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Liouville-type theorems for the Taylor–Couette–Poiseuille flow of the stationary Navier–Stokes equations

Hideo Kozono[1,2,†], **Yutaka Terasawa**[3] and **Yuta Wakasugi**[4]

[1]Department of Mathematics, Faculty of Science and Engineering, Waseda University, Tokyo 169-8555, Japan

[2]Mathematical Research Center for Co-creative Society, Tohoku University, Sendai 980-8578, Japan

[3]Graduate School of Mathematics, Nagoya University, Furocho Chikusaku Nagoya 464-8602, Japan

[4]Graduate School of Advanced Science and Engineering, Hiroshima University, Higashi-Hiroshima 739-8527, Japan

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We study the stationary Navier–Stokes equations in the region between two rotating concentric cylinders. We first prove that, for a small Reynolds number, if the fluid flow is axisymmetric and if its velocity is sufficiently small in the L^∞-norm, then it is necessarily the Taylor–Couette–Poiseuille flow. If, in addition, the associated pressure is bounded or periodic in the z axis, then it coincides with the well-known canonical Taylor–Couette flow. We discuss the relation between uniqueness and stability of such a flow in terms of the Taylor number in the case of narrow gap of two cylinders. The investigation in comparison with two Reynolds numbers based on inner and outer cylinder rotational velocities is also conducted. Next, we give a certain bound of the Reynolds number and the L^∞-norm of the velocity such that the fluid is, indeed, necessarily axisymmetric. As a result, it is clarified that smallness of Reynolds number of the fluid in the two rotating concentric cylinders governs both axisymmetry and the Taylor–Couette–Poiseuille flow with the exact form of the pressure.

Key words: Taylor–Couette flow, Navier–Stokes equations

1. Introduction

This paper concerns the three-dimensional stationary incompressible Navier–Stokes equations

$$\left.\begin{aligned} \boldsymbol{v}\cdot\nabla\boldsymbol{v} + p &= \nu\Delta\boldsymbol{v}, \\ \nabla\cdot\boldsymbol{v} &= 0, \end{aligned}\right\} \tag{1.1}$$

† Email address for correspondence: kozono@waseda.jp

where $\boldsymbol{v} = \boldsymbol{v}(x) = (v_1(x), v_2(x), v_3(x))$ is the velocity vector, $p = p(x)$ denotes the scalar pressure and $\nu > 0$ is the viscosity constant. For the Navier–Stokes equations (1.1) in the whole space $\mathbb{R}^3$, it has been an open problem whether $\boldsymbol{v} \equiv 0$ is the only solution under the conditions that $\boldsymbol{v}$ has finite Dirichlet integral and vanishes at spatial infinity (see Galdi 2011, remark X.9.4). Seregin (2018) reformulated the problem in such a way that any bounded solution $\boldsymbol{v}$ must be constant. There are many partial answers to this problem and, for instance, we refer readers to Carrillo, Pan & Zhang (2020), Chae (2014), Chae & Wolf (2016), Chamorro, Jarrín & Lemarié-Rieusset (2021), Koch *et al.* (2009) and Kozono, Terasawa & Wakasugi (2017) and references therein.

Recently, the Liouville-type theorems in non-compact domains in $\mathbb{R}^3$ have also been studied. Carrillo *et al.* (2020) showed that a smooth solution with the finite Dirichlet integral to the Navier–Stokes equations (1.1) in a slab domain $\mathbb{R}^2 \times [0, 1]$ with the no-slip boundary condition must be zero. Among other results, they also treated the axially symmetric case with the periodic boundary condition and proved the Liouville-type result under the finite Dirichlet integral. The assumption of the finite Dirichlet integral was relaxed by Tsai (2021) and Bang *et al.* (2023). In particular, Bang *et al.* (2023) obtained the Liouville-type theorem on the Poiseuille flow of the Navier–Stokes equations (1.1) in a slab domain $\mathbb{R}^2 \times [0, 1]$ with no-slip boundary condition. Indeed, they showed that if $(\boldsymbol{v}, p)$ is a smooth solution satisfying $\|\boldsymbol{v}\|_{L^\infty} < \pi$, then $\boldsymbol{v}$ must be the Poiseuille flow like $\boldsymbol{v} = (ax_3(1 - x_3), bx_3(1 - x_3), 0)$ with some constants $a, b \in \mathbb{R}$. Their result may be regarded as the generalized Liouville-type theorem on non-trivial flow.

Motivated by these results, we have reached a natural question as to whether Liouville-type theorems hold for other non-trivial exact solutions of the Navier–Stokes equations. In this paper, we study the Liouville-type theorem on the Taylor–Couette–Poiseuille flow in a region between two rotating concentric cylinders. We also compare our mathematical uniqueness result with stability in the fluid mechanics in terms of Reynolds number and Taylor number in the case when the ratio of the radii of the two cylinders is sufficiently close to one.

1.1. *Axially symmetric case*

Let $0 < R_1 < R_2$ be constants and let $\varOmega = \{(x_1, x_2, x_3) \in \mathbb{R}^3; R_1 < \sqrt{x_1^2 + x_2^2} < R_2\}$, that is, a region between two concentric cylinders. In $\varOmega$, we consider the axially symmetric incompressible stationary Navier–Stokes equations in cylindrical coordinates:

$$\left.\begin{aligned} (v^r\partial_r + v^z\partial_z)v^r - \frac{(v^\theta)^2}{r} + \partial_r p &= \nu\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2}\right)v^r, \\ (v^r\partial_r + v^z\partial_z)v^\theta + \frac{v^r v^\theta}{r} &= \nu\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2}\right)v^\theta, \\ (v^r\partial_r + v^z\partial_z)v^z + \partial_z p &= \nu\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)v^z, \\ \frac{1}{r}\partial_r(rv^r) + \partial_z v^z &= 0, \end{aligned}\right\} \tag{1.2}$$

where $r \in (R_1, R_2)$, $z \in \mathbb{R}$, $\boldsymbol{v} = \boldsymbol{v}(r, z) = v^r\boldsymbol{e}_r + v^\theta\boldsymbol{e}_\theta + v^z\boldsymbol{e}_z$ with $\boldsymbol{e}_r = (\cos\theta, \sin\theta, 0)$, $\boldsymbol{e}_\theta = (-\sin\theta, \cos\theta, 0)$, $\boldsymbol{e}_z = (0, 0, 1)$ denoting the basis of the cylindrical coordinate and

$p = p(r, z)$. Moreover, we impose on $\boldsymbol{v}$ the boundary conditions

$$\left.\begin{aligned} v^r(R_j, z) = v^z(R_j, z) = 0 \quad (j = 1, 2), \\ v^\theta(R_1, z) = R_1\omega_1, \quad v^\theta(R_2, z) = R_2\omega_2, \end{aligned}\right\} \tag{1.3}$$

with some $\omega_1, \omega_2 \in \mathbb{R}$, that is, the inner and outer cylinders rotate with angular velocities ω_1 and ω_2, respectively.

It is well known that there exists an exact solution to (1.2) called the Taylor–Couette flow:

$$\boldsymbol{v} = (0, v^\theta, 0) \quad \text{with } v^\theta = Ar + B\frac{1}{r}, \tag{1.4}$$

where

$$A = \begin{cases} \dfrac{\mu - \eta^2}{1 - \eta^2}\omega_1, & \omega_1 \neq 0, \\ \dfrac{1}{1 - \eta^2}\omega_2, & \omega_1 = 0, \end{cases} \qquad B = \begin{cases} \dfrac{1 - \mu}{1 - \eta^2}\omega_1 R_1^2, & \omega_1 \neq 0, \\ -\dfrac{1}{1 - \eta^2}\omega_2 R_1^2, & \omega_1 = 0, \end{cases} \tag{1.5a,b}$$

with non-dimensional quantities μ and η given by

$$\mu = \frac{\omega_2}{\omega_1} \quad \text{for } \omega_1 \neq 0, \qquad \eta = \frac{R_1}{R_2}. \tag{1.6a,b}$$

It is also known that the Taylor–Couette flow is stable if ω_1 is sufficiently small. However, if ω_1 exceeds a certain critical value, then the Taylor–Couette flow becomes unstable and a fluid motion called the Taylor vortex appears (see e.g. Kirchgässner & Sorger 1969; Chossat & Iooss 1994). For a recent result on the compressible fluid motion, we refer to Kagei & Teramoto (2020).

In this paper, we show a Liouville-type theorem on the more generalized Taylor–Couette–Poiseuille flow including (1.4), provided that the velocity is not too large. Although the Taylor–Couette–Poiseuille flow below (equation (1.9)) is not altogether new (see e.g. Ma & Wang 2009; Guy Raguin & Georgiadis 2024), our derivation itself seems to be new, because it is obtained from the fact that $\partial_z \boldsymbol{v} \equiv 0$. The first main theorem reads as follows.

THEOREM 1.1. *Let* $(\boldsymbol{v}, p)$ *be an axially symmetric smooth solution of* (1.2) *in* $\varOmega$ *with the boundary conditions* (1.3). *There exists a constant* $C_1(\nu, R_1, R_2) > 0$ *such that if* ω_1, ω_2 *and* $\|\boldsymbol{v}\|_{L^\infty}$ *satisfy*

$$\max\{R_1|\omega_1|, R_2|\omega_2|\} < C_1(\nu, R_1, R_2) \tag{1.7}$$

and

$$\|\boldsymbol{v}\|_{L^\infty(\varOmega)} < C_1(\nu, R_1, R_2), \tag{1.8}$$

respectively, then $(\boldsymbol{v}, p)$ *must be the generalized Taylor–Couette–Poiseuille flow:*

$$\left.\begin{aligned} v^r &\equiv 0, \\ v^\theta &= Ar + B\frac{1}{r}, \\ v^z &= \frac{a}{4\nu}R_1^2\left[\left(\frac{r}{R_1}\right)^2 - 1 + \frac{1 - \eta^2}{\eta^2 \log \eta}\log\left(\frac{r}{R_1}\right)\right], \\ p &= az + b + \frac{A^2}{2}r^2 + 2AB\log r - \frac{B^2}{2}\frac{1}{r^2}, \end{aligned}\right\} \tag{1.9}$$

with some constants $a, b \in \mathbb{R}$, *where the constants A and B are the same as in* (1.5*a,b*). *In particular, if the pressure p is bounded or periodic in z, then the constant a in* (1.9) *must be zero, and hence we have* $v^z \equiv 0$, *which means that* $\boldsymbol{v}$ *coincides with the well-known canonical Taylor–Couette flow given by* (1.4).

Remark 1.1. (i) Since the boundary condition (1.3) implies $\max\{R_1|\omega_1|, R_2|\omega_2|\} \leq \|\boldsymbol{v}\|_{L^\infty}$, the condition (1.7) is necessary for (1.8).

(ii) From the proof of theorem 1.1, we may take $C_1(\nu, R_1, R_2)$ as

$$C_1(\nu, R_1, R_2) = \frac{\nu}{2\sqrt{C_P}}, \tag{1.10}$$

where $C_P := R_2(R_2 - R_1)^2/R_1\pi^2$ is related to the Poincaré inequality. This implies that if the viscosity ν is large in comparison with the radii R_1 and R_2, then the fluid motion remains as laminar flow, i.e. the generalized Taylor–Couette–Poiseuille flow (1.9). It should be noted that our assumptions (1.7) and (1.8) do not need to impose any smallness on the pressure p of (1.2).

(iii) Taylor (1923) introduced a Taylor number Ta in a thin gap $\eta \approx 1$ defined by

$$Ta \equiv -\frac{2A\omega_1 R_2^4}{\nu^2}(1-\eta)^4(1+\mu). \tag{1.11}$$

There is a critical value Ta_c such that if $Ta < Ta_c$, then the Taylor–Couette flow (1.4) is stable. For simplicity, we consider the case when $\omega_2 = 0$, i.e. $\mu = 0$. Then it holds that

$$Ta = \frac{2}{\nu^2}\omega_1^2 R_2^4 \frac{\eta^2}{1+\eta}(1-\eta)^3 = \frac{2}{\nu^2}R_2^2(R_1\omega_1)^2\frac{1}{1+\eta}(1-\eta)^3. \tag{1.12}$$

By (1.7) and (1.10) we have

$$Ta < \frac{\pi^2}{2}\frac{\eta}{1+\eta}(1-\eta). \tag{1.13}$$

Hence it follows from theorem 1.1 that under hypotheses of (1.13) and (1.8), the Taylor–Couette–Poiseuille flow (1.9) is the unique solution of (1.2) with (1.3). On the other hand, Taylor (1923) showed that $Ta_c = 1708$, from which the Taylor–Couette flow (1.4) is stable under the condition that

$$Ta < 1708. \tag{1.14}$$

In comparison with (1.13) and (1.14), our uniqueness result seems to be quite restrictive from the viewpoint of the Taylor number Ta. However, we should emphasize that uniqueness of solutions necessarily requires more restrictive conditions than those of stability.

(iv) In the non-dimensional form of the (1.2), the Reynolds numbers Re_j are defined by

$$Re_j = \frac{R_j\omega_j(R_2 - R_1)}{\nu} \quad (j = 1, 2). \tag{1.15}$$

Then, (1.10) implies that the assumption (1.7) is written as

$$\max\{Re_1, Re_2\} < \frac{R_2 - R_1}{\nu}C_1(\nu, R_1, R_2) = \frac{\pi\sqrt{R_1}}{2\sqrt{R_2}} = \frac{\pi}{2}\sqrt{\eta}. \tag{1.16}$$

Namely, theorem 1.1 implies that if the Reynolds numbers of the inner and outer cylinders and the velocity are bounded by a certain constant determined only by means of radii R_1 and R_2, then the axisymmetric flow must be necessarily the generalized Taylor–Couette–Poiseuille flow (1.9). Matsukawa & Tsukahara (2022) performed direct numerical simulation for $\eta = 0.833$, and showed that in the set $(Re_1, Re_2) = (400, -1000)$, the Taylor–Couette–Poiseuille flow becomes turbulent. For such an η, we have $\frac{\pi}{2}\sqrt{\eta} = 1.476$ and, hence, under the hypothesis $\max\{Re_1, Re_2\} < 1.476$ with (1.8) the Taylor–Couette–Poiseuille flow (1.9) is the unique solution of (1.2) with (1.3). Since our result is on uniqueness of solutions, it may be reasonable that the Reynolds numbers are by far the smaller in comparison with occurrence of instability.

(v) Kagei & Nishida (2015) and Kagei & Nishida (2019) investigated the plane Poiseuille flow for the compressible Navier–Stokes equations, and gave a mathematical proof of its instability for Reynolds numbers much less than the critical Reynolds number when the Mach number is suitably large.

(vi) The assumption that the pressure p is bounded or periodic in z seems physically reasonable, and hence in a possible physical situation, the laminar axially symmetric flow $\boldsymbol{v}$ in the two rotating concentric cylinder is necessarily the canonical Taylor–Couette flow (1.4).

(vii) Temam (1977, ch. II, § 4) studied the uniqueness and the non-uniqueness of the problem (1.2) in the case of $\omega_2 = 0$ provided that the flow $\boldsymbol{v}$ of (1.2) is periodic in z. Introducing the disturbance $\boldsymbol{u} = (u^r, u^\theta, u^z)$ such that $\boldsymbol{v}$ has the form $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{u}$ with $\boldsymbol{v}_0$ denoting the Taylor–Couette flow (1.4), he reduced such a question on uniqueness as to whether $\boldsymbol{u} \equiv 0$ in $\varOmega$. It was proved in Temam (1977, ch. II, proposition 4.2) that, under smallness hypotheses of the Reynolds number, $\boldsymbol{u} \equiv 0$ provided that (u^r, u^z) is written by the stream function ψ in the coordinate (r, z) satisfying $\partial_r \psi(r_1, z) = \partial_r \psi(r_2, z) = 0$. In comparison with Temam's result, we remove the assumption of periodicity in z and avoid making use of such a stream function, although we impose on $\boldsymbol{v}$ the smallness condition (1.8).

1.2. *General case*

Next, we treat the general case in which axial symmetry is not necessarily assumed. Consider the same region $\varOmega$ as in § 1.1, and the incompressible stationary Navier–Stokes equations in cylindrical coordinates in $\varOmega$:

$$\left.\begin{aligned}
(\boldsymbol{v}\cdot\nabla)v^r - \frac{(v^\theta)^2}{r} + \partial_r p &= \nu\left(\Delta - \frac{1}{r^2}\right)v^r - \nu\frac{2}{r^2}\partial_\theta v^\theta,\\
(\boldsymbol{v}\cdot\nabla)v^\theta + \frac{v^r v^\theta}{r} + \frac{1}{r}\partial_\theta p &= \nu\left(\Delta - \frac{1}{r^2}\right)v^\theta + \nu\frac{2}{r^2}\partial_\theta v^r,\\
(\boldsymbol{v}\cdot\nabla)v^z + \partial_z p &= \nu\Delta v^z,\\
\frac{1}{r}\partial_r(rv^r) + \frac{1}{r}\partial_\theta v^\theta + \partial_z v^z &= 0,
\end{aligned}\right\} \tag{1.17}$$

where we have used the notations

$$(\boldsymbol{v}\cdot\nabla) = v^r\partial_r + \frac{v^\theta}{r}\partial_\theta + v^z\partial_z, \tag{1.18}$$

$$\varDelta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2. \tag{1.19}$$

Moreover, we also impose on $\boldsymbol{v}$ the same boundary conditions as (1.3), that is,

$$\left.\begin{aligned} v^r(R_j,\theta,z) = v^z(R_j,\theta,z) = 0 \quad (j=1,2), \\ v^\theta(R_1,\theta,z) = R_1\omega_1, \quad v^\theta(R_2,\theta,z) = R_2\omega_2, \end{aligned}\right\} \tag{1.20}$$

with some $\omega_1, \omega_2 \in \mathbb{R}$.

For the general case, under similar assumptions to (1.7) and (1.8), we have the following Liouville-type theorem for $(\partial_\theta \boldsymbol{v}, \partial_\theta p)$ which shows axial symmetry of the solutions to (1.17).

THEOREM 1.2. *Let $(\boldsymbol{v}, p)$ be a smooth solution of (1.17) in Ω with the boundary conditions (1.20). There exists a constant $C_2(\nu, R_1, R_2) > 0$ such that if ω_1, ω_2 and $\|\boldsymbol{v}\|_{L^\infty}$ satisfy*

$$\max\{R_1|\omega_1|, R_2|\omega_2|\} < C_2(\nu, R_1, R_2) \tag{1.21}$$

and

$$\|\boldsymbol{v}\|_{L^\infty(\Omega)} < C_2(\nu, R_1, R_2), \tag{1.22}$$

respectively, then it holds that $\partial_\theta \boldsymbol{v} \equiv 0$ and $\partial_\theta p \equiv 0$ in Ω, that is, $(\boldsymbol{v}, p)$ is axially symmetric.

Remark 1.2. From the proof of theorem 1.2, we may take the constant $C_2(\nu, R_1, R_2)$ as

$$C_2(\nu, R_1, R_2) = \nu \left(\sqrt{C_P}\left(2 + \frac{C_P}{R_1^2}\right) + \frac{3C_P}{2R_1} \right)^{-1}, \tag{1.23}$$

where $C_P := R_2(R_2 - R_1)^2/R_1\pi^2$. We should emphasize that any assumption on smallness of the pressure p of (1.17) is redundant.

Combining theorems 1.1 and 1.2, we immediately reach the following Liouville-type theorem for the general case.

COROLLARY 1.3. *Let $(\boldsymbol{v}, p)$ be a smooth solution of (1.17) in Ω with the boundary conditions (1.20). Let $C_1(\nu, R_1, R_2)$ and $C_2(\nu, R_1, R_2)$ be the same constants as in theorems 1.1 and 1.2, respectively. We set $C_*(\nu, R_1, R_2) \equiv \min\{C_1(\nu, R_1, R_2), C_2(\nu, R_1, R_2)\}$. Suppose that $(\boldsymbol{v}, p)$ is a smooth solution of (1.17) in Ω with the boundary conditions (1.20). If ω_1, ω_2 and $\boldsymbol{v}$ satisfy*

$$\max\{R_1|\omega_1|, R_2|\omega_2|\} < C_*(\nu, R_1, R_2) \tag{1.24}$$

and

$$\|\boldsymbol{v}\|_{L^\infty(\Omega)} < C_*(\nu, R_1, R_2), \tag{1.25}$$

respectively, then $(\boldsymbol{v}, p)$ is axially symmetric and coincides with the generalized Taylor–Couette–Poiseuille flow given by (1.9). In particular, if p is bounded or periodic in z, then $\boldsymbol{v}$ is necessarily the canonical Taylor–Couette flow (1.4).

Remark 1.3. It is easy to see that the generalized Taylor–Couette–Poiseuille flow (1.9) is also a solution of the Stokes equations in Ω with the same boundary condition (1.3). Hence without any assumption on smallness of $\|\boldsymbol{v}\|_{L^\infty(\Omega)}$ as in (1.7) it holds that any bounded smooth solution $\boldsymbol{v}$ of the Stokes equations uniquely coincides with the generalized Taylor–Couette–Poiseuille flow (1.9) . In particular, if the pressure p is bounded or periodic in z, then $\boldsymbol{v}$ is necessarily the canonical Taylor–Couette flow (1.4). This may be regarded as the Liouville-type theorem on the Stokes equations.

2. Preliminaries

In what follows, C denotes generic constants which may change from line to line. Also, the operators $\nabla_{r,z}$ and $\nabla_{r,\theta,z}$ stand for $\nabla_{r,z} f(r, z) = (\partial_r f, \partial_z f)(r, z)$ and $\nabla_{r,\theta,z} f(r, \theta, z) = (\partial_r f, \partial_\theta f, \partial_z f)(r, \theta, z)$, respectively.

We first state the boundedness of derivatives of solutions. This can be proved in the same way as in Bang *et al.* (2023, lemma 2.3) in which the three-dimensional slab domain is treated, since all estimates in the proof are local and do not depend on the shape of $\varOmega$.

LEMMA 2.1. *Let* $(\boldsymbol{v}, p)$ *be a smooth solution of the Navier–Stokes equations* (1.1) *in* $\varOmega$ *with the boundary conditions* (1.3). *Assume that* v *is bounded. Then,* $\nabla_{r,\theta,z}\boldsymbol{v}$, $\nabla^2_{r,\theta,z}\boldsymbol{v}$ *and* $\nabla_{r,\theta,z} p$ *are also bounded.*

Next, we prepare the test function used in this paper. Let $L > 1$ and define

$$\varphi_L(z) = \begin{cases} 1 & (|z| < L - 1), \\ L - |z| & (L - 1 \le |z| \le L), \\ 0 & (|z| > L). \end{cases} \tag{2.1}$$

Also, we put

$$\varSigma_L = \{x \in \varOmega \mid L - 1 \le |z| \le L\}. \tag{2.2}$$

Note that $\operatorname{supp}\partial_z\varphi_L \subset \varSigma_L$.

Finally, we prove a Poincaré-type inequality for the r direction in $\varOmega$ and $\varSigma_L$, which will be used for $\partial_z\boldsymbol{v}$ and $\partial_\theta\boldsymbol{v}$.

LEMMA 2.2. *Let* $f = f(r, \theta, z)$ *be a smooth function on* $\bar{\varOmega}$ *satisfying the boundary condition* $f(R_j, \theta, z) = 0$ $(j = 1, 2)$. *Let* $L > 1$. *Then, we have*

$$\|f\sqrt{\varphi_L}\|_{L^2(D)} \le \sqrt{C_P}\|\partial_r f\sqrt{\varphi_L}\|_{L^2(D)}, \tag{2.3}$$

where D *denotes* $\varOmega$ *or* $\varSigma_L$ *and*

$$C_P = \frac{R_2(R_2 - R_1)^2}{R_1\pi^2}. \tag{2.4}$$

Proof. When $D = \varOmega$, using cylindrical coordinates and applying the Poincaré inequality in the r direction, we calculate

$$\begin{aligned} \|f\sqrt{\varphi_L}\|^2_{L^2(\varOmega)} &= \int_{\mathbb{R}}\int_0^{2\pi} \|f\sqrt{r}\|^2_{L^2(R_1,R_2)}\varphi_L(z)\,\mathrm{d}\theta\,\mathrm{d}z \\ &\le R_2\int_{\mathbb{R}}\int_0^{2\pi} \|f\|^2_{L^2(R_1,R_2)}\varphi_L(z)\,\mathrm{d}\theta\,\mathrm{d}z \\ &\le R_2\frac{(R_2 - R_1)^2}{\pi^2}\int_{\mathbb{R}}\int_0^{2\pi} \|\partial_r f\|^2_{L^2(R_1,R_2)}\varphi_L(z)\,\mathrm{d}\theta\,\mathrm{d}z \\ &\le \frac{R_2(R_2 - R_1)^2}{R_1\pi^2}\int_{\mathbb{R}}\int_0^{2\pi} \|\partial_r f\sqrt{r}\|^2_{L^2(R_1,R_2)}\varphi_L(z)\,\mathrm{d}\theta\,\mathrm{d}z \\ &= C_P\|\partial_r f\sqrt{\varphi_L}\|^2_{L^2(\varOmega)}. \end{aligned} \tag{2.5}$$

The case $D = \varSigma_L$ can be proved in completely the same way.

3. Proof of theorem 1.1

Let us prove theorem 1.1. Assume that $(\boldsymbol{v}, p)$ is an axially symmetric smooth solution of (1.2) in $\varOmega$. Following the argument of Bang *et al.* (2023), we first show

$$\partial_z \boldsymbol{v} \equiv 0. \tag{3.1}$$

To this end, we differentiate (1.2) with respect to z and obtain

$$\left.\begin{aligned}
&(\partial_z v^r \partial_r + \partial_z v^z \partial_z) v^r + (v^r \partial_r + v^z \partial_z)\partial_z v^r - \frac{2 v^\theta \partial_z v^\theta}{r} + \partial_z \partial_r p \\
&\qquad = \nu \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2} \right) \partial_z v^r, \\
&(\partial_z v^r \partial_r + \partial_z v^z \partial_z) v^\theta + (v^r \partial_r + v^z \partial_z)\partial_z v^\theta + \frac{v^\theta \partial_z v^r + v^r \partial_z v^\theta}{r} \\
&\qquad = \nu \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2} \right) \partial_z v^\theta, \\
&(\partial_z v^r \partial_r + \partial_z v^z \partial_z) v^z + (v^r \partial_r + v^z \partial_z)\partial_z v^z + \partial_z^2 p = \nu \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 \right) \partial_z v^z, \\
&\qquad\qquad \partial_z \partial_r v^r + \frac{\partial_z v^r}{r} + \partial_z^2 v^z = 0.
\end{aligned}\right\} \tag{3.2}$$

Moreover, we have the boundary conditions for $\partial_z v$:

$$\partial_z v(R_j, z) = 0 \quad (j = 1, 2). \tag{3.3}$$

Let $L > 1$ and take the test function φ_L and the region $\varSigma_L$ defined by (2.1) and (2.2), respectively. We multiply the equations of v^r, v^θ, v^z in (3.2) by $\partial_z v^r \varphi_L(z)$, $\partial_z v^\theta \varphi_L(z)$, $\partial_z v^z \varphi_L(z)$, respectively, and sum them and integrate them over $\varOmega$. Then, we have the integral identity

$$I = II + III + IV + V. \tag{3.4}$$

Here, $I = I^r + I^\theta + I^z$ is the sum related to the viscous terms of the right-hand side of (3.2) defined by

$$I^\lambda = \nu \int_\varOmega \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2} \right) \partial_z v^\lambda \partial_z v^\lambda \varphi_L(z)\, \mathrm{d}x \quad (\lambda = r, \theta), \tag{3.5}$$

$$I^z = \nu \int_\varOmega \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 \right) \partial_z v^z \partial_z v^z \varphi_L(z)\, \mathrm{d}x. \tag{3.6}$$

Terms II, III and IV are related to the nonlinear terms of the left-hand side of (3.2) defined by

$$II = \sum_{\lambda=r,\theta,z} II^\lambda = \sum_{\lambda=r,\theta,z} \int_\Omega \left(\partial_z v^r \partial_r + \partial_z v^z \partial_z\right) v^\lambda \partial_z v^\lambda \varphi_L(z)\,\mathrm{d}x, \tag{3.7}$$

$$III = \sum_{\lambda=r,\theta,z} III^\lambda = \sum_{\lambda=r,\theta,z} \int_\Omega \left(v^r \partial_r + v^z \partial_z\right) \partial_z v^\lambda \partial_z v^\lambda \varphi_L(z)\,\mathrm{d}x, \tag{3.8}$$

$$\begin{aligned} IV &= -2\int_\Omega \frac{v^\theta \partial_z v^\theta}{r} \partial_z v^r \varphi_L(z)\,\mathrm{d}x + \int_\Omega \frac{v^\theta \partial_z v^r + v^r \partial_z v^\theta}{r} \partial_z v^\theta \varphi_L(z)\,\mathrm{d}x \\ &= \int_\Omega \frac{1}{r}\left(v^r \partial_z v^\theta - v^\theta \partial_z v^r\right) \partial_z v^\theta \varphi_L(z)\,\mathrm{d}x, \end{aligned} \tag{3.9}$$

respectively. Finally, term V is the sum related to the pressure terms of (3.2) defined by

$$V = \int_\Omega (\partial_z \partial_r p \partial_z v^r + \partial_z^2 p \partial_z v^z) \varphi_L(z)\,\mathrm{d}x. \tag{3.10}$$

First, we compute the viscous terms I. For $\lambda = r, \theta$, integration by parts with noting that $r(\partial_r^2 + (1/r)\partial_r) = \partial_r(r\partial_r)$ implies

$$\begin{aligned} I^\lambda &= \nu \int_\Omega \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2}\right) \partial_z v^\lambda \partial_z v^\lambda \varphi_L(z)\,\mathrm{d}x \\ &= -2\pi\nu \int_{\mathbb{R}} \int_{R_1}^{R_2} \left(|\partial_r \partial_z v^\lambda|^2 + |\partial_z^2 v^\lambda|^2 + \frac{|\partial_z v^\lambda|^2}{r^2}\right) \varphi_L(z) r\,\mathrm{d}r\,\mathrm{d}z \\ &\quad - 2\pi\nu \int_{\mathbb{R}} \int_{R_1}^{R_2} \partial_z^2 v^\lambda \partial_z v^\lambda \partial_z \varphi_L(z) r\,\mathrm{d}r\,\mathrm{d}z \\ &=: I_1^\lambda + I_2^\lambda. \end{aligned} \tag{3.11}$$

In the same way, for I^z, we have

$$\begin{aligned} I^z &= \nu \int_\Omega \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right) \partial_z v^z \partial_z v^z \varphi_L(z)\,\mathrm{d}x \\ &= -2\pi\nu \int_{\mathbb{R}} \int_{R_1}^{R_2} (|\partial_r \partial_z v^z|^2 + |\partial_z^2 v^z|^2) \varphi_L(z) r\,\mathrm{d}r\,\mathrm{d}z \\ &\quad - 2\pi\nu \int_{\mathbb{R}} \int_{R_1}^{R_2} \partial_z^2 v^z \partial_z v^z \partial_z \varphi_L(z) r\,\mathrm{d}r\,\mathrm{d}z \\ &=: I_1^z + I_2^z. \end{aligned} \tag{3.12}$$

Now, for a later purpose, we express the sum of good terms by $Y(L)$:

$$\begin{aligned} Y(L) &:= -(I_1^r + I_1^\theta + I_1^z) \\ &= \nu \left(\|\partial_r \partial_z \boldsymbol{v} \sqrt{\varphi_L}\|_{L^2(\Omega)}^2 + \|\partial_z^2 \boldsymbol{v} \sqrt{\varphi_L}\|_{L^2(\Omega)}^2 + \sum_{\lambda=r,\theta} \|r^{-1} \partial_z v^\lambda \sqrt{\varphi_L}\|_{L^2(\Omega)}^2 \right). \end{aligned} \tag{3.13}$$

We remark that the definition of $\varphi_L(z)$ (see (2.1)) implies

$$Y'(L) = \nu \left(\|\partial_r \partial_z \boldsymbol{v}\|^2_{L^2(\varSigma_L)} + \|\partial_z^2 \boldsymbol{v}\|^2_{L^2(\varSigma_L)} + \sum_{\lambda=r,\theta} \|r^{-1}\partial_z v^\lambda\|^2_{L^2(\varSigma_L)} \right). \tag{3.14}$$

By using this $Y(L)$, term I can be written as

$$I = -Y(L) + \sum_{\lambda=r,\theta,z} I_2^\lambda. \tag{3.15}$$

Let us estimate the remainder terms I_2^λ for $\lambda = r, \theta, z$. Since $\partial_z v^\lambda \in L^\infty(\varOmega)$ by lemma 2.1 and $|\varSigma_L| \le C$ with some constant C independent of L, we estimate

$$|I_2^\lambda| \le C\|\partial_z^2 v^\lambda\|_{L^2(\varSigma_L)}\|\partial_z v^\lambda\|_{L^2(\varSigma_L)} \le C\|\partial_z^2 \boldsymbol{v}\|_{L^2(\varSigma_L)} \le C\sqrt{Y'(L)}, \tag{3.16}$$

where the constant C is independent of L.

Next, we consider the nonlinear term II. By integration by parts and the divergence-free condition $\partial_r(r\partial_z v^r) + \partial_z(r\partial_z v^z) = 0$, terms II^λ for $\lambda = r, \theta, z$ are written as

$$\begin{aligned} II^\lambda &= -2\pi \int_{\mathbb{R}} \int_{R_1}^{R_2} (\partial_z v^r \partial_r + \partial_z v^z \partial_z)\partial_z v^\lambda v^\lambda \varphi_L(z) r \,\mathrm{d}r\,\mathrm{d}z \\ &\quad - 2\pi \int_{\mathbb{R}} \int_{R_1}^{R_2} \partial_z v^z v^\lambda \partial_z v^\lambda \partial_z \varphi_L(z) r \,\mathrm{d}r\,\mathrm{d}z \\ &=: II_1^\lambda + II_2^\lambda. \end{aligned} \tag{3.17}$$

The Hölder inequality implies

$$\begin{aligned} |II_1^\lambda| &\le \|\partial_z v^r \sqrt{\varphi_L}\|_{L^2(\varOmega)} \|\partial_r \partial_z v^\lambda \sqrt{\varphi_L}\|_{L^2(\varOmega)} \|\boldsymbol{v}\|_{L^\infty(\varOmega)} \\ &\quad + \|\partial_z v^z \sqrt{\varphi_L}\|_{L^2(\varOmega)} \|\partial_z^2 v^\lambda \sqrt{\varphi_L}\|_{L^2(\varOmega)} \|\boldsymbol{v}\|_{L^\infty(\varOmega)}. \end{aligned} \tag{3.18}$$

Moreover, by noting the boundary condition (3.3) and applying lemma 2.2, we further estimate

$$\begin{aligned} |II_1^\lambda| &\le \|\boldsymbol{v}\|_{L^\infty(\varOmega)} \sqrt{C_P} (\|\partial_r \partial_z v^r \sqrt{\varphi_L}\|_{L^2(\varOmega)} \|\partial_r \partial_z v^\lambda \sqrt{\varphi_L}\|_{L^2(\varOmega)} \\ &\quad + \|\partial_r \partial_z v^z \sqrt{\varphi_L}\|_{L^2(\varOmega)} \|\partial_z^2 v^\lambda \sqrt{\varphi_L}\|_{L^2(\varOmega)}). \end{aligned} \tag{3.19}$$

Hence, to term $II_1 := II_1^r + II_1^\theta + II_1^z$, we apply the Schwarz inequality to conclude

$$\begin{aligned} |II_1| &\le \|\boldsymbol{v}\|_{L^\infty(\varOmega)} \sqrt{C_P} \left(2\|\partial_r \partial_z v^r \sqrt{\varphi_L}\|^2_{L^2(\varOmega)} + \frac{1}{2} \sum_{\lambda=r,\theta,z} \|\partial_r \partial_z v^\lambda \sqrt{\varphi_L}\|^2_{L^2(\varOmega)} \right. \\ &\quad \left. + 2\|\partial_r \partial_z v^z \sqrt{\varphi_L}\|^2_{L^2(\varOmega)} + \frac{1}{2} \sum_{\lambda=r,\theta,z} \|\partial_z^2 v^\lambda \sqrt{\varphi_L}\|^2_{L^2(\varOmega)} \right). \end{aligned} \tag{3.20}$$

Note that the terms in parentheses are members of $Y(L)$. Similarly to (3.16) we have by lemma 2.1 that

$$\|\partial_z \boldsymbol{v}\|_{L^2(\varSigma_L)} \le \|\partial_z \boldsymbol{v}\|_{L^\infty(\varOmega)} |\varSigma_L|^{1/2} \le C, \tag{3.21}$$

with the constant C independent of L. Hence, applying the Poincáre inequality in $\varSigma_L$ with the aid of (3.3) to the estimate for term $II_2 := II_2^r + II_2^\theta + II_2^z$, we have by the Hölder inequality that

$$|II_2| \le C\|\partial_z \boldsymbol{v}\|^2_{L^2(\Sigma_L)}\|\boldsymbol{v}\|_{L^\infty(\Omega)} \le C\|\partial_r\partial_z \boldsymbol{v}\|_{L^2(\Sigma_L)} \le C\sqrt{Y'(L)}. \tag{3.22}$$

Let us estimate term III. For $\lambda = r, \theta, z$, we write III^λ as

$$III^\lambda = 2\pi \int_{\mathbb{R}}\int_{R_1}^{R_2} (v^r\partial_r + v^z\partial_z)\left(\frac{1}{2}|\partial_z v^\lambda|^2\right)\varphi_L(z) r\,\mathrm{d}r\,\mathrm{d}z. \tag{3.23}$$

Since the divergence-free condition means that $\partial_r(rv^r) + \partial_z(rv^z) = 0$, we have by integration by parts with the aid of (3.3) that

$$III^\lambda = -\pi \int_{\mathbb{R}}\int_{R_1}^{R_2} v^z\partial_z\varphi_L(z)|\partial_z v^\lambda|^2 r\,\mathrm{d}r\,\mathrm{d}z. \tag{3.24}$$

Then, in the same way as for term II_2, we obtain

$$|III| \le C\|\partial_r\partial_z \boldsymbol{v}\|_{L^2(\Sigma_L)} \le C\sqrt{Y'(L)}. \tag{3.25}$$

The remaining nonlinear term IV can be treated similarly to I. Indeed, using the Hölder inequality and lemma 2.2, we have

$$\begin{aligned}
|IV| &\le \|r^{-1}\partial_z v^r\sqrt{\varphi_L}\|_{L^2(\Omega)}\|\partial_z v^\theta\sqrt{\varphi_L}\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^\infty(\Omega)} \\
&\quad + \|r^{-1}\partial_z v^\theta\sqrt{\varphi_L}\|_{L^2(\Omega)}\|\partial_z v^\theta\sqrt{\varphi_L}\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^\infty(\Omega)} \\
&\le \|\boldsymbol{v}\|_{L^\infty(\Omega)}\sqrt{C_P}(\|r^{-1}\partial_z v^r\sqrt{\varphi_L}\|_{L^2(\Omega)} + \|r^{-1}\partial_z v^\theta\sqrt{\varphi_L}\|_{L^2(\Omega)}) \\
&\quad \times \|\partial_r\partial_z v^\theta\sqrt{\varphi_L}\|_{L^2(\Omega)} \\
&\le \|\boldsymbol{v}\|_{L^\infty(\Omega)}\sqrt{C_P} \\
&\quad \times \left(\frac{1}{2}\sum_{\lambda=r,\theta}\|r^{-1}\partial_z v^\lambda\sqrt{\varphi_L}\|^2_{L^2(\Omega)} + \|\partial_r\partial_z v^\theta\sqrt{\varphi_L}\|^2_{L^2(\Omega)}\right).
\end{aligned} \tag{3.26}$$

We again note that the terms in parentheses are members of $Y(L)$.

Finally, we estimate the pressure term V. Since the divergence-free condition yields $\partial_r(r\partial_z v^r) + \partial_z(r\partial_z v^z) = 0$, we have by integration by parts with (3.3) that

$$\begin{aligned}
V &= 2\pi \int_{\mathbb{R}}\int_{R_1}^{R_2} (\partial_z\partial_r p\partial_z v^r + \partial_z^2 p\partial_z v^z)\varphi_L(z) r\,\mathrm{d}r\,\mathrm{d}z \\
&= -\int_{\mathbb{R}}\int_{R_1} \partial_z p\partial_z v^z\partial_z\varphi_L(z) r\,\mathrm{d}r\,\mathrm{d}z.
\end{aligned} \tag{3.27}$$

By $\partial_z p \in L^\infty(\Omega)$, which follows from lemma 2.1, $|\Sigma_L| \le C$ with some constant C independent of L and lemma 2.2, we obtain

$$|V| \le C\|\partial_z p\|_{L^2(\Sigma_L)}\|\partial_z \boldsymbol{v}\|_{L^2(\Sigma_L)} \le C\|\partial_r\partial_z \boldsymbol{v}\|_{L^2(\Sigma_L)} \le C\sqrt{Y'(L)}. \tag{3.28}$$

Putting the estimates (3.15)–(3.26) and (3.28) together into the original integral identity (3.4), we have

$$Y(L) \le \frac{2\sqrt{C_P}}{\nu}\|\boldsymbol{v}\|_{L^\infty(\Omega)} Y(L) + C\sqrt{Y'(L)}. \tag{3.29}$$

Therefore, putting $C_1(\nu, R_1, R_2) := \nu/2\sqrt{C_P}$ and using the assumption (1.8) on $\|\boldsymbol{v}\|_{L^\infty(\Omega)}$, we see that the first term on the right-hand side can be absorbed to the left-hand side.

Hence, we conclude

$$Y(L) \leq C\sqrt{Y'(L)}. \tag{3.30}$$

Note that the constant C on the right-hand side is independent of L.

The differential inequality (3.30) enables us to reach the first goal (3.1). Indeed, by (3.30) it holds that

$$Y(L) = 0 \quad \text{for all } L > 1. \tag{3.31}$$

Suppose the contrary. Then there exists some $L_0 > 1$ such that $Y(L_0) > 0$. Since $Y(L)$ is a non-decreasing function of L, we have $Y(L) > 0$ for all $L \geq L_0$. Thus, from (3.30), we deduce for $L > L_0$ that

$$1 \leq C^2 Y(L)^{-2} Y'(L) = C^2(-Y(L)^{-1})'. \tag{3.32}$$

Integrating it over $[L_0, L]$ leads to

$$L - L_0 \leq C^2(-Y(L)^{-1} + Y(L_0)^{-1}) \leq C^2 Y(L_0)^{-1}. \tag{3.33}$$

However, letting L be sufficiently large, we reach a contradiction to conclude that $Y(L) = 0$ for all $L > 1$. Now, we have that $\nabla_{r,z}\partial_z \boldsymbol{v} \equiv 0$, which means that the function $\partial_z \boldsymbol{v}$ is a constant vector. Combining this with the boundary condition (3.3) implies that $\partial_z \boldsymbol{v} \equiv 0$. Thus, we have (3.1).

Finally, we show that the solution $(\boldsymbol{v}, p)$ has the form described in the statement of the theorem. First, by $\partial_z v \equiv 0$ and the divergence-free condition, we have

$$\partial_r v^r + \frac{v^r}{r} = \frac{1}{r}\partial_r(rv^r) = 0, \tag{3.34}$$

which shows $\partial_r(rv^r) \equiv 0$, that is, rv^r is a constant. However, the boundary condition $v^r = 0$ on $\partial\Omega$ again implies $v^r \equiv 0$.

Going back to the system (3.2), we have

$$\partial_z\partial_r p = \partial_z^2 p = 0, \tag{3.35}$$

which implies that $\partial_z p = a$ with some constant $a \in \mathbb{R}$. Integrating it gives

$$p(r, z) = az + h(r), \tag{3.36}$$

with some smooth function $h(r)$.

We further go back to the original system (1.2) and determine v^θ, v^z and $h(r)$. First, by noting that $v^r = 0$ and $\boldsymbol{v}$ is independent of z, the second equation of (1.2) yields that v^θ is subject to the equation

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) v^\theta = 0. \tag{3.37}$$

This is the Euler–Cauchy equation and we find the general solution of the form

$$v^\theta = Ar + \frac{B}{r}. \tag{3.38}$$

The boundary condition gives

$$R_2\omega_2 = v^\theta(R_2) = AR_2 + \frac{B}{R_2}, \tag{3.39}$$

$$R_1\omega_1 = v^\theta(R_1) = AR_1 + \frac{B}{R_1}. \tag{3.40}$$

Solving this, we determine the constants A, B and obtain

$$v^\theta(r) = \frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2} r + \frac{R_1^2 R_2^2(-\omega_2 + \omega_1)}{R_2^2 - R_1^2} \frac{1}{r}. \tag{3.41}$$

Next, from the third equation of (1.2) and the formula (3.36), we have the equation of v^z:

$$\nu \left(\partial_r^2 + \frac{1}{r}\partial_r\right) v^z = a, \tag{3.42}$$

that is,

$$\frac{1}{r}\partial_r \left(r\partial_r v^z\right) = \frac{a}{\nu}. \tag{3.43}$$

This implies

$$r\partial_r v^z(r) = D + \frac{a}{2\nu} r^2, \tag{3.44}$$

with some constant D. Integrating it over $[R_1, r]$ and using the boundary condition $v^z(R_1) = 0$ by (1.3), we have

$$v^z(r) = D \log \frac{r}{R_1} + \frac{a}{4\nu}(r^2 - R_1^2). \tag{3.45}$$

From the boundary condition $v^z(R_2) = 0$ by (1.3), the constant D is determined as

$$D = -\frac{a}{4\nu} \frac{R_2^2 - R_1^2}{\log R_2/R_1}. \tag{3.46}$$

Thus, we conclude that

$$v^z(r) = \frac{a}{4\nu}\left[(r^2 - R_1^2) - \frac{R_2^2 - R_1^2}{\log(R_2/R_1)} \log \frac{r}{R_1}\right]. \tag{3.47}$$

Finally, from the first equation of (1.2), we deduce

$$-\frac{(v^\theta)^2}{r} + \partial_r p = 0. \tag{3.48}$$

This and the formulas (3.36) and (3.41) lead to

$$\begin{aligned} h'(r) &= \frac{1}{r}\left[\frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2} r + \frac{R_1^2 R_2^2(-\omega_2 + \omega_1)}{R_2^2 - R_1^2} \frac{1}{r}\right]^2 \\ &= \left(\frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2}\right)^2 r + \frac{2R_1^2 R_2^2(-\omega_2 + \omega_1)(R_2^2\omega_2 - R_1\omega_1)}{(R_2^2 - R_1^2)^2} \frac{1}{r} \\ &\quad + \left(\frac{R_1^2 R_2^2(-\omega_2 + \omega_1)}{R_2^2 - R_1^2}\right)^2 \frac{1}{r^3}. \end{aligned} \tag{3.49}$$

Integrating it, we have

$$h(r) = b + \frac{1}{2}\left(\frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2}\right)^2 r^2 + \frac{2R_1^2R_2^2(-\omega_2+\omega_1)(R_2^2\omega_2 - R_1^2\omega_1)}{(R_2^2-R_1^2)^2}\log r - \frac{1}{2}\left(\frac{R_1^2R_2^2(-\omega_2+\omega_1)}{R_2^2-R_1^2}\right)^2\frac{1}{r^2}, \tag{3.50}$$

with some constant $b \in \mathbb{R}$, that is, the pressure is given by

$$p(r,z) = az + b + \frac{1}{2}\left(\frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2}\right)^2 r^2 + \frac{2R_1^2R_2^2(-\omega_2+\omega_1)(R_2^2\omega_2 - R_1^2\omega_1)}{(R_2^2-R_1^2)^2}\log r - \frac{1}{2}\left(\frac{R_1^2R_2^2(-\omega_2+\omega_1)}{R_2^2-R_1^2}\right)^2\frac{1}{r^2}. \tag{3.51}$$

Rewriting (3.41), (3.47) and (3.51) by using μ and η defined by (1.5*a*,*b*) completes the proof of theorem 1.1.

4. Proof of theorem 1.2

Let us prove theorem 1.2. Assume that $(\boldsymbol{v}, p)$ is a smooth solution to (1.17). We differentiate (1.17) with respect to θ to obtain

$$\left.\begin{aligned}
&(\partial_\theta \boldsymbol{v}\cdot\nabla)v^r + (\boldsymbol{v}\cdot\nabla)\partial_\theta v^r - \frac{2v^\theta\partial_\theta v^\theta}{r} + \partial_\theta\partial_r p \\
&\quad = \nu\left(\Delta - \frac{1}{r^2}\right)\partial_\theta v^r - \nu\frac{2}{r^2}\partial_\theta^2 v^\theta, \\
&(\partial_\theta \boldsymbol{v}\cdot\nabla)v^\theta + (\boldsymbol{v}\cdot\nabla)\partial_\theta v^\theta + \frac{v^r\partial_\theta v^\theta + v^\theta\partial_\theta v^r}{r} + \frac{1}{r}\partial_\theta^2 p \\
&\quad = \nu\left(\Delta - \frac{1}{r^2}\right)\partial_\theta v^\theta + \nu\frac{2}{r^2}\partial_\theta^2 v^r, \\
&(\partial_\theta \boldsymbol{v}\cdot\nabla)v^z + (\boldsymbol{v}\cdot\nabla)\partial_\theta v^z + \partial_\theta\partial_z p = \nu\Delta\partial_\theta v^z, \\
&\frac{1}{r}\partial_r(r\partial_\theta v^r) + \frac{1}{r}\partial_\theta^2 v^\theta + \partial_z\partial_\theta v^z = 0.
\end{aligned}\right\} \tag{4.1}$$

Here, we recall that the operators $(\boldsymbol{v}\cdot\nabla)$ and Δ are defined by (1.18) and (1.19), respectively. We also have the boundary conditions

$$\partial_\theta \boldsymbol{v}(R_j,\theta,z) = 0 \quad (j=1,2). \tag{4.2}$$

Let $L > 1$ and take the test function $\varphi_L(z)$ and the region Σ_L defined by (2.1) and (2.2), respectively. Similarly to the previous section, we multiply the equations of v^r, v^θ, v^z in (4.1) by $\partial_\theta v^r\varphi_L(z)$, $\partial_\theta v^\theta\varphi_L(z)$, $\partial_\theta v^z\varphi_L(z)$, respectively, and sum up and integrate them over Ω. As a result, we have the integral identity

$$I = II + III + IV + V. \tag{4.3}$$

Here, $I = I^r + I^\theta + I^z$ is the sum of the viscous term defined by

$$I^r = \nu \int_\Omega \left[\left(\Delta - \frac{1}{r^2} \right) \partial_\theta v^r \partial_\theta v^r - \frac{2}{r^2} \partial_\theta^2 v^\theta \partial_\theta v^r \right] \varphi_L(z)\, \mathrm{d}x, \tag{4.4}$$

$$I^\theta = \nu \int_\Omega \left[\left(\Delta - \frac{1}{r^2} \right) \partial_\theta v^\theta \partial_\theta v^\theta + \frac{2}{r^2} \partial_\theta^2 v^r \partial_\theta v^\theta \right] \varphi_L(z)\, \mathrm{d}x, \tag{4.5}$$

$$I^z = \nu \int_\Omega \Delta \partial_\theta v^z \partial_\theta v^z \varphi_L(z)\, \mathrm{d}x. \tag{4.6}$$

Also, II, III and IV are the nonlinear terms defined by

$$II = \sum_{\lambda=r,\theta,z} II^\lambda = \sum_{\lambda=r,\theta,z} \int_\Omega (\partial_\theta \boldsymbol{v} \cdot \nabla) v^\lambda \partial_\theta v^\lambda \varphi_L(z)\, \mathrm{d}x, \tag{4.7}$$

$$III = \sum_{\lambda=r,\theta,z} III^\lambda = \sum_{\lambda=r,\theta,z} \int_\Omega (\boldsymbol{v} \cdot \nabla) \partial_\theta v^\lambda \partial_\theta v^\lambda \varphi_L(z)\, \mathrm{d}x, \tag{4.8}$$

$$\begin{aligned} IV &= \int_\Omega \left(-\frac{2v^\theta \partial_\theta v^\theta}{r} \partial_\theta v^r + \frac{v^r \partial_\theta v^\theta + v^\theta \partial_\theta v^r}{r} \partial_\theta v^\theta \right) \varphi_L(z)\, \mathrm{d}x \\ &= \int_\Omega \frac{1}{r} (v^r \partial_\theta v^\theta - v^\theta \partial_\theta v^r) \partial_\theta v^\theta \varphi_L(z)\, \mathrm{d}x. \end{aligned} \tag{4.9}$$

Finally, V is the sum of the pressure terms defined by

$$V = \int_\Omega \left(\partial_\theta \partial_r p \partial_\theta v^r + \frac{1}{r} \partial_\theta^2 p \partial_\theta v^\theta + \partial_\theta \partial_z p \partial_\theta v^z \right) \varphi_L(z)\, \mathrm{d}x. \tag{4.10}$$

First, we consider term I. By integration by parts with the aid of the boundary condition $\partial_\theta \boldsymbol{v}(R_j, \theta, z) = 0$ for $j = 1, 2$, we infer that

$$\begin{aligned} I^r + I^\theta &= -\nu \int_\Omega \left[|\partial_r \partial_\theta v^r|^2 + \frac{1}{r^2} |\partial_\theta^2 v^r|^2 + |\partial_z \partial_\theta v^r|^2 + \frac{1}{r^2} |\partial_\theta v^r|^2 + \frac{2}{r^2} \partial_\theta^2 v^\theta \partial_\theta v^r \right. \\ &\quad \left. + |\partial_r \partial_\theta v^\theta|^2 + \frac{1}{r^2} |\partial_\theta^2 v^\theta|^2 + |\partial_z \partial_\theta v^\theta|^2 + \frac{1}{r^2} |\partial_\theta v^\theta|^2 - \frac{2}{r^2} \partial_\theta^2 v^r \partial_\theta v^\theta \right] \varphi_L(z)\, \mathrm{d}x \\ &\quad - \nu \int_\Omega (\partial_z \partial_\theta v^r \partial_\theta v^r + \partial_z \partial_\theta v^\theta \partial_\theta v^\theta) \partial_z \varphi_L(z)\, \mathrm{d}x \\ &= -\nu \int_\Omega \left[|\partial_r \partial_\theta v^r|^2 + |\partial_z \partial_\theta v^r|^2 + |\partial_r \partial_\theta v^\theta|^2 + |\partial_z \partial_\theta v^\theta|^2 \right. \\ &\quad \left. + \frac{1}{r^2} |\partial_\theta^2 v^\theta + \partial_\theta v^r|^2 + \frac{1}{r^2} |\partial_\theta^2 v^r - \partial_\theta v^\theta|^2 \right] \varphi_L(z)\, \mathrm{d}x \\ &\quad - \nu \int_\Omega (\partial_z \partial_\theta v^r \partial_\theta v^r + \partial_z \partial_\theta v^\theta \partial_\theta v^\theta) \partial_z \varphi_L(z)\, \mathrm{d}x \\ &=: I_1^{r,\theta} + I_2^{r,\theta} \end{aligned} \tag{4.11}$$

and

$$
\begin{aligned}
I^z &= \nu \int_\Omega \Delta \partial_\theta v^z \partial_\theta v^z \varphi_L(z)\, \mathrm{d}x \\
&= -\nu \int_\Omega \left[|\partial_r \partial_\theta v^z|^2 + \frac{1}{r^2} |\partial_\theta^2 v^z|^2 + |\partial_z \partial_\theta v^z|^2 \right] \varphi_L(z)\, \mathrm{d}x \\
&\quad - \nu \int_\Omega \partial_z \partial_\theta v^z \partial_\theta v^z \partial_z \varphi_L(z)\, \mathrm{d}x \\
&=: I_1^z + I_2^z.
\end{aligned} \tag{4.12}
$$

Similarly to the previous section, we define

$$
\begin{aligned}
Y(L) &:= -(I_1^{r,\theta} + I_1^z) \\
&= \nu \Bigg(\|\partial_r \partial_\theta \boldsymbol{v} \sqrt{\varphi_L}\|_{L^2(\Omega)}^2 + \|\partial_z \partial_\theta \boldsymbol{v} \sqrt{\varphi_L}\|_{L^2(\Omega)}^2 + \left\| \frac{1}{r} (\partial_\theta^2 v^\theta + \partial_\theta v^r) \sqrt{\varphi_L} \right\|_{L^2(\Omega)}^2 \\
&\quad + \left\| \frac{1}{r} (\partial_\theta^2 v^r - \partial_\theta v^\theta) \sqrt{\varphi_L} \right\|_{L^2(\Omega)}^2 + \left\| \frac{1}{r} \partial_\theta^2 v^z \sqrt{\varphi_L} \right\|_{L^2(\Omega)}^2 \Bigg).
\end{aligned} \tag{4.13}
$$

We remark that the definition of $\varphi_L(z)$ (see (2.1)) implies

$$
\begin{aligned}
Y'(L) &:= \nu \Bigg(\|\partial_r \partial_\theta \boldsymbol{v}\|_{L^2(\Sigma_L)}^2 + \|\partial_z \partial_\theta \boldsymbol{v}\|_{L^2(\Sigma_L)}^2 \\
&\quad + \left\| \frac{1}{r} (\partial_\theta^2 v^\theta + \partial_\theta v^r) \right\|_{L^2(\Sigma_L)}^2 + \left\| \frac{1}{r} (\partial_\theta^2 v^r - \partial_\theta v^\theta) \right\|_{L^2(\Sigma_L)}^2 + \left\| \frac{1}{r} \partial_\theta^2 v^z \right\|_{L^2(\Sigma_L)}^2 \Bigg).
\end{aligned} \tag{4.14}
$$

Using the above $Y(L)$, we have

$$
I = -Y(L) + I_2^{r,\theta} + I_2^z. \tag{4.15}
$$

Since $|\Sigma_L| = 2\pi(R_2^2 - R_1^2)$ is independent of L, it follows from the the Schwarz inequality and lemma 2.1 that the remainder term I_2^z is estimated as

$$
\begin{aligned}
|I_2^z| &\le \nu \|\partial_z \partial_\theta v^z\|_{L^2(\Sigma_L)} \|\partial_\theta v^z\|_{L^2(\Sigma_L)} \\
&\le \nu \|\nabla \boldsymbol{v}\|_{L^\infty(\Omega)} |\Sigma_L|^{1/2} \|\partial_z \partial_\theta \boldsymbol{v}\|_{L^2(\Sigma_L)} \\
&\le C \|\partial_z \partial_\theta \boldsymbol{v}\|_{L^2(\Sigma_L)} \\
&\le C \sqrt{Y'(L)},
\end{aligned} \tag{4.16}
$$

with some constant $C > 0$ independent of L. Similarly, it is easy to see that $I_2^{r,\theta}$ has the same bound:

$$
|I_2^{r,\theta}| \le C \sqrt{Y'(L)}. \tag{4.17}
$$

We next estimate the nonlinear term II. For $\lambda = r, \theta, z$, integration by parts and the divergence-free condition imply

$$
\begin{aligned}
II^\lambda &= -\int_\Omega (\partial_\theta \boldsymbol{v} \cdot \nabla) \partial_\theta v^\lambda v^\lambda \varphi_L(z)\, \mathrm{d}x - \int_\Omega \partial_\theta v^z v^\lambda \partial_\theta v^\lambda \partial_z \varphi_L(z)\, \mathrm{d}x \\
&=: II_1^\lambda + II_2^\lambda.
\end{aligned} \tag{4.18}
$$

By the Hölder inequality, term II_1^λ is estimated as

$$
\begin{aligned}
|II_1^\lambda| \le{}& \|\partial_\theta v^r\sqrt{\varphi_L}\|_{L^2(\Omega)}\|\partial_r\partial_\theta v^\lambda\sqrt{\varphi_L}\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^\infty(\Omega)}\\
&+\|\partial_\theta v^\theta\sqrt{\varphi_L}\|_{L^2(\Omega)}\left\|\frac{\partial_\theta^2 v^\lambda}{r}\sqrt{\varphi_L}\right\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^\infty(\Omega)}\\
&+\|\partial_\theta v^z\sqrt{\varphi_L}\|_{L^2(\Omega)}\|\partial_z\partial_\theta v^\lambda\sqrt{\varphi_L}\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^\infty(\Omega)}.
\end{aligned}
\tag{4.19}
$$

Let us further estimate the right-hand side. From lemma 2.2, we obtain for $\kappa = r, \theta, z$ that

$$
\|\partial_\theta v^\kappa\sqrt{\varphi_L}\|_{L^2(\Omega)} \le \sqrt{C_P}\|\partial_r\partial_\theta v^\kappa\sqrt{\varphi_L}\|_{L^2(\Omega)}. \tag{4.20}
$$

Moreover, for the term $\|(\partial_\theta^2 v^\lambda/r)\sqrt{\varphi_L}\|_{L^2(\Omega)}$, by lemma 2.2, we have, for the case $\lambda = r$,

$$
\begin{aligned}
\left\|\frac{\partial_\theta^2 v^r}{r}\sqrt{\varphi_L}\right\|_{L^2(\Omega)} &\le \left\|\frac{1}{r}(\partial_\theta^2 v^r-\partial_\theta v^\theta)\sqrt{\varphi_L}\right\|_{L^2(\Omega)}+\left\|\frac{1}{r}\partial_\theta v^\theta\sqrt{\varphi_L}\right\|_{L^2(\Omega)}\\
&= \left\|\frac{1}{r}(\partial_\theta^2 v^r-\partial_\theta v^\theta)\sqrt{\varphi_L}\right\|_{L^2(\Omega)}+\frac{\sqrt{C_P}}{R_1}\|\partial_r\partial_\theta v^\theta\sqrt{\varphi_L}\|_{L^2(\Omega)};
\end{aligned}
\tag{4.21}
$$

and for the case $\lambda = \theta$,

$$
\left\|\frac{\partial_\theta^2 v^\theta}{r}\sqrt{\varphi_L}\right\|_{L^2(\Omega)} \le \left\|\frac{1}{r}(\partial_\theta^2 v^\theta+\partial_\theta v^r)\sqrt{\varphi_L}\right\|_{L^2(\Omega)}+\frac{\sqrt{C_P}}{R_1}\|\partial_r\partial_\theta v^r\sqrt{\varphi_L}\|_{L^2(\Omega)}. \tag{4.22}
$$

Therefore, combining (4.19)–(4.22), and applying the Schwarz inequality, we deduce

$$
\begin{aligned}
|II_1^r| \le{}& \frac{\|\boldsymbol{v}\|_{L^\infty(\Omega)}\sqrt{C_P}}{2}\left\{2\|\partial_r\partial_\theta v^r\sqrt{\varphi_L}\|^2_{L^2(\Omega)}+\left(1+\frac{2C_P}{R_1^2}\right)\|\partial_r\partial_\theta v^\theta\sqrt{\varphi_L}\|^2_{L^2(\Omega)}\right.\\
&\left.+\|\partial_r\partial_\theta v^z\sqrt{\varphi_L}\|^2_{L^2(\Omega)}+2\left\|\frac{1}{r}(\partial_\theta^2 v^r-\partial_\theta v^\theta)\sqrt{\varphi_L}\right\|^2_{L^2(\Omega)}+\|\partial_z\partial_\theta v^r\sqrt{\varphi_L}\|^2_{L^2(\Omega)}\right\},
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
|II_1^\theta| \le{}& \frac{\|\boldsymbol{v}\|_{L^\infty(\Omega)}\sqrt{C_P}}{2}\left\{\left(1+\frac{2C_P}{R_1^2}\right)\|\partial_r\partial_\theta v^r\sqrt{\varphi_L}\|^2_{L^2(\Omega)}+2\|\partial_r\partial_\theta v^\theta\sqrt{\varphi_L}\|^2_{L^2(\Omega)}\right.\\
&\left.+\|\partial_r\ \partial_\theta v^z\sqrt{\varphi_L}\|^2_{L^2(\Omega)}+2\left\|\frac{1}{r}(\partial_\theta^2 v^\theta+\partial_r v^\theta)\sqrt{\varphi_L}\right\|^2_{L^2(\Omega)}+\|\partial_z\partial_\theta v^\theta\sqrt{\varphi_L}\|^2_{L^2(\Omega)}\right\},
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
|II_1^z| \le{}& \frac{\|\boldsymbol{v}\|_{L^\infty(\Omega)}\sqrt{C_P}}{2}\left\{\|\partial_r\partial_\theta v^r\sqrt{\varphi_L}\|^2_{L^2(\Omega)}+\|\partial_r\partial_\theta v^\theta\sqrt{\varphi_L}\|^2_{L^2(\Omega)}\right.\\
&\left.+2\|\partial_r\partial_\theta v^z\sqrt{\varphi_L}\|^2_{L^2(\Omega)}+\left\|\frac{1}{r}\partial_\theta^2 v^z\sqrt{\varphi_L}\right\|^2_{L^2(\Omega)}+\|\partial_z\partial_\theta v^z\sqrt{\varphi_L}\|^2_{L^2(\Omega)}\right\}.
\end{aligned}
\tag{4.25}
$$

Finally, adding the above estimates, we conclude that term $II_1 := II_1^r + II_1^\theta + II_1^z$ satisfies the estimate

$$|II_1| \le \|\boldsymbol{v}\|_{L^\infty(\Omega)} \frac{C_{II}}{\nu} Y(L), \tag{4.26}$$

with

$$C_{II} := \sqrt{C_P}\left(2 + \frac{C_P}{R_1^2}\right). \tag{4.27}$$

By lemma 2.1 and the Poincáre inequality in Σ_L, we see that the remainder term II_2^λ can be estimated as

$$\begin{aligned} |II_2^\lambda| &\le \|\partial_\theta \boldsymbol{v}\|_{L^2(\Sigma_L)}^2 \|\boldsymbol{v}\|_{L^\infty(\Sigma_L)} \\ &\le \|\partial_\theta \boldsymbol{v}\|_{L^\infty(\Omega)} |\Sigma_L|^{1/2} \|\boldsymbol{v}\|_{L^\infty(\Omega)} \|\partial_\theta \boldsymbol{v}\|_{L^2(\Sigma_L)} \\ &\le C\|\partial_r \partial_\theta \boldsymbol{v}\|_{L^2(\Sigma_L)} \\ &\le C\sqrt{Y'(L)}, \end{aligned} \tag{4.28}$$

for $\lambda = r, \theta, z$, where C is a constant independent of L.

Thirdly, we consider the nonlinear term III. Integration by parts and the divergence-free condition lead to

$$III^\lambda = \int_\Omega (\boldsymbol{v}\cdot\nabla)\left(\frac{1}{2}|\partial_\theta v^\lambda|^2\right)\varphi_L(z)\,\mathrm{d}x = -\frac{1}{2}\int_\Omega v^z |\partial_\theta v^\lambda|^2 \partial_z \varphi_L(z)\,\mathrm{d}x, \tag{4.29}$$

for $\lambda = r, \theta, z$. Moreover, similarly to (4.28) we have that

$$|III^\lambda| \le C\|\partial_\theta \boldsymbol{v}\|_{L^2(\Sigma_L)}^2 \|\boldsymbol{v}\|_{L^\infty(\Sigma_L)} \le C\sqrt{Y'(L)}, \tag{4.30}$$

for $\lambda = r, \theta, z$, where C is a constant independent of L. Next, we treat the remaining nonlinear term IV. By lemma 2.2 and the Schwarz inequality, we have

$$\begin{aligned} |IV| &\le R_1^{-1}(\|\partial_\theta v^r \sqrt{\varphi_L}\|_{L^2(\Omega)} + \|\partial_\theta v^\theta \sqrt{\varphi_L}\|_{L^2(\Omega)}) \\ &\quad \times \|\partial_\theta v^\theta \sqrt{\varphi_L}\|_{L^2(\Omega)} \|\boldsymbol{v}\|_{L^\infty(\Omega)} \\ &\le \|\boldsymbol{v}\|_{L^\infty(\Omega)} \frac{C_P}{R_1}\left(\frac{1}{2}\|\partial_r\partial_\theta v^r \sqrt{\varphi_L}\|_{L^2(\Omega)}^2 + \frac{3}{2}\|\partial_r\partial_\theta v^\theta \sqrt{\varphi_L}\|_{L^2(\Omega)}^2\right) \\ &\le \|\boldsymbol{v}\|_{L^\infty(\Omega)} \frac{C_{IV}}{\nu} Y(L), \end{aligned} \tag{4.31}$$

where

$$C_{IV} := \frac{3C_P}{2R_1}. \tag{4.32}$$

Finally, we estimate the pressure terms V. Integration by parts and the divergence-free condition lead to

$$V = -\int_\Omega \partial_\theta p \partial_\theta v^z \partial_z \varphi_L(z)\,\mathrm{d}x. \tag{4.33}$$

Therefore, from lemma 2.1 and the Poincáre inequality in Σ_L, we obtain

$$|V| \le C\|\partial_\theta p\|_{L^2(\Sigma_L)} \|\partial_\theta v^z\|_{L^2(\Sigma_L)} \le C\|\partial_r\partial_\theta v^z\|_{L^2(\Sigma_L)} \le C\sqrt{Y'(L)}. \tag{4.34}$$

Now we put the estimates (4.15), (4.16), (4.17), (4.26), (4.28), (4.30), (4.31) and (4.34) together into (4.3) and conclude that

$$Y(L) \le \|\boldsymbol{v}\|_{L^\infty(\Omega)} \frac{C_{II} + C_{IV}}{\nu} Y(L) + C\sqrt{Y'(L)}. \tag{4.35}$$

We define

$$C_2(\nu, R_1, R_2) := \left[\frac{C_{II} + C_{IV}}{\nu}\right]^{-1}. \tag{4.36}$$

Then, by the assumption $\|\boldsymbol{v}\|_{L^\infty(\Omega)} < C_2(\nu, R_1, R_2)$, we reach the differential inequality

$$Y(L) \le C\sqrt{Y'(L)}. \tag{4.37}$$

Therefore, with completely the same argument as in the previous section, we have $Y(L) = 0$ for all $L > 1$. This implies that $\partial_r \partial_\theta \boldsymbol{v} \equiv 0$, that is, $\partial_\theta \boldsymbol{v}$ is independent of $r \in [R_1, R_2]$. However, the boundary condition requires $\partial_\theta \boldsymbol{v}(R_j, \theta, z) = 0$ $(j = 1, 2)$ for any $(\theta, z) \in [0, 2\pi] \times \mathbb{R}$. Thus, $\partial_\theta \boldsymbol{v}$ must be identically zero. Then, by (4.1), we see that $\partial_r \partial_\theta p = \partial_\theta^2 p = \partial_z \partial_\theta p = 0$, that is, $\partial_\theta p \equiv c$ in Ω with some constant $c \in \mathbb{R}$. However, since p must be periodic in θ, we conclude $c = 0$, that is, $\partial_\theta p \equiv 0$. This completes the proof of theorem 1.2.

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Author ORCIDs.
Hideo Kozono https://orcid.org/0000-0002-9196-6526.

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