SUBMODULES OF COMMUTATIVE $C^*$-ALGEBRAS

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Abstract. In this paper we generalise a result of Izuchi and Suárez (K. Izuchi and D. Suárez, Norm-closed invariant subspaces in $L^\infty$ and $H^\infty$, Glasgow Math. J. 46 (2004), 399–404) on the shift invariant subspaces of $L^\infty$ and $H^\infty$, to the non-commutative setting. Considering these subspaces as $C(\mathbb{T})$-modules contained in $L^\infty$, we show that under some restrictions, a similar description can be given for the $\mathcal{B}$-submodules of $\mathcal{A}$, where $\mathcal{A}$ is a $C^*$-algebra and $\mathcal{B}$ is a commutative $C^*$-subalgebra of $\mathcal{A}$. We use this to give a description of the $\mathbb{M}_n(\mathcal{B})$-submodules of $\mathbb{M}_n(\mathcal{A})$.

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1. Introduction. Let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. A subspace $S$ of $L^p(\mathbb{T})$ is said to be shift invariant if for every $f \in S$ we have that the function $z \mapsto zf(z)$ is also in $S$. As is usual, particular importance over the years has been placed on the cases $p = 2$ and $p = \infty$. These subspaces, as well as arising naturally in an abundance of purely operator theoretic contexts, have proved important in the study of linear time invariant systems in control theory. A very lucid account of this is given in [7, Chap. 3].

Shift invariant subspaces come in two forms. If $S$ is shift invariant and, in addition, we have that the function $z \mapsto \overline{zf}(z)$ is in $S$ whenever $f \in S$, then $S$ is called doubly invariant or 2-invariant, otherwise it is called simply invariant or 1-invariant. Equivalently, simply invariant subspaces are the shift invariant subspaces $S$ such that $zS \neq S$ and doubly invariant subspaces are those with $zS = S$.

When $p < \infty$, the classification of the closed doubly invariant subspaces is given by Wiener’s theorem [7, Theorem 3.1.1]. The classification of the closed simply invariant subspaces is slightly more difficult and is the content of the Beurling–Helson theorem [7, Theorem 3.1.2]. These theorems were then used to derive analogous results for the weak-$^*$ closed shift invariant subspaces of $L^\infty(\mathbb{T})$. Although this provided a satisfactory classification of these subspaces, much less is known in general about the norm closed shift invariant subspaces of $L^\infty(\mathbb{T})$. The most significant progress made so far are the results of Izuchi and Suárez in [5]. In their paper the authors characterised the maximal norm closed simply invariant subspaces of $L^\infty(\mathbb{T})$ and all norm closed doubly invariant subspaces of $L^\infty(\mathbb{T})$. In this paper, we will only be considering the latter. For completeness, we will present this result of Izuchi and Suárez [5], but first some definitions will be required.

As usual $\Delta(L^\infty(\mathbb{T}))$ is the spectrum of $L^\infty(\mathbb{T})$ and $\hat{f} \in C(\Delta(L^\infty(\mathbb{T})))$ is the Gelfand transform of $f \in L^\infty(\mathbb{T})$. We regard each bounded Borel measure $\mu$ on $\Delta(L^\infty(\mathbb{T}))$ as a linear functional on $L^\infty(\mathbb{T})$ and so we write $\ker \mu$ for the collection of all $f \in L^\infty(\mathbb{T})$
such that
\[ \int_{\Delta(L^\infty(\mathbb{T}))} \hat{f} \, d\mu = 0. \]

Let \( z \) denote the identity function on \( \mathbb{T} \). Then for each \( \lambda \in \mathbb{T} \) we define \( \mathcal{F}_\lambda \subseteq \Delta(L^\infty(\mathbb{T})) \) to be the set of all characters \( \varphi \in \Delta(L^\infty(\mathbb{T})) \) such that \( \hat{z}(\varphi) = \lambda \). Then \( \Delta(L^\infty(\mathbb{T})) = \bigcup_{\lambda \in \mathbb{T}} \mathcal{F}_\lambda \). Define \( \Pi \) to be the set of all bounded Borel measures \( \mu \) on \( \Delta(L^\infty(\mathbb{T})) \) such that \( \text{supp} \mu \subseteq \mathcal{F}_\lambda \) for some \( \lambda \in \mathbb{T} \). We are now in a position to state the theorem.

**Theorem 1.** ([5]). A closed subspace \( S \) of \( L^\infty(\mathbb{T}) \) is doubly invariant if and only if there is some collection of measures \( \Lambda \subseteq \Pi \) such that
\[ S = \bigcap_{\mu \in \Lambda} \ker \mu. \]

We aim to show that this result is, in fact, a special case of more general results describing some of the modules of certain commutative \( C^* \)-algebras. Before proceeding, we will first fix some notation that will be adopted throughout, most of which is standard.

### 1.1. Notation.
Let \( \mathcal{H} \) be a Hilbert space. \( I \) will denote the identity in \( \mathcal{B}(\mathcal{H}) \). For a subalgebra \( \mathfrak{A} \subseteq \mathcal{B}(\mathcal{H}) \), we denote by \( (\mathfrak{A}) \) its closed unit ball, \( \mathfrak{A}^\omega \) its weak closure, \( Z(\mathfrak{A}) \) its centre and \( \mathfrak{A}' \) its commutant in \( \mathcal{B}(\mathcal{H}) \) – that is \( \mathfrak{A}' = \{ T \in \mathcal{B}(\mathcal{H}) : TA = AT \text{ for every } A \in \mathfrak{A} \} \). Given \( \psi \in \mathfrak{A}' \) and \( A \in \mathfrak{A} \), we will write \( A\psi \) to denote the functional \( \mathcal{B} \mapsto \psi(AB) \) on \( \mathfrak{A} \). \( M_n(\mathfrak{A}) \) will denote the algebra of all \( n \times n \) matrices with entries in \( \mathfrak{A} \), although we will simply write \( M_n \) rather than \( M_n(\mathfrak{C}) \). \( E_{ij} \) denotes the element of \( M_n \) with \( i, j \)th entry equal to 1 and all other entries 0.

### 2. A non-commutative generalisation.
It is easily observed that the problem of determining the closed doubly invariant subspaces of \( L^\infty(\mathbb{T}) \) can be thought of as one of determining the closed \( C(\mathbb{T}) \)-modules contained in \( L^\infty(\mathbb{T}) \). It is then natural to ask, if rather than \( C(\mathbb{T}) \) and \( L^\infty(\mathbb{T}) \) we have two \( C^* \)-algebras \( \mathfrak{B} \) and \( \mathfrak{A} \) with \( \mathfrak{B} \subseteq \mathfrak{A} \), whether we can still give a description of the closed \( \mathfrak{B} \)-submodules of \( \mathfrak{A} \). We show that under some restrictions on the algebras, a similar description can be given which generalises the result of Izuchi and Suárez [5]. In particular, we will always require \( \mathfrak{B} \) to be commutative. In stating and proving the main results, we use many aspects from the non-commutative theory of antisymmetric algebras developed many years ago by Szymanski in [8, 9]. We begin by recalling a basic definition from this theory.

Let \( \mathcal{H} \) be a Hilbert space and let \( \mathfrak{A} \subseteq \mathcal{B}(\mathcal{H}) \) be an operator algebra. A projection \( P \in \mathfrak{A}' \) is called \( \mathfrak{A} \)-antisymmetric if for every \( A \in \mathfrak{A} \) such that \( PA = PA^* \), there exists some \( r \in \mathbb{R} \) such that \( PA = rP \). An \( \mathfrak{A} \)-antisymmetric projection \( P \) is maximal if whenever \( Q \) is an \( \mathfrak{A} \)-antisymmetric projection such that \( Q \geq P \), with the standard ordering of projections, we have \( Q = P \). It is a straightforward application of Zorn’s Lemma to show that every \( \mathfrak{A} \)-antisymmetric projection is dominated by a maximal one. We denote by \( \mathcal{M}(\mathfrak{A}) \) the set of all maximal \( \mathfrak{A} \)-antisymmetric projections. It was shown in [9] that if \( \mathfrak{A} \) acts non-degenerately on \( \mathcal{H} \), \( \mathcal{M}(\mathfrak{A}) \) is contained in the centre of \( \mathfrak{A}^{\omega} \) and that the elements of \( \mathcal{M}(\mathfrak{A}) \) are all orthogonal. For a general operator algebra, indeed even for
a $C^*$-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, there need not be any $\mathcal{A}$-antisymmetric projections (consider, for example, $\mathcal{A} = \mathcal{B}(\mathcal{H}) = \mathbb{M}_2$).

We say that $\mathcal{M}(\mathcal{A})$ is full or that $\mathcal{A}$ has a full set of antisymmetric projections if $\mathcal{M}(\mathcal{A})$ is non-empty and

$$\sum_{P \in \mathcal{M}(\mathcal{A})} P = I,$$

where the sum converges in the strong operator topology. It is easily verified that a necessary (but not sufficient) condition for $\mathcal{A}$ to have a full set of antisymmetric projections is that $\mathcal{A}$ is commutative.

Throughout the remainder of this section, we fix a Hilbert space $\mathcal{H}$ and two $C^*$-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ with $\mathcal{B} \subseteq \mathcal{A}$. We shall also assume the following:

1. $\mathcal{B}$ is commutative with a full set of antisymmetric projections.
2. $\mathcal{B}$ (and hence $\mathcal{A}$) acts non-degenerately on $\mathcal{H}$. We will say that a functional $\psi \in \mathcal{A}^*$ is antisymmetrically supported if for each $P \in \mathcal{M}(\mathcal{B})$ either $P\psi = \psi$ or $P\psi = 0$. We say that a set $\Lambda \subseteq \mathcal{A}^*$ is antisymmetrically supported if every $\psi \in \Lambda$ is antisymmetrically supported.

We can now give a generalisation of Theorem 1.

**Theorem 2.** If every norm continuous linear functional on $\mathcal{A}$ is ultraweakly continuous, then $M \subseteq \mathcal{A}$ is a closed left $\mathcal{B}$-module if and only if there exists an antisymmetrically supported set $\Lambda \subseteq \mathcal{A}^*$ such that

$$M = \bigcap_{\psi \in \Lambda} \ker \psi.$$

In order to prove Theorem 2 we will require the following lemma. This is a non-commutative analogue of a result in the commutative theory of uniform algebras, a detailed account of which can be found in [2].

**Lemma 3.** Assume that every norm continuous linear functional on $\mathcal{A}$ is ultraweakly continuous and let $M \subseteq \mathcal{A}$ be a closed left $\mathcal{B}$-module.

(a) For every $\psi \in \mathcal{A}^*$ we have that if $\psi \in M^\perp$ then $\psi \in (PM)^\perp$ for each $P \in \mathcal{M}(\mathcal{B})$.

(b) If $A \in \mathcal{A}$ and $PA \in PM$ for every $P \in \mathcal{M}(\mathcal{B})$ then $A \in M$.

**Proof.** (a) Fix $\psi \in M^\perp$ and $P \in \mathcal{M}(\mathcal{B})$. Since $P \in \overline{\mathcal{B}}_w^*$, there exists a net $(B_\lambda) \subseteq \mathcal{B}$ with $B_\lambda \to P$ in the ultraweak topology. As every $\psi \in \mathcal{A}^*$ is ultraweakly continuous, for each $A \in M$ we have $\psi(PA) = \lim_\lambda \psi(B_\lambda A) = 0$.

(b) Fix $\psi \in M^\perp$, $A \in (\mathcal{A})_1$ and suppose that $PA \in PM$ for every $P \in \mathcal{M}(\mathcal{B})$. The requirement that $\mathcal{M}(\mathcal{B})$ is full then ensures that the sum

$$\sum_{P \in \mathcal{M}(\mathcal{B})} PA$$

converges in the strong operator topology (and hence in the weak operator topology) to $A$. Since $\psi$ is ultraweakly continuous, it is weakly continuous on $(\mathcal{A})_1$.
[6, Proposition 7.4.5], and so

\[ \psi(A) = \sum_{P \in \mathcal{M}(\mathfrak{B})} \psi(PA). \]

It then follows from part (a) that \( \psi(A) = 0 \).

We can now proceed to prove Theorem 2.

**Proof of Theorem 2.** Let \( M \subseteq \mathfrak{A} \) be a closed left \( \mathfrak{B} \)-module. Define \( \Lambda \subseteq M^\perp \) to be the set of all \( \psi \in M^\perp \), which are antisymmetrically supported. Firstly, we have the trivial inclusion

\[ M \subseteq \bigcap_{\psi \in \Lambda} \ker \psi. \]

We also see that if \( \psi \in (PM)^\perp \) for some \( P \in \mathcal{M}(\mathfrak{B}) \) then \( P\psi \in M^\perp \). Since \( P \) is a projection, we also clearly have that \( P\psi \) is antisymmetrically supported and so \( P\psi \in \Lambda \). So if \( \psi(A) = 0 \) for every \( \psi \in \Lambda \), then \( \psi(PA) = 0 \) for every \( \psi \in (PM)^\perp \). It then follows from Lemma 3(b) that \( A \in M \), and therefore

\[ M = \bigcap_{\psi \in \Lambda} \ker \psi. \]

Conversely, fix \( \psi \in \mathfrak{A}^* \) and \( P \in \mathcal{M}(A) \) such that \( P\psi = \psi \). Then for each \( B \in \ker \psi \) and \( A \in A \),

\[ \psi(AB) = \psi(PAB) = \lambda \psi(B) = 0 \]

for some \( \lambda \in \mathbb{C} \). So \( \ker \psi \) is a left \( \mathfrak{B} \)-module, and hence an intersection of such things will also be a left \( \mathfrak{B} \)-module. \( \square \)

**Example 4.** Let \( \mathcal{H} \) be a separable Hilbert space with orthonormal basis \( (e_n) \). We denote by \( \mathcal{K}(\mathcal{H}) \) and \( \mathcal{D}_0(\mathcal{H}) \) the algebras of compact operators and compact diagonal operators (with respect to the basis \( (e_n) \)) respectively. It is straightforward to verify that the \( \mathcal{D}_0(\mathcal{H}) \)-antisymmetric projections are rank 1 projections onto subspaces spanned by the basis vectors and that these are in fact maximal so that \( \mathcal{M}(\mathcal{D}_0(\mathcal{H})) = \{ P_n : n \in \mathbb{N} \} \), where \( P_n \) is the projection onto the subspace spanned by \( e_n \). Since every continuous linear functional on \( \mathcal{K}(\mathcal{H}) \) is induced by a trace class operator, it has an extension to \( \mathcal{B}(\mathcal{H}) \), which is ultraweakly continuous ([11, p. 96]). Fix \( S \in S_1(\mathcal{H}) \), where \( S_1(\mathcal{H}) \) denotes the trace class operators on \( \mathcal{H} \). We will use \( \hat{S} \) to denote the functional \( T \mapsto \text{tr}ST \). An elementary calculation shows that \( P_n \hat{S} = \hat{S} \hat{P}_n \) if and only if \( e_k \in \ker S \) for every \( k \neq n \). Equivalently, \( P_n \hat{S} = \hat{S} \hat{P}_n \) if and only if the matrix for \( S \) only has non-zero entries in the \( n \)-th column. Then \( T \in \ker S \) if and only if the \( n \)-th entry of \( ST \) is 0. Consequently, every \( \mathcal{D}_0(\mathcal{H}) \)-submodule of \( \mathcal{K}(\mathcal{H}) \) can be constructed by starting with some collection \( \{ S_i \} \subseteq S_1(\mathcal{H}) \), each member of which will only have non-zero entries in one column, the \( n_i \)-th column say, and then taking all \( T \in \mathcal{K}(\mathcal{H}) \) such that the \( n_i, n_i \)-th entry of \( S_i T \) vanishes for all \( \lambda \).

**Corollary 5.** Let \( X \) be a closed subalgebra of \( \mathfrak{A} \) containing \( \mathfrak{B} \). If every norm continuous linear functional on \( \mathfrak{B} \) is ultraweakly continuous, then \( M \) is a closed left \( \mathfrak{B} \)-submodule of \( X \) if and only if there exists an antisymmetrically supported set \( \Lambda \subseteq \mathfrak{A}^* \).
such that

\[ M = \bigcap_{\psi \in \Lambda} \ker \psi \cap X. \]

**Proof.** Fix \( P \in \mathcal{M}(\mathfrak{B}) \) and suppose that we have \( \psi \in \mathfrak{A}^* \) with \( P\psi = \psi \). For every \( B \in \mathfrak{B} \) and \( A \in X \) we have that \( BA - \lambda A \in \ker \psi \), where \( \lambda \in \mathbb{C} \) is such that \( PB = \lambda P \). If we also have that \( A \in \ker \psi \), then we must have that \( BA \in \ker \psi \). Hence, \( \ker \psi \cap X \) is a closed left \( \mathfrak{B} \)-submodule of \( X \). Conversely, every closed left \( \mathfrak{B} \)-submodule of \( X \) is trivially a closed left \( \mathfrak{B} \)-submodule of \( \mathfrak{A} \), and so the result follows from Theorem 2. \( \square \)

**Example 6.** Let \( X \) be a closed subalgebra of \( L^\infty(\mathbb{T}) \) which contains \( H^\infty(\mathbb{T}) \). Such algebras are called *Douglas algebras* and a detailed account of these is given in [3, Chap. 9]. If we further suppose that \( X \) strictly contains \( H^\infty(\mathbb{T}) \), then by Theorems 1.4 and 2.2 of [3] we have that \( X \) contains \( C(\mathbb{T}) \). Corollary 5 then implies that the closed shift invariant subspaces of \( X \) are all of the form \( S \cap X \), where \( S \) is a closed shift invariant subspace of \( L^\infty(\mathbb{T}) \).

We now wish to extend Theorem 2 to give a description of the \( \mathcal{M}_n(\mathfrak{B}) \)-submodules of \( \mathcal{M}_n(\mathfrak{A}) \). However, we will first consider \( \mathfrak{A}^n = \mathfrak{A} \oplus \cdots \oplus \mathfrak{A} \) acting on \( \mathcal{H}^n \). We can regard \( \mathfrak{B} \) as a subalgebra of \( \mathfrak{A}^n \) by identifying \( B \in \mathfrak{B} \) with \( (B, \ldots, B) \in \mathfrak{A}^n \). It is clear that if every bounded linear functional on \( \mathfrak{A} \) is ultraweakly continuous then the same is true for \( \mathfrak{A}^n \). The definition of antisymmetrically supported elements and subsets of \( \mathfrak{A}^* \) extends to \( \mathfrak{A}^{*n} \) without change, but noting that if \( \psi = (\psi_1, \ldots, \psi_n) \in \mathfrak{A}^{*n} \) then \( P\psi = (P\psi_1, \ldots, P\psi_n) \). We are now left with the task of determining the maximal \( \mathfrak{B} \)-antisymmetric projections in \( \mathcal{B}(\mathcal{H}^n) \). It should be noted that despite considering \( \mathfrak{B} \) acting on \( \mathcal{H}^n \) we will reserve \( \mathcal{M}(\mathfrak{B}) \) exclusively for denoting the maximal \( \mathfrak{B} \)-antisymmetric projections in \( \mathcal{B}(\mathcal{H}) \). Let \( Q \in \mathcal{B}(\mathcal{H}^n) \) be any maximal \( \mathfrak{B} \)-antisymmetric projection. Since \( Q \) is contained in the weak closure of \( \mathfrak{A}^n \), we can write \( Q = (Q_1, \ldots, Q_n) \), where each \( Q_j \) acts on \( \mathcal{H} \). Then it is easy to check that each \( Q_j \) is a maximal \( \mathfrak{B} \)-antisymmetric projection in \( \mathcal{B}(\mathcal{H}) \). Suppose there are indices \( j \) and \( k \) with \( Q_j \neq Q_k \). Then there is some \( B \in \mathfrak{B} \) and distinct complex numbers \( \lambda_j \) and \( \lambda_k \) with \( BQ_j = \lambda_j B \) and \( BQ_k = \lambda_k Q_k \). Then \( B(Q_j, Q_k) = (BQ_j, BQ_k) = (\lambda_j Q_j, \lambda_k Q_k) \neq (\lambda_j Q_j, Q_k) \) for any \( \lambda \in \mathbb{C} \). It follows that any projection in \( \mathcal{B}(\mathcal{H}^n) \) having a subprojection equivalent to \( (Q_j, Q_k) \) cannot be \( \mathfrak{B} \)-antisymmetric. So in particular \( Q \) is not \( \mathfrak{B} \)-antisymmetric. We conclude from this that all the \( Q_j \) are equal and hence \( Q = (P, \ldots, P) \) for some \( P \in \mathcal{M}(\mathfrak{B}) \).

We will now turn our attention to the left \( \mathcal{M}_n(\mathfrak{B}) \)-submodules of \( \mathcal{M}_n(\mathfrak{A}) \). We occasionally identify \( \mathcal{M}_n(\mathfrak{A}) \) and \( \mathcal{M}_n(\mathfrak{B}) \) with \( \mathfrak{A} \otimes \mathcal{M}_n \) and \( \mathfrak{B} \otimes \mathcal{M}_n \), respectively, when it is convenient to do so. Before continuing, let us agree on a useful convention. We will regard elements of \( \mathfrak{A}^{*n} \) as column vectors and if \( A = (A_\ell) \in \mathcal{M}_n(\mathfrak{A}) \), \( \lambda = (\lambda_\ell) \in \mathcal{M}_n \) and \( \psi \in \mathfrak{A}^{*n} \) then the ‘products’ \( A\psi \) and \( \lambda A \) are the usual ones; however, in this instance we interpret terms, such as \( A_\ell \psi_k \), to mean \( \psi_k(A_\ell) \). With this understood, we define for each \( \psi \in \mathfrak{A}^{*n} \) a linear map \( R_\psi : \mathcal{M}_n(\mathfrak{A}) \to \mathbb{C}^n \) by setting \( R_\psi A = A\psi \) for every \( A \in \mathcal{M}_n(\mathfrak{A}) \).

**Theorem 7.** If every norm continuous linear functional on \( \mathfrak{A} \) is ultraweakly continuous then \( M \subseteq \mathcal{M}_n(\mathfrak{A}) \) is a closed left \( \mathcal{M}_n(\mathfrak{B}) \)-module if and only if there is an antisymmetrically supported set \( \Lambda \subseteq \mathfrak{A}^{*n} \) such that

\[ M = \bigcap_{\psi \in \Lambda} \ker R_\psi. \]
Proof. Assume that $M \subseteq M_n(\mathfrak{A})$ is a closed left $M_n(\mathfrak{B})$-module. For $i = 1, \ldots, n$, let $M_i$ be the set of all $(A_1, \ldots, A_n) \in \mathfrak{A}^n$ such that $A = (A_j) \in M$. It is clear that each $M_i$ is a closed left $\mathfrak{B}$-module and so there exist antisymmetrically supported sets $\Lambda_1, \ldots, \Lambda_n \subseteq \mathfrak{A}^{*n}$ such that

$$M_i = \bigcap_{\psi \in \Lambda_i} \ker \psi.$$  

We claim that all the $\Lambda_i$ are equal. Fix $i$ and $j$ with $1 \leq i, j \leq n$. Since $M$ is a left $M_n(\mathfrak{B})$-module, we have in particular that for every $B \in \mathfrak{B}$ and every $A \in M$, $(B \otimes E_{ji})A \in M$. The $j$th row of $(B \otimes E_{ji})A$ is $(BA_1, \ldots, BA_n)$. It follows that $\mathfrak{B}M_i \subseteq M_j$. Suppose there is some $A \in M_i \setminus M_j$. Then there is some $\psi \in \Lambda_j$ and $P \in M(\mathfrak{B})$ such that $P\psi = \psi$ and $\psi(A) \neq 0$. However, if we choose a net $(B_\lambda) \subseteq \mathfrak{B}$ with $B_\lambda \to P$ in the ultraweak topology (which we can always do as $P \in \mathfrak{B}^{*w}$), then we see that

$$\psi(A) = \psi(PA) = \lim_{\lambda} \psi(B_\lambda A) = 0.$$  

This proves that such an $A$ cannot exist and hence $M_i \subseteq M_j$. Swapping $i$ and $j$ in the previous analysis gives $M_i = M_j$. Once we set $\Lambda = \Lambda_i$ for some (and hence all) $i$, the inclusion

$$M \subseteq \bigcap_{\psi \in \Lambda} \ker R_\psi$$  

follows immediately.

Given any $A \in \cap \ker R_\psi$, there must exist $A^{(1)}, \ldots, A^{(n)} \in M$ such that for each $j = 1, \ldots, n$, $E_{jj}A^{(j)} = E_{jj}A$. Again, using that $M$ is a left $M_n(\mathfrak{B})$-module, we have that $(B \otimes E_{jj})A^{(j)} \in M$ for every $B \in \mathfrak{B}$. Since $M$ is ultraweakly closed, it follows that $P \otimes E_{jj}A^{(j)} \in M$ for every $P \in M(\mathfrak{B})$. As $M(\mathfrak{B})$ is full, the sum

$$\sum_{P \in M(\mathfrak{B})} P \otimes E_{jj}A^{(j)}$$

converges in the strong operator topology to $E_{jj}A^{(j)}$, and since all the partial sums are bounded, it also converges in the ultraweak topology. So $E_{jj}A^{(j)} \in M$. Writing

$$A = \sum_{j=1}^n E_{jj}A^{(j)},$$

we see that

$$M = \bigcap_{\psi \in \Lambda} \ker R_\psi.$$  

The converse is straightforward. Fix $\psi \in \mathfrak{A}^{*n}$ with $P\psi = \psi$ for some $P \in M(\mathfrak{B})$. If $A \in \ker R_\psi$ and $B \in M_n(\mathfrak{B})$ then there is a matrix $\lambda = (\lambda_\psi) \in M_n$ such that

$$BA\psi = \lambda A\psi = 0.$$
It follows that for any antisymmetrically supported set $\Lambda \subseteq \mathbb{A}^n$,

$$\bigcap_{\psi \in \Lambda} \ker R_\psi$$

is a left $M_\rho(\mathcal{B})$-module. \qed

3. The case where $\mathcal{B} \subseteq \mathcal{Z}(\mathcal{A})$. In the following we fix two unital $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, as before with $\mathcal{B} \subseteq \mathcal{A}$, but here we insist that $\mathcal{B} \subseteq \mathcal{Z}(\mathcal{A})$ and that $\mathcal{B}$ contains the identity in $\mathcal{A}$. As usual, $\hat{\mathcal{A}}$ will denote the spectrum of $\mathcal{A}$ (i.e. the set of equivalence classes of irreducible representations of $\mathcal{A}$) equipped with the usual topology. Let $\Psi$ be the reduced atomic representation of $\mathcal{A}$. That is,

$$\Psi = \bigoplus_{[\pi] \in \hat{\mathcal{A}}} \pi,$$

where each $\pi: \mathcal{A} \to \mathcal{H}_\pi$ is a representative of the equivalence class $[\pi] \in \hat{\mathcal{A}}$. From now on we work only with this set of representatives and make no reference to the equivalence classes. We will show that this representation ensures that $\mathcal{B}$ has a full set of antisymmetric projections and each $P \in M(\mathcal{B})$ has a particularly simple form.

For $\pi \in \hat{\mathcal{A}}$, let $E_\pi$ denote the projection in $\Psi(\mathcal{A})'$ defined by setting

$$\rho(E_\pi) = \begin{cases} I & \text{if } \rho = \pi \\ 0 & \text{if } \rho \neq \pi \end{cases}$$

for every $\rho \in \hat{\mathcal{A}}$. Since $\mathcal{B}$ is contained in the centre of $\mathcal{A}$, we have that for any irreducible representation $\pi$ of $\mathcal{A}$, $\pi(\mathcal{B}) = CI$. Since every irreducible representation of $\mathcal{B}$ extends to an irreducible representation of $\mathcal{A}$ (on a necessarily larger Hilbert space), we see that the map $\pi \mapsto \pi_\mathcal{B}$, where $\pi_\mathcal{B}(A) = (\pi(A)\xi|\xi)$ for any unit vector $\xi \in \mathcal{H}_\pi$, defines a continuous surjection of $\hat{\mathcal{A}}$ onto $\Delta(\mathcal{B})$. Following the ideas of Izuchi and Suárez [5] we will define for each $\varphi \in \Delta(\mathcal{B})$, the fibre above $\varphi$ to be the set

$$\mathcal{F}_\varphi = \{\pi \in \hat{\mathcal{A}} : \pi_\mathcal{B} = \varphi\}.$$

Set

$$P_\varphi = \sum_{\pi \in \mathcal{F}_\varphi} E_\pi$$

with the sum converging in the strong operator topology. Since for each $B \in \mathcal{B}$ there is some $\lambda \in \mathbb{C}$ with $\pi(B) = \lambda I$ for every $\pi \in \mathcal{F}_\varphi$, it is clear that the projection $P_\varphi$ is $\mathcal{B}$-antisymmetric. The fact that

$$\sum_{\varphi \in \Delta(\mathcal{B})} P_\varphi = I$$

follows from the surjectivity of the map $\pi \mapsto \pi_\mathcal{B}$. So to show that each $P_\varphi$ is maximal it is only necessary to show that for any two distinct characters $\varphi, \chi \in \Delta(\mathcal{B})$ there exists some $B \in \mathcal{B}$ and distinct complex numbers $\lambda_1$ and $\lambda_2$ such that $\Psi(B)P_\varphi = \lambda_1 P_\varphi$ and $\Psi(B)P_\chi = \lambda_2 P_\chi$. This follows easily since by the Gelfand representation there
must exist \( B \in \mathcal{B} \) with \( \varphi(B) = 1 \) and \( \chi(B) = 0 \), and so \( \Psi(B)P_\varphi = \varphi(B)P_\varphi = P_\varphi \) and \( \Psi(B)P_\chi = \chi(B)P_\chi = 0 \).

We are still not in a position to apply Theorem 2 because we have not shown that each \( \psi \in \mathfrak{A}^* \) is ultraweakly continuous on \( \Psi(\mathfrak{A}) \), and indeed this is in general not the case. Despite this, we will show, using the idea of Glicksberg in [4] for the commutative case, that the conclusions of Lemma 3 still hold. Before doing this, however, we must start by fixing some terminology. If \( \psi \in \mathfrak{A}^* \), the null space of \( \psi \) is the ideal \( \mathcal{N}(\psi) \subseteq \mathfrak{A} \) consisting of all \( A \in \mathfrak{A} \) such that \( \psi(BAC) = 0 \) for every \( B, C \in \mathfrak{A} \) and the support of \( \psi \) is the projection

\[
S_\psi = \sum_{\mathcal{N}(\psi) \subseteq \ker \pi} E_\pi.
\]

So a functional \( \psi \in \mathfrak{A}^* \) is antisymmetrically supported if and only if \( S_\psi \subseteq P_\psi \) for some \( \varphi \in \Delta(\mathcal{B}) \). Since for every \( A \in \mathfrak{A} \) and \( \pi \in \mathfrak{A} \) with \( \mathcal{N}(\psi) \subseteq \ker \pi \), \( \pi((I - S_\psi)A) = 0 \), it follows from [1, Proposition 2.11.2] that the norm of \( (I - S_\psi)A \in \mathfrak{A}/\mathcal{N}(\psi) \) is 0 and so \( S_\psi \psi = \psi \).

**Lemma 8.** Assume \( \mathcal{B} \subseteq Z(\mathfrak{A}) \) and let \( M \subseteq \mathfrak{A} \) be a closed left \( \mathcal{B} \)-module. If \( A \in \mathfrak{A} \) and \( P_\psi A \in P_\psi M \) for every \( \psi \in \Delta(\mathcal{B}) \) then \( A \in M \).

**Proof.** Let \( \psi \in (\mathfrak{A}^*)_1 \) be an extreme point of \((M^\perp)_1 \). We will show that \( S_\psi \) is a \( \mathcal{B} \)-antisymmetric projection.

Let us first note that a projection \( P \in \Psi(\mathfrak{A}) \) is \( \mathcal{B} \)-antisymmetric if and only if the ideal \( \mathcal{B} \cap (I - P)\mathcal{B} \) is maximal in \( \mathfrak{B} \). This is because if \( P \) is \( \mathcal{B} \)-antisymmetric then for each \( B \in \mathfrak{B} \) there is some \( \lambda(B) \in \mathbb{C} \) such that \( P\Psi(B) = \lambda(B)P \), so the map \( B \mapsto \lambda(B) \) is character on \( \mathcal{B} \) with kernel \( \mathcal{B} \cap (I - P)\mathcal{B} \). Conversely, if \( \mathcal{B} \cap (I - P)\mathcal{B} \) is maximal, then \( P \mathcal{B} \cong \mathcal{B}/\mathcal{B} \cap (I - P)\mathcal{B} \) is and so \( P \mathcal{B} = \mathcal{C}P \).

Let \( \tau : \mathcal{B} \to \mathcal{B}/(\mathfrak{B} \cap (I - S_\psi)\mathfrak{B}) \) be the quotient map. Choose a positive element \( B \in (\mathcal{B}^*)_1 \) such that \( \tau(B) \neq 0 \) and \( \tau(B) \) is not invertible. Then by [1, Proposition 2.11.2] and the Gelfand–Naimark theorem, there must exist some \( \pi \in \mathfrak{A} \) with \( \mathcal{N}(\psi) \subseteq \ker \pi \) and \( \pi(B) = 0 \). This implies that \( \psi \) and \( B\psi \) are linearly independent, otherwise there would be some non-zero \( \lambda \in \mathbb{C} \) with \( (B - \lambda I)\psi = 0 \), and since \( B \in Z(\mathfrak{A}) \), \( B - \lambda I \in \mathcal{N}(\psi) \subseteq \ker \pi \). We also have for any \( A, C \in (\mathfrak{A}^*)_1 \),

\[
|B\psi(A) + (I - B)\psi(C)| = |\psi(BA + (I - B)C)| \\
\leq \|S_\psi(BA + (I - B)C)\| \\
= \sup_{E_\pi \subseteq S_\psi} \|\pi(BA + (I - B)C)\| \\
\leq \sup_{E_\pi \subseteq S_\psi} (\pi_\mathcal{B}(B)\|A\| + (1 - \pi_\mathcal{B}(B))\|C\|) \leq 1.
\]

Consequently, we have \( 1 = \|\psi\| \leq \|B\psi\| + \|(I - B)\psi\| \leq 1 \). Writing

\[
\psi = \|B\psi\| \left(\frac{B\psi}{\|B\psi\|} + (I - B)\psi\right) \left(\frac{(I - B)\psi}{\|(I - B)\psi\|}\right)
\]

we have expressed \( \psi \) as a nontrivial convex sum of elements in \( \mathfrak{A}^* \cap M^\perp \), which is a contradiction. We conclude that every non-zero positive element of \( \mathcal{B}/(\mathfrak{B} \cap (I - S_\psi)\mathfrak{B}) \) is invertible. It follows from the Gelfand–Mazur theorem that \( \mathcal{B}/(\mathfrak{B} \cap (I - S_\psi)\mathfrak{B}) \) has co-dimension 1, which completes the proof. \( \square \)
From this we can state versions of Theorems 2 and 7 for this setting.

**Theorem 9.** If \( \mathcal{B} \subseteq Z(\mathcal{A}) \) then \( M \subseteq \mathcal{A} \) is a closed left \( \mathcal{B} \)-module if and only if there exists an antisymmetrically supported set \( \Lambda \subseteq \mathcal{A}^* \) such that

\[
M = \bigcap_{\psi \in \Lambda} \ker \psi.
\]

**Theorem 10.** If \( \mathcal{B} \subseteq Z(\mathcal{A}) \) then \( M \subseteq \mathcal{M}_n(\mathcal{A}) \) is a closed left \( \mathcal{M}_n(\mathcal{B}) \)-module if and only if there is an antisymmetrically supported set \( \Lambda \subseteq \mathcal{A}^{*n} \) such that

\[
M = \bigcap_{\psi \in \Lambda} \ker R \psi.
\]

The proofs of Theorems 9 and 10 are almost identical to those of Theorems 2 and 7 and so we omit them. There is however one difference that should be pointed out: The appeal to the ultraweak continuity of bounded linear functionals in the proof of Theorem 7 is not necessary for Theorem 10 because \( \mathcal{B} \) contains the identity.

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