97.11 The rational distance problem for polygons

Introduction

For \( n \geq 3 \), let \( P_n \) denote a regular \( n \)-gon with rational side-length. Attractive questions are:

(P1) If \( U, V, W \) are 3 consecutive vertices of \( P_n \), do there exist points in the plane of \( P_n \), other than \( V \), that are at rational distances from \( U, V, \) and \( W \)?

(P2) Do there exist points in the plane of \( P_n \), that are at rational distances from the \( n \) vertices of \( P_n \)?

In this note, we provide a complete answer to both problems, except for \( n = 4 \) in (P2).

First, notice the following:

- Both problems are invariant by a rational rescaling. Hence, without loss of generality, we may assume that the common side-length of \( P_n \) is 1.
- For a given \( n \geq 3 \), if (P1) has a negative answer, then (P2) has a negative answer as well.

The main results are the following:

**Theorem 1**: Let \( n \geq 3 \). The answer to (P1) is positive for \( n \in \{3, 4, 6, 12\} \) and negative in all other cases.

**Theorem 2**: Let \( n \geq 3, n \neq 4 \). The answer to (P2) is positive for \( n \in \{3, 6\} \) and negative in all other cases.

We first need some simple preliminaries.

The letters \( \mathbb{Q}, \mathbb{Q}^+, \mathbb{R} \) denote respectively the set of rational, non-negative rational, and real numbers.

**Lemma 1**: Let \( a, \beta, \gamma \in \mathbb{Q}, \gamma \neq 0 \). Then \( x = (a\sqrt{5} + \beta)\sqrt{\gamma} \) is not a root of the equation \( x^4 - 5x^2 + 5 = 0 \).

**Proof**: Otherwise, \( y = x^2 \) would be a root of \( y^2 - 5y + 5 = 0 \), so \( x^2 = \frac{5}{2} \pm \frac{1}{2}\sqrt{5} \), leading to \( 5a^2 + \beta^2 = \frac{5}{2} \) and \( 2a\beta\gamma = \pm\frac{1}{2} \), so \( a\beta\gamma \neq 0 \).

Eliminating \( \beta^2 = \frac{1}{16\gamma^2a^2} \) in the first equation yields \( 80\gamma^2a^4 + 1 = 40\gamma a^2 \).

Therefore, \( t = 4\gamma a^2 \) would be a rational root of the equation in \( t \), \( 5t^2 - 10t + 1 = 0 \), a contradiction since \( \Delta = 4\sqrt{5} \notin \mathbb{Q} \), where \( \Delta \) is the discriminant.

Next, as usual, let \( \phi \) denote Euler's function. If \( n \) is a positive integer, \( \phi(n) \) counts the number of integers \( m, 1 \leq m \leq n \), such that \( m \) is relatively prime to \( n \).

**Lemma 2**: Let \( n \in \{7, 9, 11\} \) or \( n > 12 \). Then, \( \phi(n) \geq 6 \).
Proof: The result follows from the elementary property (left as an exercise): the solution sets of the equations \( \phi(n) = k \), \( k = 1, 2, 3, 4, 5 \) are respectively \( \{1, 2\}, \{3, 4, 6\}, \phi, \{5, 8, 10, 12\} \) and \( \phi \).

Finally, recall that a real number \( \theta \) is called algebraic if \( \theta \) is a zero of rational polynomials with positive degree; the least such degree is called the algebraic degree of \( \theta \) and is denoted by \( [\theta : \mathbb{Q}] \). For example, \( [\sqrt{5} : \mathbb{Q}] = 2 \) and \( [\sqrt{10} : \mathbb{Q}] = 3 \). According to this definition, the following is obvious:

If a real number \( \theta \) is a zero of some rational polynomial with degree \( n_0 \geq 0 \), then \( \theta \) is algebraic and

\[
[\theta : \mathbb{Q}] \leq n_0.
\]

Results for isosceles triangles with rational common side

Let \( \nabla \) denote an isosceles triangle with side-lengths 1, 1, \( x \). It is known that if \( x \) or \( x^2 \in \mathbb{Q} \), then there exist infinitely many points in the plane of \( \nabla \) that are at rational distance from the 3 vertices of \( \nabla \), (see [1] and [2]). Here we rather need a characterisation for arbitrary \( x \).

Proposition 1: Let \( AOB \) be a non-degenerate triangle with \( OA = OB = 1 \) and \( \omega = \angle AOB \), \( (0 < \omega < \pi) \). Set \( a = -\cos\omega \), \( b = \sin\omega \), \( (b > 0) \). Then, the following statements are equivalent:

(i) There is a point \( M \) in the plane of \( AOB \), \( M \neq O \), such that \( MA, MO, MB \) are all rational.

(ii) \( \exists q \in \mathbb{R}, \exists R, S, T \in \mathbb{Q}^+, R > 0 \), such that, if \( 2p = R^2 - S^2 + 1 \), \( 2\lambda = R^2 - T^2 + 1 \), \( (p, \lambda \in \mathbb{Q}) \), we have

\[
p^2 + q^2 = R^2 \quad (2)
\]

and

\[
qb - pa = \lambda. \quad (3)
\]

Proof: Consider the \( x-y \) axes with origin \( O \), such that \( A = (1, 0) \) and \( B = (-a, b) \). (ii) \( \Rightarrow \) (i) Define \( M = (p, q) \). We have

\[
MO^2 = p^2 + q^2 = R^2, \quad \text{so} \quad MO = R \in \mathbb{Q}. \quad \text{Further,} \quad M \neq O \quad \text{as} \quad R > 0.
\]

Next, \( MA^2 = (p - 1)^2 + q^2 = R^2 - 1 - 2p = R^2 + 1 - (R^2 - S^2 + 1) = S^2 \).

Hence, \( MA = S \in \mathbb{Q} \). Finally,

\[
MB^2 = (p + a)^2 + (q - b)^2 = (p^2 + q^2) + (a^2 + b^2) + 2pa - 2qb = R^2 + 1 + 2pa - 2(\lambda + \lambda) = R^2 + 1 - (R^2 - T^2 + 1) = T^2.
\]

Hence, \( MB = T \in \mathbb{Q} \).

(i) \( \Rightarrow \) (ii) Let \( M = (p, q), M \neq O \), be a point in the plane of \( AOB \) such that \( MO = R, MA = S, MB = T \) are all rational. As \( M \neq O \), then \( R > 0 \).

The Pythagorean relations provide

\[
p^2 + q^2 = R^2, \quad (p - 1)^2 + q^2 = S^2, \quad (p + a)^2 + (q - b)^2 = T^2.
\]
From the first two relations we get $2p = R^2 - S^2 + 1$ (so $p \in \mathbb{Q}$). Expanding the last relation yields

$$(p^2 + q^2) + (a^2 + b^2) + 2pa - 2qb = R^2 + 1 + 2pa - 2qb = T^2.$$ 

By setting $2\lambda = R^2 - T^2 + 1$ (so $\lambda \in \mathbb{Q}$), we obtain $qb - pa = \lambda$.

As a corollary of Proposition 1, we obtain the *useful* elegant sufficient condition:

**Proposition 2:** With the hypothesis of Proposition 1, suppose that the equation

$$b = (aa + \beta)\sqrt{\gamma}$$

is insolvable with $\alpha, \beta, \gamma \in \mathbb{Q}, \gamma > 0$. Then, there is no point $M$ in the plane of $AOB, M \neq O$, such that $MA, MO, MB$ are all rational.

**Proof:** Otherwise, let $p, q, R, \lambda, \ldots$ ($R > 0$) be as in Proposition 1, with $p^2 + q^2 = R^2$ and $qB - pa = \lambda$. Set $\gamma = q^2 = R^2 - p^2$. Then, $\gamma \in \mathbb{Q}^+$ and $q = \pm\sqrt{\gamma}$.

Case 1: $q \neq 0$. We may write $b = \frac{pa + \lambda}{q} = \frac{pa + \lambda}{q^2} = \frac{(pa + \lambda)}{\gamma}(\pm\sqrt{\gamma})$.

Now clearly, $b$ has the form $b = (aa + \beta)\sqrt{\gamma}$ with $a, \beta, \gamma \in \mathbb{Q}, \gamma > 0$, a contradiction.

Case 2: $q = 0$. Then, $p^2 = R^2$, so $p = \pm R$ and $p \neq 0$ (as $R > 0$). From $0 = qb - pa + \lambda$, we get $a = -\lambda/p \in \mathbb{Q}$. Therefore, if we set $\mu = b^2 = 1 - a^2$, we see that $\mu \in \mathbb{Q}^+$. In fact $\mu > 0$ since $\mu = b^2$ and $b = \sin \omega > 0$. Now, $b = \sqrt{\mu} = (0.a + 1)\sqrt{\mu}$, with $0, 1, \mu \in \mathbb{Q}$, $\mu > 0$, a contradiction.

**Proof of Theorem 1**

Case $n = 3$: Let $ABC$ be a unit equilateral triangle. Disregarding the vertices, the point $M \in AB$ such that $MA = \frac{1}{3}$ satisfies $MA = \frac{1}{8}, MB = \frac{5}{8}$ and $MC = \frac{7}{8}$. In fact, as proved in [1] and [2], the suitable points $M$ are infinite in number.

Case $n = 4$: Let $ABCD$ be a unit square. The point $M_0 = B$ satisfies $M_0A = 1, M_0B = 0, M_0C = 1$. The point $M \in AB$ such that $MA = \frac{1}{4}$ satisfies $MA = \frac{1}{4}, MB = \frac{3}{4}$ and $MC = \frac{7}{8}$. In fact, since $ABC$ has side-lengths 1, 1, $\sqrt{2}$, the suitable points $M$ are infinite in number, as proved in [2].

Case $n = 6$: The centroid of $P_6 = A_1A_2 \ldots A_6$ (unitary) is at distance 1 from $A_1, A_2, A_3$. The point $M$ is on $A_1A_2$ produced such that $A_1M = 2A_1A_2$ and so satisfies $MA_1 = 2, MA_2 = 1, MA_3 = 1$, etc. In fact, since $A_1A_2A_3$ has side-lengths 1, 1, $\sqrt{3}$, the suitable points $M$ are infinite.
Case \( n = 12 \): Let \( P_{12} = A_1A_2 \ldots A_{12} \) be a unit regular 12-gon. The point \( M \) on \( A_2A_9 \) such that \( A_2M = \frac{1}{\sqrt{2}} \) satisfies \( MA_1 = \frac{13}{12} \), \( MA_2 = \frac{1}{\sqrt{2}} \), \( MA_3 = \frac{1}{\sqrt{2}} \). Other points can be found. An attempt to find all the suitable points \( M \) is possible, but this is another question.

Case \( n = 8 \): The answer to (P1) is negative. Otherwise, let \( p, q, R, \lambda, \ldots \) \((R > 0)\) be as in Proposition 1, with \( p^2 + q^2 = R^2 \) (2) and \( q\beta - pa = \lambda \) (3). Here, \( \omega = \frac{3}{4} \pi \), so \( a = \beta = \frac{1}{\sqrt{2}} \). Hence by (3),

\[
q = p + \lambda \sqrt{2}.
\]

(5)

(2) and (5) yield \( p^2 + (p + \lambda \sqrt{2})^2 = R^2 \) hence \( 2p\lambda \sqrt{2} = R^2 - 2p^2 - 2\lambda^2 \in \mathbb{Q} \).

This requires \( p\lambda = 0 \). If \( p = 0 \), then by (5), \( q = \lambda \sqrt{2} \), so by (2), \( 2\lambda^2 = R^2 \). Since \( R > 0 \), then \( \lambda \neq 0 \) and hence \( 2 = (R/\lambda)^2 \), where \( \frac{R}{\lambda} \in \mathbb{Q} \) a contradiction.

If \( \lambda = 0 \), then by (5), \( q = p \), so by (2), \( 2p^2 = R^2 \), leading to the same contradiction.

Case \( n \in \{5, 10\} \): The answer to (P1) is negative. By virtue of Proposition 2, we only need to show that the equation \( b = (aa + \beta)\sqrt{\gamma} \), or equivalently,

\[
2b = (aa + \beta)\sqrt{\gamma}
\]

is insolvable with \( a, \beta, \gamma \in \mathbb{Q}, \gamma > 0 \).

For \( n = 5 \), \( \omega = \frac{3}{4} \pi \), so \( a = \frac{\sqrt{5} - 1}{4} \) and \( 2b = \sqrt{\frac{5 + \sqrt{5}}{2}} \). Observe that \( 2b \) is a root of \( x^4 - 5x^2 + 5 = 0 \), while \( (aa + \beta)\sqrt{\gamma} = \left(\frac{\sqrt{5} - 1}{4} + \beta\right)\sqrt{\gamma} \), which clearly has the form \( (A\sqrt{5} + B)\sqrt{\gamma} \), \( A, B, \gamma \in \mathbb{Q}, \gamma > 0 \), is not a root of this equation by Lemma 1. We conclude that relation (6) never holds.

For \( n = 10 \), the proof is similar with \( \omega = \frac{3}{4} \pi \), \( a = \frac{\sqrt{5} + 1}{4} \) and

\[
2b = \sqrt{\frac{5 - \sqrt{5}}{2}}.
\]

Finally, Case \( n \in \{7, 9, 11\} \) or \( n > 12 \): The answer to (P1) is negative. By Lemma 2, \( \varphi(n) \geq 6 \), so

\[
\frac{1}{n} \varphi(n) \geq 3.
\]

By virtue of Proposition 2, we only need to show that equation (4)

\[
b = (aa + \beta)\sqrt{\gamma} \]

is insolvable with \( a, \beta, \gamma \in \mathbb{Q}, \gamma > 0 \). Here,

\[
a = -\cos \left(\frac{(n - 2)\pi}{n}\right) = \cos \frac{2\pi}{n} \quad \text{and} \quad b = \sin \left(\frac{(n - 2)\pi}{n}\right) = \sin \frac{2\pi}{n} \quad (b > 0)
\]
as \( n \geq 7 \). For the purpose of contradiction, suppose that (4) holds. Then,

\[
b^2 = \gamma(\alpha^2 + \beta)^2, \]

that is, \( 1 - a^2 = \gamma a^2 + \beta^2 + 2\alpha\beta a \). Hence, \( a \) is a
zero of the rational polynomial \( f(X) = (1 + \gamma \alpha^2)X^2 + (2a \beta \gamma)X + (\gamma \beta^2 - 1) \). Since \( 1 + \gamma \alpha^2 > 1 > 0 \), this polynomial has degree 2, and hence by (1), \( \alpha \) is algebraic and

\[
[a : \mathbb{Q}] < 2. \tag{8}
\]

But, there is a general result relative to the number \( a = \cos(2\pi / n) \) (see [3]):

- The number \( a = \cos(2\pi / n) \) is algebraic and for \( n \geq 3 \),

\[
\left[ \cos \frac{2\pi}{n} : \mathbb{Q} \right] = \frac{1}{2} \varphi(n). \tag{9}
\]

Combining (9) and (7) we obtain \([a : \mathbb{Q}] = [\cos \frac{2\pi}{n} : \mathbb{Q}] = \frac{1}{2} \varphi(n) \geq 3\), in contradiction to (8).

**Proof of Theorem 2**

Let \( n \geq 3 \), \( n \neq 4 \). If \( n \neq 3, 6, 12 \), the answer to (P2) is negative since, by Theorem 1, it is negative for (P1).

For \( n \in \{3, 6\} \), the answer to (P2) is clearly positive (for \( n = 3 \), any vertex will do, and for \( n = 6 \), the centroid of \( P_6 \) will do). It remains only to prove that the answer to (P2) is negative for \( n = 12 \).

**Lemma 3:** Let \( P_{12} \) be a regular 12-gon with unit side. Then, if \( A, O, B \) denote 3 consecutive vertices of \( P_{12} \), the points that are at rational distance from \( A, O \) and \( B \), all lie in \( L \cup L' \), where the line \( L \) (respectively \( L' \)) denote the perpendicular at \( O \) to \( OA \) (respectively \( OB \)).

**Proof:** We refer to Proposition 1 and its proof. Consider the \( x-y \) axes with origin \( O \), such that \( A = (1, 0) \) and \( B = (-a, b) \), with \( a = -\cos(5\pi/6) = \frac{1}{2} \sqrt{3} \) and \( b = \sin(5\pi/6) = \frac{1}{2} \). Let \( M = (p, q) \) be a point \( \neq O \) such that \( MO = R \) (\( R > 0 \)), \( MA = S \) and \( MB = T \) are all rational (recall \( p \in \mathbb{Q} \)). With \( 2\lambda = R^2 - T^2 + 1 \), we have \( qb - pa = \lambda \), that is, \( \frac{1}{2}q - \frac{1}{2}\sqrt{3}p = \lambda \), or, \( q = p\sqrt{3} + 2\lambda (p, \lambda \in \mathbb{Q}) \). Squaring provides \( 4p\lambda \sqrt{3} = q^2 - 3p^2 - 4\lambda^2 \in \mathbb{Q} \). This requires \( p\lambda = 0 \). If \( p = 0 \), then \( M = (0, q) \) lies on the \( y \)-axis, which is the perpendicular line at \( O \) to \( OA \). If \( \lambda = 0 \), then \( q = p\sqrt{3} \). Hence, the point \( M = (p, q) \) lies on the line \( y = \sqrt{3}x \), which is precisely the perpendicular line at \( O \) to \( OB \).

Finally we prove that the answer to (P1) is negative for \( n = 12 \). More precisely, let \( P_{12} = A_1A_2A_3 \ldots A_{12} \) denote a regular 12-gon with unit side. Then, no point in the plane of \( P_{12} \) is at rational distance from 5 consecutive vertices of \( P_{12} \). Indeed, for the purpose of contradiction, suppose that some point \( M \) in the plane of \( P_{12} \) is at rational distance from \( A_{12}, A_1, A_2, A_3, A_4 \), say.

Consider first the vertices \( A_{12}, A_1, A_2 \). Check that the perpendicular line at \( A_1 \) to \( A_1A_{12} \), respectively to \( A_1A_2 \), is precisely the line \( A_1A_6 \), respectively the line \( A_1A_8 \) (see figure below). Hence by Lemma 3,

\[
M \in X = A_1A_6 \cup A_1A_8.
\]
Similarly, by considering $A_1$, $A_2$, $A_3$ and then $A_2$, $A_3$, $A_4$ we see that

$$M \in Y = A_2A_7 \cup A_2A_9 \quad \text{and} \quad M \in Z = A_3A_8 \cup A_3A_{10}.$$ 

Therefore, $M \in X \cap Y \cap Z$. But in fact, it is not difficult to notice (see figure above) that $X \cap Y \cap Z$ is an empty set. We obtain a contradiction, and the proof is complete.

**Open Problem:** Do there exist points in the plane of a unitary regular hexagon, other than the centroid, that are at rational distance from the 6 vertices?

**Acknowledgement:** The author would like to thank the referee for several helpful suggestions and comments and for pointing out an appropriate reference.

**References**


ROY BARBARA

PO Box 90, 357 Jdeidet-El-Metn, Lebanon

e-mail: roybarbara.math@gmail.com