G) \), and by [15, Section 5.3.5] \( G \) is in \( \mathfrak{A} \). Now let \( n > 1 \) and consider \( G/A^{(n-1)} \).
Then
\[
\frac{G/A(n-1)}{A/A(n-1)} \in \mathcal{L} \text{ and } A/A(n-1) \in \mathcal{S}_{n-1}.
\]

By induction hypothesis \(G/A(n-1)\in \mathcal{S}_{n-1}\) and thus \(G(n-1) = A'(n-1)\). This implies that \(G^{(n)} = 1\) and \(G \in \mathcal{S}_n\), as desired. \(\square\)

**Theorem 3.7.** Let \(G\) be a locally graded \(\mathfrak{T}\)-group and suppose that every proper subgroup of \(G\) is in \(\mathcal{S}_n\mathcal{C}\) for some fixed positive integer \(n\). If \(G\) contains a normal \(\mathfrak{N}\)-subgroup \(N\) such that \(G/N \in \mathfrak{C}\), then \(G \in \mathcal{S}_n\mathcal{C}\).

**Proof.** Assume for a contradiction that \(G\) is an \(MN\mathcal{S}_n\mathcal{C}\)-group. Since \(G\) has no proper subgroup of finite index, we have \(G/N \in \mathcal{L}\) or \(G = N\). Hence, \(G' \neq G\) and thus \(1 \neq G/G' \in \mathcal{L}\). If \(N = G\), then we have the contradiction that \(G\) is in \(\mathfrak{A}\) by \([18, \text{Section 5.2.5}]\), since \(G\) is in \(\mathfrak{T} \cap \mathfrak{N}\). So \(N \neq G\) and hence \(N\) has a normal subgroup \(S \in \mathcal{S}_n\) such that \(N/S \in \mathcal{C}\). So \(N/S\) contains a maximal \(\mathcal{L}\)-subgroup \(R/S\) such that \(N/R \in \mathfrak{S}\). By Lemma 3.6, \(R \in \mathcal{S}_n\). Therefore, we can assume by Lemma 2.1 that \(R\) is characteristic in \(N\) and hence \(R\) is normal in \(G\). So \(G/R \in \mathfrak{C}\). Consequently, \(G \in \mathcal{S}_n\mathcal{C}\), a contradiction, and the proof is complete. \(\square\)

**4. \(\mathfrak{N}\mathcal{C}\)-groups with certain characteristic subgroups.**

**Lemma 4.1.** Let \(G\) be in \(\mathfrak{T} \cap \mathfrak{N}\), \(N\) a normal subgroup of \(G\), \(\omega\) an outer commutator word of weight \(t \geq 2\) such that \(\omega(N) = 1\). If \(G/N\) is in \(\mathfrak{A} \cap \mathfrak{D}\), then \(\omega(G) = 1\).

**Proof.** We proceed by induction on \(t\). If \(t = 2\), then \(\omega(x, y) = [x, y]\) and

\[
\omega(N) = [N, N] = 1,
\]
i.e. \(N\) is in \(\mathfrak{A}\). So \(G/C_G(N)\) is in \(\mathfrak{A} \cap \mathfrak{D}\) and isomorphic to a subgroup of \(\text{Aut } N\). By \([3, \text{Lemma 1}]\) \(G = C_G(N)\) and thus \(N \leq Z(G)\). Applying \([18, 5.3.5]\) we have that \(G\) is in \(\mathfrak{A}\). Let \(t > 2\); then \(\omega = [\psi, \phi]\) for some outer commutator words \(\psi, \phi\) of weight \(1 \leq t_1, t_2 < t\) such that \(t = t_1 + t_2\). Now \(G/N\) is in \(\mathcal{L}\) and \(\psi(N/\psi(N)) = 1\). If \(t_1 > 1\), then by induction hypothesis \(\psi(G/\psi(N)) = 1\), i.e. \(\psi(G) \leq \psi(N)\). Clearly \(\psi(N) \leq \psi(G)\) and it follows that \(\psi(G) = \psi(N)\). If also \(t_2 > 1\), then similarly \(\phi(G) = \phi(N)\), and we have

\[
\omega(G) = [\psi(G), \phi(G)] = [\psi(N), \phi(N)] = 1,
\]
as required. So we may assume that \(t_2 = 1\) and hence \(t_1 > 1\), since \(t > 2\). (If \(t_1 = 1\), then a similar argument works.) Then \(\omega(N) = [\psi(N), N] = 1\) and hence \(N \leq C_G(\psi(N))\).

We also have that \(\psi(N)\) is in \(\mathfrak{A}\). Then \(G/C_G(\psi(N))\) is in \(\mathfrak{A} \cap \mathfrak{D}\) and isomorphic to a subgroup of \(\text{Aut } \psi(N)\). So by \([3, \text{Lemma 1}]\) \(\psi(N) \leq Z(G)\); in other words \([\psi(N), G] = 1\). It follows that

\[
\omega(G) = [\psi(G), G] = [\psi(N), G] = 1,
\]
and the proof is complete. \(\square\)

**Theorem 4.2.** Let \(G\) be a \(\mathfrak{T}\)-group and let \(N \in \mathfrak{N}_c \cap \mathfrak{X}_\omega\) be a normal subgroup of \(G\) such that \(G/N \in \mathfrak{C}\) for some outer commutator word \(\omega\). Then \(G\) contains a characteristic...
(even invariant under all surjective endomorphisms) subgroup $S \in \mathfrak{N}_c \cap \mathfrak{X}_\omega$ such that $G/S \in \mathfrak{C}$.

Proof. Let $W := (N^\alpha \mid \alpha \in \text{Aut } G)$, then $W$ is characteristic in $G$ and $W/N \in \mathfrak{C}$. By [8, Lemma 4.7] $W$ is in $\mathfrak{H}$. We also have that $W/N$ has a normal $\mathfrak{A} \cap \mathfrak{D}$-subgroup $R/N \in \mathfrak{H}$ such that $W/R$ is in $\mathfrak{G}$. Now by Lemma 4.1 we have $R \in \mathfrak{N}_c \cap \mathfrak{X}_\omega$. By Lemma 2.1 $W$ has characteristic (even invariant under all surjective endomorphisms) subgroups $S_1 \in \mathfrak{N}_c$ and $S_2 \in \mathfrak{X}_\omega$ such that $W/S_i$ is in $\mathfrak{G}$ for $i = 1, 2$. Put $S = S_1 \cap S_2$, then $|W : S| < \infty$ and $S$ is contained in $\mathfrak{N}_c \cap \mathfrak{X}_\omega$. Since $W$ is characteristic in $G$, we see that $S$ is characteristic in $G$, and since $G/W \in \mathfrak{C}$ and $W/S$ is finite, we have $G/S \in \mathfrak{C}$. The proof is complete.

If we take $\omega = \gamma_{c+1}$, then

$$\mathfrak{N}_c \cap \mathfrak{X}_\omega = \mathfrak{N}_c \cap \mathfrak{N}_c = \mathfrak{N}_c.$$ 

Hence, we obtain the following result.

Corollary 4.3. Let $G$ be a $\mathfrak{Z}$-group and let $N \in \mathfrak{N}_c$ be a normal subgroup of $G$ such that $G/N \in \mathfrak{C}$. Then $G$ contains a characteristic (even invariant under all surjective endomorphisms) subgroup $S \in \mathfrak{N}_c$ such that $G/S \in \mathfrak{C}$.

Corollary 4.3 sharpens [8, Lemma 4.7] and generalises [3, Lemma 3] and [9, Corollary 1(i)] in the periodic case.

In [8, p. 321] Hartley gives an example that shows that the ‘periodicity’ condition cannot be removed from the hypothesis of Corollary 4.3 and defined Chernikov-subnormality ($\mathfrak{C}$-subnormality, in short) as follows:

A subgroup $N$ of a group $G$ is called $\mathfrak{C}$-subnormal in $G$ if there is a finite series

$$N = N_0 \trianglerighteq N_1 \trianglerighteq \cdots \trianglerighteq N_r = G$$

such that $N_{i+1}/N_i \in \mathfrak{C}$ for $0 \leq i \leq r - 1$.

Corollary 4.4. Let $G$ be a $\mathfrak{Z}$-group containing a $\mathfrak{C}$-subnormal subgroup $N \in \mathfrak{N}_c$. Then $G$ contains a characteristic (even invariant under all surjective endomorphisms) subgroup $S \in \mathfrak{N}_c$ such that $G/S \in \mathfrak{C}$.

Proof. The result follows by Corollary 4.3 and a simple induction.

We can give an immediate application of Corollary 4.3 by considering the following result due to Hartley.

Theorem 4.5 [8, Theorem B]. If $G$ is a locally finite group admitting an involutory automorphism $\phi$ such that $C_G(\phi)$ is in $\mathfrak{C}$, then both $[G, \phi]$ and $G/[G, \phi]$ are in $\mathfrak{C}$.

As Shumyatsky mentions in [19, p. 160], if we take $N = C_G(\phi)([G, \phi])$, then $N \in \mathfrak{N}_2$, $G/N \in \mathfrak{C}$ and $N$ is $\phi$-invariant. So by Corollary 4.3, $G$ has a characteristic subgroup $S \in \mathfrak{N}_2$ such that $G/S \in \mathfrak{C}$.

We record here the following theorem, which is an immediate consequence of Lemma 2.1.

Theorem 4.6. Let $G$ be a group and let $N \in \mathfrak{X}_\omega$ be a normal subgroup of $G$ for some outer commutator word $\omega$ such that $G/N \in \mathfrak{C}$. Then $G$ contains a characteristic (even invariant under all surjective endomorphisms) subgroup $S \in \mathfrak{X}_{\omega \circ \theta}$ such that $G/S$ is finite.
ON CERTAIN APPLICATIONS OF THE KHUKHRO–MAKARENKO THEOREM 283

Proof. Since $G/N \in \mathcal{C}$, there exists a normal $\mathfrak{A} \cap \mathfrak{D}$-subgroup $R/N$ of $G/N$ such that $G/R$ is in $\mathfrak{F}$. Since $N \in \mathcal{X}_\omega$ and $R/N$ is in $\mathfrak{A}$, we have $R \in \mathcal{X}_\omega \mathfrak{A}$. By Lemma 2.1 $G$ has a characteristic subgroup (even invariant under all surjective endomorphisms) $S \in \mathcal{X}_\omega \mathfrak{A}$ such that $G/S$ is in $\mathfrak{F}$, and the result is established. □

Of course, if we replace the condition $G/N \in \mathcal{C}$ with $G/N \in \mathfrak{A} \mathfrak{F}$ in Theorem 4.6, then the result remains true.

REFERENCES

2. V. V. Belyaev and M. Kuzucuoglu, Locally finite barely transitive groups, Algebra Logic 42 (2003), 147–152 (translated from Algebra i Logika 42 (2003), 261–270).